NEGATIVE COEFFICIENTS IN EHRHART POLYNOMIALS

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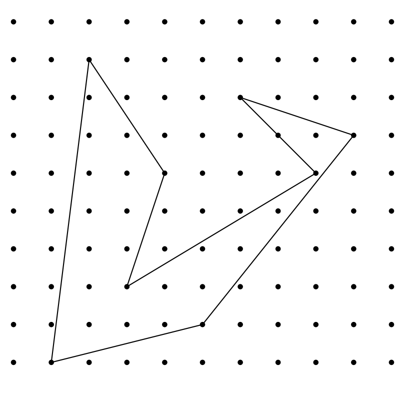
Abstract. In this paper, we will cover the fundamentals of Ehrhart theory, while also discovering polytopes that have negative coefficients in their Ehrhart polynomials. Ehrhart theory is the study of the number of lattice points within integral convex polytopes, and he proved that such values are polynomials. It turns out that not only are these values polynomials, but they can also have negative coefficients as well in sufficiently large dimensions. The study of polytopes with all-positive Ehrhart polynomial coefficients is called Ehrhart positivity; it is still an open field.

1. Introduction

We know how to find the area of a lattice polygon that is not too complex. How do we find the area of a lattice polygon that has irregular angles and is very complex, like the one pictured below?



Georg Alexander Pick decided to study these. In 1899, he discovered Pick's Theorem (Theorem 2.1), which finds the area of a polygon on a lattice plane. This theorem became very well-known and useful. It is a gem, but unfortunately, it only applies to the second dimension.

Eugène Ehrhart hence ventured to find a method that would work in all dimensions—for all polytopes, which are higher-dimensional generalizations a polygon. He decided to continue with a different approach: counting the number of interior lattice points when a polytope is scaled up by a factor of t. This turns out to be a rational polynomial, called the *Lattice Point Enumerator*, otherwise known as the *Ehrhart polynomial*.

The result of Ehrhart's study became its own field: Ehrhart theory. Ehrhart theory is the bridge between combinatorics, algebra, and geometry. There are many notions from combinatorics that are used in Ehrhart theory. For example, in situations where the Ehrhart polynomial is hard to work with, we often work with *Ehrhart series*, the generating function of an Ehrhart polynomial.

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After seeing that all coefficients are positive in the Ehrhart polynomials of polygons, one question proceeds naturally:

Question 1.1. Are all coefficients of every Ehrhart polynomial positive? In other words, is every Ehrhart polynomial Ehrhart-positive?

This question becomes of greater importance, as Ehrhart discovered that the first two coefficients of Ehrhart polynomials are specific properties of a polytope. Therefore, knowing if coefficients can be negative is crucial to understand if any subsequent coefficient can be other properties of polytopes, since properties of most polytopes are positive.

As we will see, the answer to this question is no. Negative coefficients do exist in some polytopes.

Remark 1.1. Ehrhart positivity, which is the study of the sign of coefficients in Ehrhart polynomials, is still an open field. We still do not know all polytopes that are non-Ehrhart-positive.

The goal of this paper is to provide the reader with a fundamental understanding of Ehrhart theory, as well as insight into which types of polytopes have negative Ehrhart polynomial coefficients. To do this, we will focus on convex polytopes solely in this paper. In addition, we will provide the following structure:

First, we will start by looking at Pick's Theorem in section 2, and work with some examples. Then, in section 3, we will formally define lattice polytopes. In section 4, we will define Ehrhart polynomials and their generating functions, Ehrhart series. Next, in section 5, we will explore what the coefficients in the Ehrhart polynomial of a polytope tell us about the properties of that polytope. We will then prove the existence of negative coefficients in Ehrhart polynomials. In section 6, we will explore polytopes that have negative Ehrhart polynomial coefficients. Finally, in section 7, we will go over the opposite: polytopes that have positive Ehrhart polynomial coefficients.

2. Pick's Theorem

Pick's Theorem from [Pic99] is defined as follows:

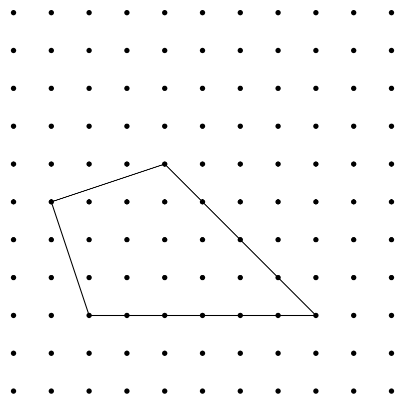
Theorem 2.1 (Pick's Theorem)**.** *Given any convex lattice polygon,*

$$A = I + \frac{B}{2} - 1$$

where A is the area of the polygon, I is the number of interior lattice points, and B is the number of points on the border of the polygon.

