Negative Coefficents in Ehrhart Polynomials

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July 10, 2025

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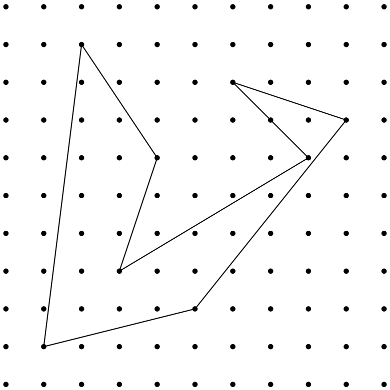
Pick's Theorem



How do we find the area?

Pick's Theorem

Georg Alexander Pick discovered Pick's Theorem in 1899.

Theorem (Pick's Theorem)

Given any convex lattice polygon,

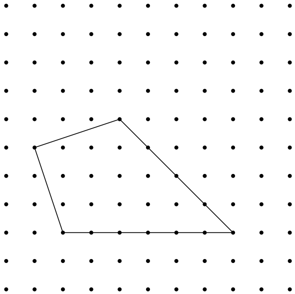
$$A = I + \frac{B}{2} - 1$$

where A is the area of the polygon, I is the number of interior lattice points, and B is the number of points on the border of the polygon.

Pick's Theorem

Example

Will will find the area of the polygon below.



Here, $I = 12$ and $B = 12$, so $A = 12 + 6 - 1 = 17$.

Pick's Theorem

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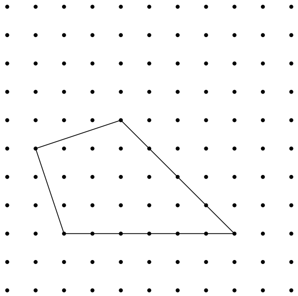


Here, $I = 12$ and $B = 12$, so $A = 12 + 6 - 1 = 17$.

Pick's theorem also works for concave polygons; however, we will focus on convex ones for this talk.

Lattice Polytopes

There are two ways to formally define lattice polytopes. The first is the vertex description; the second is the hyperplane description.

Definition (vertex description)

A *polytope* is the convex hull of finitely many points. More formally, for any polytope $\mathcal{P}$,

$$\mathcal{P} := \text{conv}\{v_1, v_2, \ldots, v_n\} = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n : \lambda_k \geq 0, \sum_{k=1}^{n} \lambda_k \leq 1\}$$

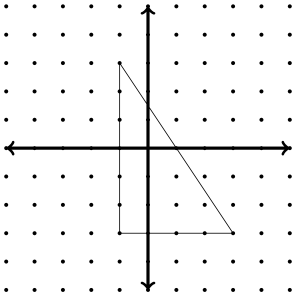
for a finite set of points $\{v_1, v_2, \ldots, v_n\} \subset \mathbb{Z}^d$.

Lattice Polytopes: Vertex Description

To more easily imagine what this means, we will provide an example in the second dimension.

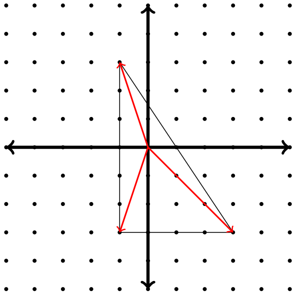
Example

Consider the polygon below.



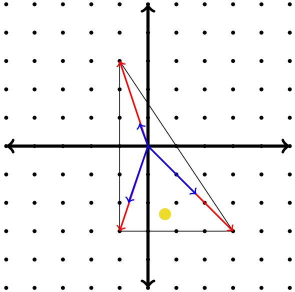
Lattice Polytopes: Vertex Description

Consider the vectors from the origin to each vertex.



Lattice Polytopes: Vertex Description

Here, any point in this triangle can be described as the sum of some fractions of these vectors. For example, the point shown below is the sum of the blue vectors.



These fractions' sum is at most 1; notice that the points that have sums equal to 1 are the ones on the border of the polygon.

Lattice Polytopes: Hyperplane Description

Definition

A *hyperplane* is a generalization of the plane to higher dimensions. In other words, it is a $(d-1)$-dimension subspace within a d-dimension space. Formally,

$$H := \{\mathbf{x} \in \mathbb{Z}^d : \mathbf{a} \cdot \mathbf{x} = b\}$$

for some $\mathbf{a} \in \mathbb{Z}^d$ and constant b.