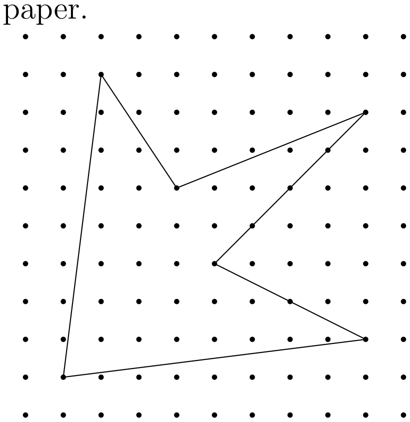
We will give a few examples.

Example 2.1. Will will find the area of the polygon below.



Here, $I = 12$ and $B = 12$, so
$$A = 12 + 6 - 1 = 17.$$

Example 2.2. Pick's theorem can also be used when a polygon is not convex, though convex polygons are the focus of this paper.



Here, $I = 27$ and $B = 10$, so
$$A = 27 + 5 - 1 = 31.$$

Now that we have a taste of Ehrhart theory in two dimensions, we will dive into more precisely defining *lattice polytopes*, so as to pave the way for studying Ehrhart theory in further detail.

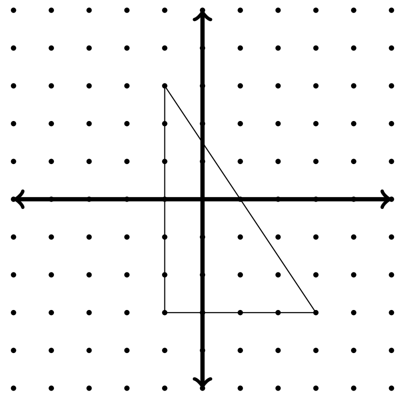
3. Lattice Polytopes

Polytopes are generalizations of the 2-dimensional polygon or the 3-dimensional polyhedron into higher dimensions. There are two formal ways to define lattice polytopes: the *vertex description* and the *hyperplane description*. The vertex description aims to define a polytope by its vertices.

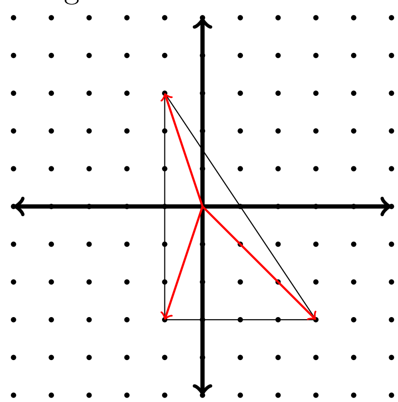
Definition 3.1 (vertex description)**.** A *polytope* is the convex hull of finitely many points. More formally, for any polytope $\mathcal{P}$,
$$\mathcal{P} := \text{conv}\{v_1, v_2, \ldots, v_n\} = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n : \lambda_k \geq 0, \sum_{k=1}^{n} \lambda_k \leq 1\}$$
for a finite set of points $\{v_1, v_2, \ldots, v_n\} \subset \mathbb{Z}^d$.

To more easily imagine what this means, we will provide an example in the second dimension.

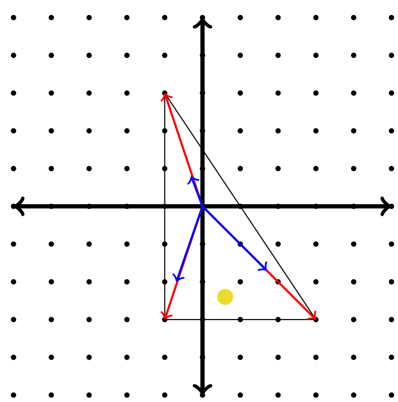
Example 3.1. Consider the polygon below.



Consider the vectors from the origin to each vertex.



Here, any point in this triangle can be described as the sum of some fractions of these vectors. For example, the point shown below is the sum of the blue vectors.



These fractions' sum is at most 1; notice that the points that have sums equal to 1 are the ones on the border of the polygon.