Lattice Polytopes: Hyperplane Description

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for some $\mathbf{a} \in \mathbb{Z}^d$ and constant b.

Definition

A *half-space* $\mathcal{H} \in \mathbb{Z}^d$ is the part of a d-dimensional space that lies on a given side of a $(d-1)$-dimensional hyperplane. More formally,

$$\mathcal{H} := \{\mathbf{x} \in \mathbb{Z}^d : \mathbf{a} \cdot \mathbf{x} \geq b\} \text{ or } \{\mathbf{x} \in \mathbb{Z}^d : \mathbf{a} \cdot \mathbf{x} \leq b\}$$

for some $\mathbf{a} \in \mathbb{Z}^d$ and constant b.

Lattice Polytopes: Hyperplane Description

Definition (hyperplane description)

A *polytope* $\mathcal{P} \subset \mathbb{Z}^d$ is the intersection of a finite number of d-dimensional half-spaces and $(d-1)$-dimensional hyperplanes.

Lattice Polytopes: Hyperplane Description

Definition (hyperplane description)

A *polytope* $\mathcal{P} \subset \mathbb{Z}^d$ is the intersection of a finite number of d-dimensional half-spaces and $(d-1)$-dimensional hyperplanes.

Definition

The *t-th dilate* of a polytope $\mathcal{P}$ is denoted as $t\mathcal{P}$, and refers to scaling $\mathcal{P}$ up by a factor of t. More formally,

$$\begin{aligned} t\mathcal{P} &= \{(tx_1, tx_2, \ldots, tx_d) : (x_1, x_2, \ldots, x_d) \in \mathcal{P}\} \\ &= \{t\mathbf{x} : \mathbf{x} \in \mathcal{P}\}. \end{aligned}$$

Ehrhart polynomials

The core idea of Ehrhart theory is the *lattice-point enumerator*, which, similar to Pick's Theorem (Theorem 1), counts the number of lattice points within a polytope. However, Ehrhart polynomials count the number of lattice points within the t-th dilate of the polytope.

Ehrhart polynomials

The core idea of Ehrhart theory is the *lattice-point enumerator*, which, similar to Pick's Theorem (Theorem 1), counts the number of lattice points within a polytope. However, Ehrhart polynomials count the number of lattice points within the t-th dilate of the polytope.

Definition

The *lattice-point enumerator* is defined as

$$L_{\mathcal{P}}(t) = \left| t\mathcal{P} \cap \mathbb{Z}^d \right|.$$

It is sometimes denoted as $i(\mathcal{P}, t)$ instead of $L_{\mathcal{P}}(t)$.

Ehrhart polynomials

It turns out that, for every polytope, this value is a rational polynomial, which Ehrhart proved in 1962. Hence, the lattice-point enumerator is sometimes also referred to as the *Ehrhart polynomial*. This is especially surprising, as there is no reason for it to be a polynomial.

Theorem (Ehrhart's theorem)

The Ehrhart polynomial of a convex lattice polytope $\mathcal{P}$ in dimension d is a rational polynomial of degree d.

Coefficients

- The leading coefficient is the area, volume, or hypervolume of the polytope, depending on its dimension.
- The second coefficient is half the sum of the volumes of each facet (higher-dimensional generalization of face).
- The constant term of Ehrhart polynomials is always 1. However, we do not know anything more about other coefficients of Ehrhart polynomials.

The Existence of Negative Coefficients in Ehrhart Polynomials

Theorem

For any $d \geq 4$, there exists a convex lattice polytope $\mathcal{P}$ whose coefficients are negative except for the coefficients of t^d and t^{d-1}.

The main idea of this proof is finding an Ehrhart polynomial that has negative coefficients.

The Existence of Negative Coefficients in Ehrhart Polynomials

Lemma

For two Ehrhart polynomials $L_{\mathcal{P}}(t)$ and $L_{\mathcal{Q}}(t)$ in dimensions d_1 and d_2, respectively, where $\mathcal{P}$ and $\mathcal{Q}$ are convex integral polytopes, there exists a convex lattice polytope of dimension $d_1 + d_2$ with Ehrhart polynomial $L_{\mathcal{P}}(t) \cdot L_{\mathcal{Q}}(t)$.

Definition

A *cartesian product* of two polytopes $\mathcal{P} = \{(x_1, x_2, \ldots, x_{d_1}) \in \mathbb{Z}^{d_1}\}$ in dimension d_1 and $\mathcal{Q} = \{(y_1, y_2, \ldots, y_{d_2}) \in \mathbb{Z}^{d_2}\}$ is

$$\mathcal{P}_1 \times \mathcal{P}_2 = \{(x_1, x_2, \ldots, x_{d_1}, y_1, y_2, \ldots, y_{d_1})\}.$$

It is well known that the cardinality of the cartesian product is the product of the cardinality of each polytope.

The Existence of Negative Coefficients in Ehrhart Polynomials

Proof.

We have two points $(a_1, a_2, \ldots, a_{d_1}) \in \mathcal{P}, (b_1, b_2, \ldots, b_{d_2}) \in \mathcal{Q}$ if and only if we can conclude that $(a_1, a_2, \ldots, a_{d_1}, b_1, b_2, \ldots, b_{d_2})$ is in $\mathcal{P} \times \mathcal{Q}$. Therefore, the Ehrhart polynomial of $\mathcal{P} \times \mathcal{Q}$ is

$$L_{\mathcal{P} \times Q}(t) = L_{\mathcal{P}}(t) \cdot L_{\mathcal{Q}}(t).$$

□

The Existence of Negative Coefficients in Ehrhart Polynomials

Let

$$I_m = \{\alpha \in \mathbb{R} : 0 \leq \alpha \leq m, m \in \mathbb{N}\}.$$

Then, I_m is a convex lattice polytope in dimension 1. Hence $L_{I_m}(t) = mt + 1$.

There exists a convex lattice polygon $\mathcal{Q}_m$ in dimension 3 whose Ehrhart polynomial is

$$L_{\mathcal{Q}_m} = \frac{m}{6}t^3 + t^2 + \frac{-m+12}{6}t + 1$$

for any $m \in \mathbb{N}$.

The Existence of Negative Coefficients in Ehrhart Polynomials

We can conclude that there exists an integral convex polytope $\mathcal{P}_m$ in the dth ($d \geq 4$) dimension with Ehrhart polynomial

$$L_{P_m}(t) = (L_{I_{d-3}}(t))^{d-3} L_{\mathcal{Q}_m}(t)$$

by using the previous two polynomials and applying the lemma multiple times. After we plug in their values,

$$L_{\mathcal{P}_m}(t) = ((d-3)t+1)^{d-3} \cdot \left(\frac{m}{6} t^3 + t^2 + \frac{-m+12}{6} t + 1 \right).$$

The Existence of Negative Coefficients in Ehrhart Polynomials

When $A_i = (d-3)^i \binom{d-3}{i}$ for $0 \le i \le d-2$, we can expand $((d-3)t+1)^3$ in $L_{\mathcal{P}_m}$ using the binomial theorem:

$$\sum_{i=0}^{d-3} \binom{d-3}{i} ((d-3)t)^i = A_0 + A_1 t + A_2 t + \cdots + A_{d-3} t^{d-3}.$$

This means that

$$L_{\mathcal{P}_m}(t) = \left(A_0 + A_1 t + A_2 t + \cdots + A_{d-3} t^{d-3}\right)$$

$$\times \left(\frac{m}{6} t^3 + t^2 + \frac{-m+12}{6} t + 1\right).$$

The Existence of Negative Coefficients in Ehrhart Polynomials

Then, let $L_{\mathcal{P}_m}(t)$ be $\sum_{i=0}^{d} c_i^{(d,m)} t^i$, where each $c_i^{(d,m)}$ is a rational number. We can expand the above equation to find that $c_1^{(d,m)} = \frac{-m+12}{6} + A_1$, $c_2^{d,m} = 1 + \frac{-m+12}{6} \cdot A_1 + A_2$, and in general,

$$c_j^{(d,m)} = \frac{m}{6} A_{j-3} + A_{j-2} + \frac{-m+12}{6} \cdot A_{j-1} + A_j$$

for $3 \leq j \leq d-2$. With sufficiently large m, we have that $c_1^{(d,m)}$ is negative. Similarly, $c_2^{(d,m)}$ is negative for a large m.