Unlike the vertex description, the hyperplane description aims to describe a polytope by its faces. We will start by defining the *hyperplane*, then a *half-space*.

Definition 3.2. A *hyperplane* is a generalization of the plane to higher dimensions. In other words, it is a $(d-1)$-dimension subspace within a d-dimension space. Formally,

$$H \coloneqq \{\mathbf{x} \in \mathbb{Z}^d : \mathbf{a} \cdot \mathbf{x} = b\}$$

for some $\mathbf{a} \in \mathbb{Z}^d$ and constant b.

Definition 3.3. A *half-space* $\mathcal{H} \in \mathbb{Z}^d$ is the part of a d-dimensional space that lies on a given side of a $(d-1)$-dimensional hyperplane. More formally,

$$\mathcal{H} \coloneqq \{\mathbf{x} \in \mathbb{Z}^d : \mathbf{a} \cdot \mathbf{x} \geq b\} \text{ or } \{\mathbf{x} \in \mathbb{Z}^d : \mathbf{a} \cdot \mathbf{x} \leq b\}$$

for some $\mathbf{a} \in \mathbb{Z}^d$ and constant b.

Definition 3.4 (hyperplane description)**.** A *polytope* $\mathcal{P} \subset \mathbb{Z}^d$ is the intersection of a finite number of half-spaces. Formally,

Ehrhart theory focuses on the t-th dilate of a polytope.

Definition 3.5. The *t-th dilate* of a polytope $\mathcal{P}$ is denoted as $t\mathcal{P}$, and refers to scaling $\mathcal{P}$ up by a factor of t. More formally,

$$\begin{aligned} t\mathcal{P} &= \{(tx_1, tx_2, \ldots, tx_d) : (x_1, x_2, \ldots, x_d) \in \mathcal{P}\} \\ &= \{t\mathbf{x} : \mathbf{x} \in \mathcal{P}\}. \end{aligned}$$

4. Ehrhart Polynomials and Series

4.1. **Ehrhart Polynomials.** The core idea of Ehrhart theory is the *lattice-point enumerator*, which, similar to Pick's Theorem (Theorem 2.1), counts the number of lattice points within a polytope. However, Ehrhart polynomials count the number of lattice points within the t-th dilate of the polytope.

Definition 4.1. The *lattice-point enumerator* is defined as

$$L_{\mathcal{P}}(t) = \left| t\mathcal{P} \cap \mathbb{Z}^d \right|.$$

It is sometimes denoted as $i(\mathcal{P}, t)$ instead of $L_{\mathcal{P}}(t)$.

It turns out that, for every polytope, this value is a rational polynomial, which Ehrhart proved in 1962. Hence, the lattice-point enumerator is sometimes also referred to as the *Ehrhart polynomial*. This is especially surprising, as there is no reason for it to be a polynomial.

Theorem 4.1 (Ehrhart's theorem)**.** *The Ehrhart polynomial of a convex lattice polytope $\mathcal{P}$ is a rational polynomial of degree d.*

The full proof of this theorem can be found in [Ehr62].

Below are a few examples of Ehrhart polynomials.

Example 4.1. Let

$$l_m = \{\alpha \in \mathbb{R} : 0 \leq \alpha \leq m, m \in \mathbb{N}\}.$$

Then, l_m is a convex lattice polytope in dimension 1. Hence $L_{l_m}(t) = mt + 1$.

Example 4.2. We will look at Ehrhart theory in dimension two.

Theorem 4.2 (Pick's Theorem, restated)**.** *For any convex lattice polygon $\mathcal{P}$,*

$$L_P(1) = A(P) + \frac{B(P)}{2} + 1.$$

Theorem 4.3 (Ehrhart's theorem in dimension 2, see [BR07])**.** *Let $\mathcal{P}$ be a convex lattice polygon and t be an integer.*

$$L_{\mathcal{P}}(t) = A(\mathcal{P})t^2 + \frac{B(\mathcal{P})t}{2} + 1.$$

Proof. After dilating $\mathcal{P}$ by a factor of t, we now have similar polygons. Therefore, the area of $t\mathcal{P}$ is t^2 times the area of $\mathcal{P}$, and the number of lattice points on the borderline is t times the number of lattice points on $\mathcal{P}$. □

Example 4.3. There exists a convex lattice polygon $\mathcal{Q}_m$ in dimension 3 whose Ehrhart polynomial is

$$L_{\mathcal{Q}_m} = \frac{m}{6}t^3 + t^2 + \frac{-m+12}{6}t + 1$$

for any $m \in \mathbb{N}$.

4.2. **Ehrhart Series.** When working with Ehrhart polynomials is inconvenient, we usually work with its *Ehrhart series.*

Definition 4.2. A *generating function* of a series $a_0, a_1, a_2, \dots$ is the polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \dots.$$

Definition 4.3. The *Ehrhart Series* $\mathrm{Ehr}_{\mathcal{P}}(x)$ of a polytope $\mathcal{P}$ is the generating function of $L_{\mathcal{P}}(t)$; that is, the sequence $L_{\mathcal{P}}(0), L_{\mathcal{P}}(1), L_{\mathcal{P}}(2), \dots$.

$$\mathrm{Ehr}_{\mathcal{P}}(x) = L_{\mathcal{P}}(0) + L_{\mathcal{P}}(1)x + L_{\mathcal{P}}(2)x^2 + \dots$$

Now that we know the fundamentals of Ehrhart theory, we can now explore Ehrhart polynomials more thoroughly.

5. Coefficients

5.1. **Properties of Polytopes.** It turns out that the first and second coefficients of Ehrhart polynomials are actually geometric properties of polytopes. The leading coefficient is the area, volume, or hypervolume of the polytope, depending on its dimension. The second coefficient is half the sum of the volumes of each facet (higher-dimensional generalization of face). The constant term of Ehrhart polynomials is always 1. However, we do not know anything more about other coefficients of Ehrhart polynomials.

To find out more about the coefficients, we first need to know if that is possible. One way to see this is to check the sign of the coefficients, since that might give us insight as to which properties are and are not represented in Ehrhart polynomials.

5.2. **The Existence of Negative Coefficients in Ehrhart Polynomials.**

Theorem 5.1. *For any $d \geq 4$, there exists a convex lattice polytope $\mathcal{P}$ whose coefficients are negative except for the coefficients of t^d and t^{d-1}.*

Proof. We will begin by presenting a lemma.

Lemma 5.1.1. *For two Ehrhart polynomials $L_{\mathcal{P}}(t)$ and $L_{\mathcal{Q}}(t)$ in dimensions d_1 and d_2, respectively, where $\mathcal{P}$ and $\mathcal{Q}$ are convex integral polytopes, there exists a convex lattice polytope of dimension $d_1 + d_2$ with Ehrhart polynomial $L_{\mathcal{P}}(t) \cdot L_{\mathcal{Q}}(t)$.*

Proof. We can multiply two polytopes together.

Definition 5.1. A *cartesian product* of two polytopes $\mathcal{P} = \{(x_1, x_2, \dots, x_{d_1}) \in \mathbb{Z}^{d_1}\}$ in dimension d_1 and $\mathcal{Q} = \{(y_1, y_2, \dots, y_{d_2}) \in \mathbb{Z}^{d_2}\}$ is

$$\mathcal{P}_1 \times \mathcal{P}_2 = \{(x_1, x_2, \dots, x_{d_1}, y_1, y_2, \dots, y_{d_1})\}.$$

It is well known that the cardinality of the cartesian product is the product of the cardinality of each polytope.

We have two points $(a_1, a_2, \dots, a_{d_1}) \in \mathcal{P}$, $(b_1, b_2, \dots, b_{d_2}) \in \mathcal{Q}$ if and only if we can conclude that $(a_1, a_2, \dots, a_{d_1}, b_1, b_2, \dots, b_{d_2})$ is in $\mathcal{P} \times \mathcal{Q}$. Therefore, the Ehrhart polynomial of $\mathcal{P} \times \mathcal{Q}$ is

$$L_{\mathcal{P}\times Q}(t) = L_{\mathcal{P}}(t) \cdot L_{\mathcal{Q}}(t).$$

□

From applying Lemma 5.1.1 multiple times and using Examples 4.1 and 4.3, we can conclude that there exists an integral convex polytope $\mathcal{P}_m$ in the dth ($d \geq 4$) dimension with Ehrhart polynomial

$$L_{P_m}(t) = (L_{l_{d-3}}(t))^{d-3} L_{\mathcal{Q}_m}(t).$$

Plugging in the values (see Example 4.1 and Example 4.3),

$$L_{\mathcal{P}_m}(t) = ((d-3)t+1)^{d-3} \cdot \left(\frac{m}{6}t^3 + t^2 + \frac{-m+12}{6}t + 1\right).$$