The Existence of Negative Coefficients in Ehrhart Polynomials

For $c_j^{(d,m)}$ in general,

$$\begin{aligned} c_j^{(d,m)} &= \frac{m}{6} A_{j-3} + A_{j-2} + \frac{-m+12}{6} \cdot A_{j-1} + A_j \\ &= \frac{m}{6} A_{j-3} + A_{j-2} + \frac{-m}{6} \cdot A_{j-1} + 2A_{j-1} + A_j \\ &= -\frac{A_{j-1} - A_{j-3}}{6} \cdot m + A_{j-2} + 2A_{j-1} + A_j \\ &= -(d-3)^{j-3} \cdot \frac{g(d,j)}{6} \cdot m + A_{j-2} + 2A_{j-1} + A_j, \end{aligned}$$

where $g(d,j) = (d-3)^2 \cdot \binom{d-3}{j-1} - \binom{d-3}{j-3}$.

The Existence of Negative Coefficients in Ehrhart Polynomials

Lemma

When $d \geq 5$ *and* $3 \leq j \leq d-2$,

$$g(d, j) > 0.$$

Proof.

We will proceed by induction.

$$g(d, 3) = (d-3)^2 \cdot \binom{d-3}{2} - 1,$$

and this indeed is greater than 0. □

The Existence of Negative Coefficients in Ehrhart Polynomials

We also know that

$$g(d, d-2) = (d-3)^2 - \binom{d-3}{2},$$

which is also greater than 0. So, we know that the condition given in the problem is true for $j = 3$ and $j = d-2$. We also specifically test $d = 5, 6$ and both of them work. We will now proceed with induction on d, for $d \geq 7$ and $4 \leq j \leq d-3$. Firstly, $(d-3)^2$ can also be expressed as $(d-4)^2 + 2d - 7$. Therefore,

$$g(d, j) = ((d-4)^2 + 2d - 7)\binom{d-3}{j-1} - \binom{d-3}{j-3}.$$

Then, by Pascal's Identity, this is equal to

$$((d-4)^2 + 2d - 7)\left(\binom{d-4}{j-1} + \binom{d-4}{j-2}\right) - \left(\binom{d-4}{j-3} + \binom{d-4}{j-4}\right).$$

The Existence of Negative Coefficients in Ehrhart Polynomials

We can now simplify:

$$
\begin{aligned}
&((d-4)^2+2d-7)\binom{d-4}{j-1}+((d-4)^2+2d-7)\binom{d-4}{j-2} \\
&-\binom{d-4}{j-3}-\binom{d-4}{j-4} \\
=&(d-4)^2\binom{d-4}{j-1}+(2d-7)\binom{d-4}{j-1}+(d-4)^2\binom{d-4}{j-2} \\
&+(2d-7)\binom{d-4}{j-2}-\binom{d-4}{j-3}-\binom{d-4}{j-4} \\
=&g(d-1,j)+g(d-1,j-1)+(2d-7)\binom{d-3}{j-1}.
\end{aligned}
$$

Therefore, $g(d-1,j)+g(d-1,j-1)>0$, and hence, $g(d,j)\geq 0$.

The Existence of Negative Coefficients in Ehrhart Polynomials

Since $g(d, j) > 0$, we know that $c_j^{(d,m)}$ can be negative for a sufficiently large m.
That means that we have successfully found an Ehrhart polynomial that has negative coefficients.

Non-Ehrhart-Positive Polytopes

One family of polytopes that can have negative Ehrhart polynomial coefficients is order polytopes.
To understand order polytopes, we first need to understand what posets are.

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Definition

A poset, or *partially ordered set*, $P = (P, \leq)$ is a set P with a relation $\leq$ on P that is reflexive, transitive, and asymmetric.

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- Transitivity happens when, if $x \leq y$ and $y \leq z$, then $x \leq z$.

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- Reflexivity refers to the condition that $x \leq x$ for all $x \in P$.
- Transitivity happens when, if $x \leq y$ and $y \leq z$, then $x \leq z$.
- Asymmetry refers to the fact that $x \leq y \implies y \leq x$ is false.