When $A_i = (d-3)^i \binom{d-3}{i}$ for $0 \leq i \leq d-2$, we can expand $((d-3)t+1)^3$ in $L_{\mathcal{P}_m}$ using the binomial theorem:

$$\sum_{i=0}^{d-3} \binom{d-3}{i} ((d-3)t)^i = A_0 + A_1 t + A_2 t + \cdots + A_{d-3} t^{d-3}.$$

This means that

$$L_{\mathcal{P}_m}(t) = \left(A_0 + A_1 t + A_2 t + \cdots + A_{d-3} t^{d-3}\right) \left(\frac{m}{6}t^3 + t^2 + \frac{-m+12}{6}t + 1\right).$$

Then, let $L_{\mathcal{P}_m}(t)$ be $\sum_{i=0}^{d} c_i^{(d,m)} t^i$, where each $c_i^{(d,m)}$ is a rational number. We can expand the above equation to find that $c_1^{(d,m)} = \frac{-m+12}{6} + A_1$, $c_2^{d,m} = 1 + \frac{-m+12}{6} \cdot A_1 + A_2$, and in general,

$$c_j^{(d,m)} = \frac{m}{6} A_{j-3} + A_{j-2} + \frac{-m+12}{6} \cdot A_{j-1} + A_j$$

for $3 \leq j \leq d-2$. With sufficiently large m, we have that $c_1^{(d,m)}$ is negative. Similarly, $c_2^{(d,m)}$ is negative for a large m. For $c_j^{(d,m)}$ in general,

$$\begin{aligned}
c_j^{(d,m)} &= \frac{m}{6} A_{j-3} + A_{j-2} + \frac{-m+12}{6} \cdot A_{j-1} + A_j \\
&= \frac{m}{6} A_{j-3} + A_{j-2} + \frac{-m}{6} \cdot A_{j-1} + 2A_{j-1} + A_j \\
&= -\frac{A_{j-1} - A_{j-3}}{6} \cdot m + A_{j-2} + 2A_{j-1} + A_j \\
&= -(d-3)^{j-3} \cdot \frac{g(d,j)}{6} \cdot m + A_{j-2} + 2A_{j-1} + A_j,
\end{aligned}$$

where $g(d,j) = (d-3)^2 \cdot \binom{d-3}{j-1} - \binom{d-3}{j-3}$. Now, we will present another lemma.

Lemma 5.1.2. *When $d \geq 5$ and $3 \leq j \leq d-2$,*

$$g(d,j) > 0.$$

Proof. We will proceed by induction.

$$g(d,3) = (d-3)^2 \cdot \binom{d-3}{2} - 1,$$

and this indeed is greater than 0. We also know that

$$g(d,d-2) = (d-3)^2 - \binom{d-3}{2},$$

which is also greater than 0. So, we know that the condition given in the problem is true for $j = 3$ and $j = d-2$. We also specifically test $d = 5, 6$ and both of them work. We will now proceed with induction on d, for $d \geq 7$ and $4 \leq j \leq d-3$. Firstly, $(d-3)^2$ can also be expressed as $(d-4)^2 + 2d - 7$. Therefore,

$$g(d,j) = ((d-4)^2 + 2d - 7)\binom{d-3}{j-1} - \binom{d-3}{j-3}.$$

Then, by Pascal's Identity, this is equal to

$$((d-4)^2 + 2d - 7)\left(\binom{d-4}{j-1} + \binom{d-4}{j-2}\right) - \left(\binom{d-4}{j-3} + \binom{d-4}{j-4}\right).$$

We can now simplify:

$$\begin{aligned}
&((d-4)^2 + 2d - 7)\binom{d-4}{j-1} + ((d-4)^2 + 2d - 7)\binom{d-4}{j-2} - \binom{d-4}{j-3} - \binom{d-4}{j-4} \\
=&(d-4)^2\binom{d-4}{j-1} + (2d-7)\binom{d-4}{j-1} + (d-4)^2\binom{d-4}{j-2} \\
&+ (2d-7)\binom{d-4}{j-2} - \binom{d-4}{j-3} - \binom{d-4}{j-4} \\
=&g(d-1,j) + g(d-1,j-1) + (2d-7)\binom{d-3}{j-1}.
\end{aligned}$$

Therefore, $g(d-1,j) + g(d-1,j-1) > 0$, and hence, $g(d,j) > 0$. □

Since $g(d,j) > 0$, we know that $c_j^{(d,m)}$ can be negative for a sufficiently large m. □

6. Families of Polytopes with Negative Ehrhart Polynomial Coefficients

To understand order polytopes, we first need to understand what posets are.