Non-Ehrhart-Positive Polytopes

Example

We will show that $(\mathbb{R}, \leq)$ is a poset. To do this, we need to show that $\leq$ is reflexive, transitive, and asymmetric.

- Reflexivity: for all x in $\mathbb{R}$, $x \leq x$.
- Transitivity: for all x, y, and z in $\mathbb{R}$, we have that if $x \leq y$ and $y \leq z$, then $x \leq z$.
- Asymmetry: for all x and y in $\mathbb{R}$, if $x \leq y$ and $y \leq x$, then $x = y$.

Therefore, $(\mathbb{R}, \leq)$ is indeed a poset.

Non-Ehrhart-Positive Polytopes

Definition

An *order polytope* $\mathscr{O}_P$ of a finite poset $(P, \leq_P)$ is the subset of $\mathbb{Z}^P = \{f : P \to \mathbb{Z}\}$ that is defined by

$$0 \leq f(i) \leq 1 \quad \forall i \in P$$

and

$$f(i) \leq f(j) \quad \text{if } i \leq_P j.$$

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and

$$f(i) \leq f(j) \quad \text{if } i \leq_P j.$$

Definition

The *ordinal sum* of two disjoint finite posets is the poset $(P \oplus Q, \leq_{P \oplus Q})$ such that $s \leq_{P \oplus Q} t$ if:

- $s, t \in P$ and $s \leq_P t$,
- $s, t \in Q$ and $s \leq_Q t$, or
- $s \in P$ and $t \in Q$.

Non-Ehrhart-Positive Polytopes

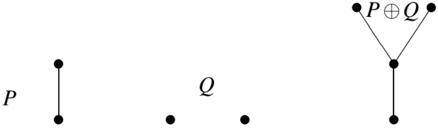

Figure: The ordinal sum of P and Q.

Non-Ehrhart-Positive Polytopes

Theorem

For any positive integer $d \geq 14$, there exists an order polytope that has negative Ehrhart polynomial coefficients.
For $d \leq 11$, any order polytope is Ehrhart-positive, meaning that their Ehrhart polynomials always have negative coefficients.
For $d \geq 21$, there is an order polytope that is non-Ehrhart positive for each d.

Non-Ehrhart Positive Polytopes

Let m, n be two positive integers, and let $P_{m,n}$ be the ordinal sum of P_m and P_n.