Definition 6.1. A poset, or *partially ordered set*, $P = (P, \leq)$ is a set P with a relation $\leq$ on P that is reflexive, transitive, and asymmetric.

- Reflexivity refers to the condition that $x \leq x$ for all $x \in P$.
- Transitivity happens when, if $x \leq y$ and $y \leq z$, then $x \leq z$.
- Asymmetry refers to the fact that $x \leq y \implies y \leq x$ is false.

We will demonstrate this with an example.

Example 6.1. We will show that $(\mathbb{R}, \leq)$ is a poset. To do this, we need to show that $\leq$ is reflexive, transitive, and asymmetric.

- Reflexivity: for all x in $\mathbb{R}$, $x \leq x$.
- Transitivity: for all x, y, and z in $\mathbb{R}$, we have that if $x \leq y$ and $y \leq z$, then $x \leq z$.

- Asymmetry: for all x and y in $\mathbb{R}$, if $x \leq y$ and $y \leq x$, then $x = y$.

Therefore, $(\mathbb{R}, \leq)$ is indeed a poset.

Now, using a poset, we will define order polytopes.

Definition 6.2. An *order polytope* $\mathscr{O}_P$ of a finite poset $(P, \leq_P)$ is the subset of $\mathbb{Z}^P = \{f : P \to \mathbb{Z}\}$ that is defined by

$$0 \leq f(i) \leq 1 \quad \forall i \in P$$

and

$$f(i) \leq f(j) \quad \text{if } i \leq_P j.$$

We will also define the *ordinal sum* of two posets.

Definition 6.3. The *ordinal sum* of two disjoint finite posets is the poset $(P \oplus Q, \leq_{P \oplus Q})$ such that $s \leq_{P \oplus Q} t$ if:

- $s, t \in P$ and $s \leq_P t$,
- $s, t \in Q$ and $s \leq_Q t$, or
- $s \in P$ and $t \in Q$.

See Figure 1 below for an example.

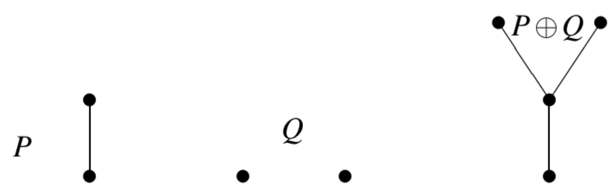


FIGURE 1. The ordinal sum of P and Q.

Order polytopes can be non-Ehrhart-positive, meaning that their Ehrhart polynomials can have negative coefficients.

Theorem 6.1. *For any positive integer $d \geq 14$, there exists an order polytope that has negative Ehrhart polynomial coefficients. t For $d \leq 11$, any order polytope is Ehrhart-positive, meaning that their Ehrhart polynomials always have negative coefficients.*

For $d \geq 21$, there is an order polytope that is non-Ehrhart positive for each d.

The main idea of the proof is that it suffices to show an example of a non-Ehrhart-positive order polytope for each d. We will let m, n be two positive integers, and let $P_{m,n}$ be the ordinal sum of P_m and P_n. It turns out that the polynomials are the ones shown in Figure 2 below; the first two are Ehrhart-positive, but the rest are not. For the full proof, see [LT19].

Other examples of polytopes that have negative Ehrhart polynomial coefficients are smooth lattice polytopes and matroids.

7. The Other Side of the Story: Ehrhart Positivity

Polytopes that do not have negative Ehrhart coefficients no matter the dimension are called *Ehrhart-positive*. Many classes of polytopes are Ehrhart-positive.

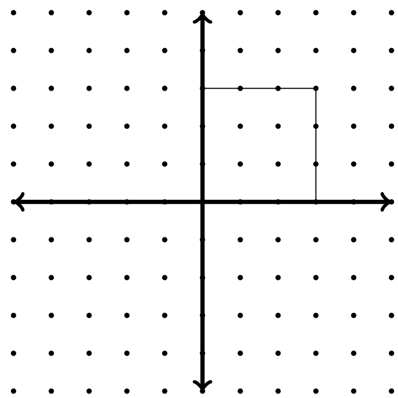
FIGURE 2. The Ehrhart polynomials of $\mathscr{O}_{P_{m,n}}$.