P	Ehrhart polynomial
$P_{6,6}$	$1+\frac{75t}{22}+\frac{824t^2}{77}+\frac{160t^3}{7}+\frac{181t^4}{6}+\frac{765t^5}{28}+\frac{127t^6}{7}+9t^7+\frac{93t^8}{28}+\frac{25t^9}{28}+\frac{1}{6}t^{10}+\frac{3t^{11}}{154}+\frac{t^{12}}{924}$
$P_{6,7}$	$1+\frac{61751t}{15015}+\frac{555t^2}{44}+\frac{928t^3}{33}+\frac{83t^4}{2}+\frac{1273t^5}{30}+\frac{255t^6}{8}+\frac{127t^7}{7}+\frac{63t^8}{8}+\frac{31t^9}{12}+\frac{5}{8}t^{10}+\frac{7t^{11}}{66}+\frac{t^{12}}{88}+\frac{t^{13}}{1716}$
$P_{7,7}$	$1-\frac{3041t}{1430}+\frac{18397t^2}{4290}+\frac{1365t^3}{44}+\frac{602t^4}{11}+\frac{301t^5}{5}+\frac{8953t^6}{180}+\frac{255t^7}{8}+\frac{127t^8}{8}+\frac{49t^9}{8}+\frac{217t^{10}}{120}+\frac{35t^{11}}{88}+\frac{49t^{12}}{792}+\frac{7t^{13}}{1144}+\frac{t^{14}}{3432}$
$P_{7,8}$	$1-\frac{1633t}{2145}+\frac{11261t^2}{2860}+\frac{208909t^3}{6435}+\frac{6125t^4}{88}+\frac{14441t^5}{165}+\frac{959t^6}{12}+\frac{5113t^7}{90}+\frac{255t^8}{8}+\frac{127t^9}{9}+\frac{49t^{10}}{10}+\frac{217t^{11}}{165}+\frac{35t^{12}}{132}+\frac{49t^{13}}{1287}+\frac{t^{14}}{286}+\frac{t^{15}}{6435}$
$P_{8,8}$	$1-\frac{9905t}{286}-\frac{81704t^2}{2145}+\frac{18740t^3}{429}+\frac{137692t^4}{1287}+\frac{1358t^5}{11}+\frac{57736t^6}{495}+\frac{1364t^7}{15}+\frac{511t^8}{9}+\frac{85t^9}{3}+\frac{508t^{10}}{45}+\frac{196t^{11}}{55}+\frac{434t^{12}}{495}+\frac{70t^{13}}{429}+\frac{28t^{14}}{1287}+\frac{4t^{15}}{2145}+\frac{t^{16}}{12870}$
$P_{8,9}$	$1-\frac{1063343t}{36465}-\frac{29713t^2}{572}+\frac{126224t^3}{6435}+\frac{17882t^4}{143}+\frac{75956t^5}{429}+\frac{1967t^6}{11}+\frac{24644t^7}{165}+\frac{1023t^8}{10}+\frac{511t^9}{9}+\frac{51t^{10}}{2}+\frac{508t^{11}}{55}+\frac{147t^{12}}{55}+\frac{434t^{13}}{715}+\frac{15t^{14}}{143}+\frac{28t^{15}}{2145}+\frac{3t^{16}}{2860}+\frac{t^{17}}{24310}$
$P_{9,9}$	$1-\frac{1285677t}{4862}-\frac{7364613t^2}{24310}+\frac{89157t^3}{572}+\frac{246946t^4}{715}+\frac{195381t^5}{715}+\frac{173242t^6}{715}+\frac{2439t^7}{11}+\frac{9204t^8}{55}+\frac{1023t^9}{10}+\frac{511t^{10}}{10}+\frac{459t^{11}}{22}+\frac{381t^{12}}{55}+\frac{1323t^{13}}{715}+\frac{279t^{14}}{715}+\frac{9t^{15}}{143}+\frac{21t^{16}}{2860}+\frac{27t^{17}}{48620}+\frac{t^{18}}{48620}$
$P_{9,10}$	$1-\frac{220154521t}{969969}-\frac{20069739t^2}{48620}-\frac{454951t^3}{12155}+\frac{453525t^4}{1144}+\frac{64414t^5}{143}+\frac{548577t^6}{1430}+\frac{1718664t^7}{5005}+\frac{3060t^8}{11}+\frac{12277t^9}{66}+\frac{1023t^{10}}{10}+\frac{511t^{11}}{11}+\frac{765t^{12}}{44}+\frac{762t^{13}}{143}+\frac{189t^{14}}{143}+\frac{186t^{15}}{715}+\frac{45t^{16}}{1144}+\frac{21t^{17}}{4862}+\frac{3t^{18}}{9724}+\frac{t^{19}}{92378}$
$P_{10,10}$	$1-\frac{135276175t}{58786}-\frac{2250043660t^2}{969969}+\frac{4024062t^3}{2431}+\frac{11364453t^4}{4862}+\frac{461265t^5}{572}+\frac{150248t^6}{429}+\frac{445884t^7}{1001}+\frac{426227t^8}{1001}+\frac{6825t^9}{22}+\frac{2047t^{10}}{11}+93t^{11}+\frac{2555t^{12}}{66}+\frac{3825t^{13}}{286}+\frac{3810t^{14}}{1001}+\frac{126t^{15}}{143}+\frac{93t^{16}}{572}+\frac{225t^{17}}{9724}+\frac{35t^{18}}{14586}+\frac{15t^{19}}{92378}+\frac{t^{20}}{184756}$

Non-Ehrhart Positive Polytopes

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- Smooth lattice polytopes (in each dimension starting from $d \geq 3$).

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- Smooth lattice polytopes (in each dimension starting from $d \geq 3$).
- Type-B generalized permutohedra (in each dimension starting from $d \geq 7$.

Non-Ehrhart Positive Polytopes

There are also many other polytopes that have negative Ehrhart polynomial coefficients:

- Smooth lattice polytopes (in each dimension starting from $d \geq 3$).
- Type-B generalized permutohedra (in each dimension starting from $d \geq 7$.
- Chain polytopes, which have the same Ehrhart polynomials as order polytopes.

Thank you!

Any questions?