P	Ehrhart polynomial
$P_{6,6}$	$1+\frac{75t}{22}+\frac{824t^2}{77}+\frac{160t^3}{7}+\frac{181t^4}{6}+\frac{765t^5}{28}+\frac{127t^6}{7}+9t^7+\frac{93t^8}{28}+\frac{25t^9}{28}+\frac{1}{6}t^{10}+\frac{3t^{11}}{154}+\frac{t^{12}}{924}$
$P_{6,7}$	$1+\frac{61751t}{15015}+\frac{555t^2}{44}+\frac{928t^3}{33}+\frac{83t^4}{2}+\frac{1273t^5}{30}+\frac{255t^6}{8}+\frac{127t^7}{7}+\frac{63t^8}{8}+\frac{31t^9}{12}+\frac{5}{8}t^{10}+\frac{7t^{11}}{66}+\frac{t^{12}}{88}+\frac{t^{13}}{1716}$
$P_{7,7}$	$1-\frac{3041t}{1430}+\frac{18397t^2}{4290}+\frac{1365t^3}{44}+\frac{602t^4}{11}+\frac{301t^5}{5}+\frac{8953t^6}{180}+\frac{255t^7}{8}+\frac{127t^8}{8}+\frac{49t^9}{8}+\frac{217t^{10}}{120}+\frac{35t^{11}}{88}+\frac{49t^{12}}{792}+\frac{7t^{13}}{1144}+\frac{t^{14}}{3432}$
$P_{7,8}$	$1-\frac{1633t}{2145}+\frac{11261t^2}{2860}+\frac{208909t^3}{6435}+\frac{6125t^4}{88}+\frac{14441t^5}{165}+\frac{959t^6}{12}+\frac{5113t^7}{90}+\frac{255t^8}{8}+\frac{127t^9}{9}+\frac{49t^{10}}{10}+\frac{217t^{11}}{165}+\frac{35t^{12}}{132}+\frac{49t^{13}}{1287}+\frac{t^{14}}{286}+\frac{t^{15}}{6435}$
$P_{8,8}$	$1-\frac{9905t}{286}-\frac{81704t^2}{2145}+\frac{18740t^3}{429}+\frac{137692t^4}{1287}+\frac{1358t^5}{11}+\frac{57736t^6}{495}+\frac{1364t^7}{15}+\frac{511t^8}{9}+\frac{85t^9}{3}+\frac{508t^{10}}{45}+\frac{196t^{11}}{55}+\frac{434t^{12}}{495}+\frac{70t^{13}}{429}+\frac{28t^{14}}{1287}+\frac{4t^{15}}{2145}+\frac{t^{16}}{12870}$
$P_{8,9}$	$1-\frac{1063343t}{36465}-\frac{29713t^2}{572}+\frac{126224t^3}{6435}+\frac{17882t^4}{143}+\frac{75956t^5}{429}+\frac{1967t^6}{11}+\frac{24644t^7}{165}+\frac{1023t^8}{10}+\frac{511t^9}{9}+\frac{51t^{10}}{2}+\frac{508t^{11}}{55}+\frac{147t^{12}}{55}+\frac{434t^{13}}{715}+\frac{15t^{14}}{143}+\frac{28t^{15}}{2145}+\frac{3t^{16}}{2860}+\frac{t^{17}}{24310}$
$P_{9,9}$	$1-\frac{1285677t}{4862}-\frac{7364613t^2}{24310}+\frac{89157t^3}{572}+\frac{246946t^4}{715}+\frac{195381t^5}{715}+\frac{173242t^6}{715}+\frac{2439t^7}{11}+\frac{9204t^8}{55}+\frac{1023t^9}{10}+\frac{511t^{10}}{10}+\frac{459t^{11}}{22}+\frac{381t^{12}}{55}+\frac{1323t^{13}}{715}+\frac{279t^{14}}{715}+\frac{9t^{15}}{143}+\frac{21t^{16}}{2860}+\frac{27t^{17}}{48620}+\frac{t^{18}}{48620}$
$P_{9,10}$	$1-\frac{220154521t}{969969}-\frac{20069739t^2}{48620}-\frac{454951t^3}{12155}+\frac{453525t^4}{1144}+\frac{64414t^5}{143}+\frac{548577t^6}{1430}+\frac{1718664t^7}{5005}+\frac{3060t^8}{11}+\frac{12277t^9}{66}+\frac{1023t^{10}}{10}+\frac{511t^{11}}{11}+\frac{765t^{12}}{44}+\frac{762t^{13}}{143}+\frac{189t^{14}}{143}+\frac{186t^{15}}{715}+\frac{45t^{16}}{1144}+\frac{21t^{17}}{4862}+\frac{3t^{18}}{9724}+\frac{t^{19}}{92378}$
$P_{10,10}$	$1-\frac{135276175t}{58786}-\frac{2250043660t^2}{969969}+\frac{4024062t^3}{2431}+\frac{11364453t^4}{4862}+\frac{461265t^5}{572}+\frac{150248t^6}{429}+\frac{445884t^7}{1001}+\frac{426227t^8}{1001}+\frac{6825t^9}{22}+\frac{2047t^{10}}{11}+93t^{11}+\frac{2555t^{12}}{66}+\frac{3825t^{13}}{286}+\frac{3810t^{14}}{1001}+\frac{126t^{15}}{143}+\frac{93t^{16}}{572}+\frac{225t^{17}}{9724}+\frac{35t^{18}}{14586}+\frac{15t^{19}}{92378}+\frac{t^{20}}{184756}$

7.1. **The Unit d-Cube.** The unit d-cube is a generalization of the 2D square and the 3D cube into higher dimensions; it is denoted as $\square_d$.

Theorem 7.1. *The Ehrhart polynomial of the unit d-cube is $(t+1)^d$ for a dimension d.*

Proof. We will work on a 2D plane, then generalize into higher dimensions. Below is the t-th dilate of the unit square.



Each coordinate of every point within this square can have $x = 0, 1, 2, \ldots, t$. Similarly, it has $y = 0, 1, 2, \ldots, t$. Therefore, there are $(t+1)^2$ ways to choose the coordinates, and the Ehrhart polynomial is $t^2 + 2t + 1$.

For a unit d-cube, every point has d coordinates, and each of these have $t+1$ values they can be. Therefore, the lattice-point enumerator for a unit d-cube is $(t+1)^d$.

□

Since the expansion of $(t+1)^d$ must always be positive by the binomial theorem, unit d-cubes are Ehrhart-positive.

7.2. **The Standard d-Simplex.** While the unit d-cube is the generalization of squares into higher dimensions, the standard d-simplex is the generalization of the 2D triangle and the 3D tetrahedron. It is denoted as $\triangle_d$.

Definition 7.1. Formally, $\triangle_d$ defined as the convex hull of the d unit vectors $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_d}$

$$\begin{aligned} \triangle_d &:= \operatorname{conv}\{\mathbf{0}, \mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_d}\} \\ &= \{(x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d : x_1 + x_2 + \cdots + x_d \leq 1 \text{ and } x_i \geq 0 \text{ for } 1 \leq i \leq d\}. \end{aligned}$$

Theorem 7.2. *The Ehrhart polynomial of the standard d-simplex is* $\binom{d+t}{d}$.

Proof. Points of the form $(x_1, x_2, \ldots, x_d)$ in a standard d-simplex satisfy

$$x_1 + x_2 + \cdots + x_d \leq t$$

where x_i for all i. We can make this an equality by writing

$$x_1 + x_2 + \cdots + x_d + x_{d+1} = t.$$

The number of solutions $(x_1, x_2, \ldots, x_d, x_{d+1})$ can be counted using Stars and Bars; we are dividing t indistinguishable stars into $d+1$ distinguishable sections with bars. So, the number of solutions is $\binom{t+(d+1)-1}{(d+1)-1} = \binom{t+d}{d}$. □

We claim that the standard d-simplex is also Ehrhart-positive.

Theorem 7.3. *The standard d-simplex is Ehrhart positive.*

Proof. Writing $\binom{t+d}{d}$ as a polynomial,

$$\frac{(t+d)!}{t! \cdot d!} = \frac{(t+d)(t+d-1)(t+d-2)\ldots(t+1)}{d!}.$$

Since $d!$ is positive, we only need to see if the coefficients of the numerator are positive. Since every factor in the numerator has positive coefficients, the expansion is also positive. □

Of course, many other polytopes have Ehrhart positivity, such as d-cross polytopes. However, Ehrhart polynomials of d-cross polytopes are hard to work with, so the Ehrhart series is used.

Remark 7.1. There is no standard way of proving Ehrhart positivity as of now. Therefore, it is difficult yet captivating to prove Ehrhart positivity, or non-Ehrhart positivity for a each polytope.

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