Discrete Morse Theory

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Euler Circle

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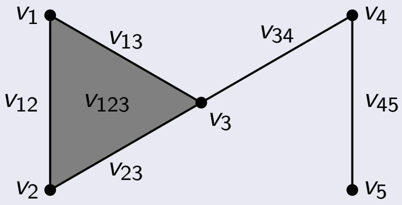
Figure 1: A simplicial complex



Introduction to Simplicial Complexes

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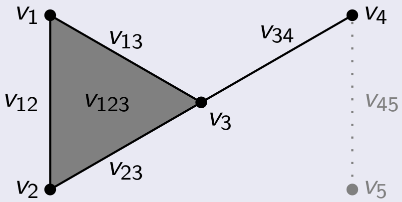

Introduction to Simplicial Complexes

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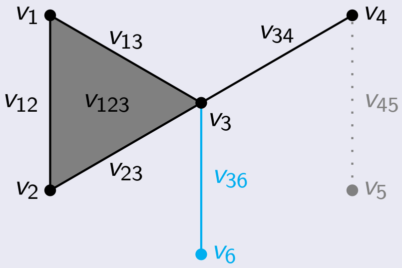


Discrete Morse Functions

Discrete Morse Functions

Definition: Discrete Morse Functions

A *discrete Morse function f* is a function $f : K \to \mathbb{R}$ that satisfies the following properties:

$$|\{\tau^{(i-1)} < \sigma : f(\tau) \geq f(\sigma)\}| \leq 1$$

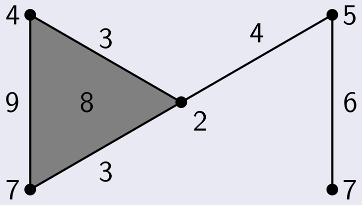
and

$$|\{\tau^{(i+1)} > \sigma : f(\tau) \leq f(\sigma)\}| \leq 1$$

for every $\sigma^{(i)} \in K$.

Discrete Morse Functions

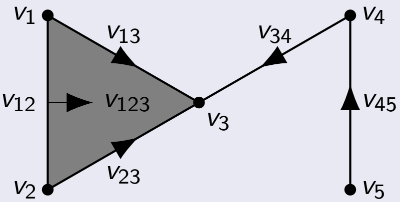
Figure 2: An example of a discrete Morse function



Each simplex has at most one "exception"; we can represent these exceptions with an induced gradient vector field.

Discrete Morse Functions

Figure 3: The induced gradient vector field of the previous discrete Morse function



Notice how the vectors describe possible collapses that reduce the simplicial complex.

The Collapse Theorem

The Collapse Theorem

Definition: Level Subcomplexes

Let $f : K \to \mathbb{R}$ be a discrete Morse function. For any $c \in \mathbb{R}$, the *level subcomplex* $K(c)$ is the subcomplex consisting of any simplices τ along with its faces such that $f(\tau) \leq c$.

Definition: Intervals

If σ and τ are subcomplexes of K such that $\sigma < \tau$, then the *interval* $[\sigma, \tau]$ is the set of all simplices in K that contain σ and are contained in τ.

The Collapse Theorem

Theorem: The Collapse Theorem

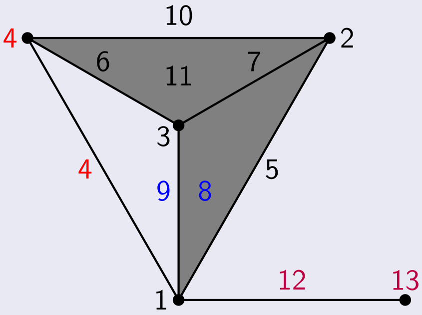
Let $f : K \to \mathbb{R}$ be a discrete Morse function and $[\sigma, \tau] \subseteq \mathbb{R}$ an interval that contains no critical values. Then $K(\tau) \searrow K(\sigma)$.

Remark

This tells us that when we are trying to simplify a simplicial complex, we only have to consider level subcomplexes with critical simplices.

The Collapse Theorem

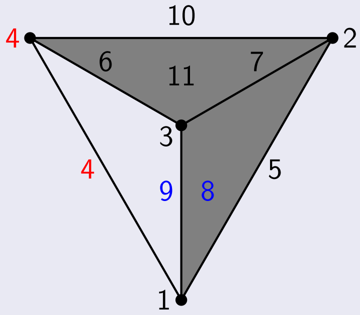
Figure 4: The Collapse Theorem in practice



The level subcomplexes generated by 13 and 12 can be collapsed.

The Collapse Theorem

Figure 4: The Collapse Theorem in practice



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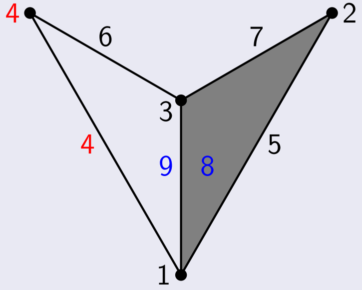

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Applications in Homology

Applications in Homology

Definition: Betti Numbers

Briefly speaking, the homology of a simplicial complex refers to the number of "holes" in the complex. The *Betti numbers* b_n refer to the number of n-dimensional holes in a simplicial complex.

Applications in Homology

Theorem: Weak Discrete Morse Inequalities

Let $f : K \to \mathbb{R}$ be a discrete Morse function with m_i critical values in dimension i for $i = 0, 1, 2, \ldots, n := \dim K$. Then we have the following inequalities:

i. For all $i = 0, 1, 2, \ldots, n$, $b_i \leq m_i$.
ii. The Euler characteristic $\chi(K)$ is equal to $\sum_{i=0}^{n} (-1)^i m_i$.

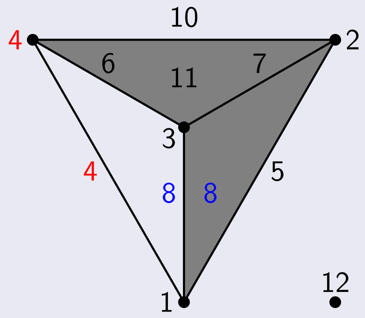
Theorem: Strong Discrete Morse Inequalities

Let $f : K \to \mathbb{R}$ be a discrete Morse function. For each $p = 0, 1, 2, \ldots, n, n+1$, we have

$$b_p - b_{p-1} + b_{p-2} - \ldots + (-1)^p b_0 \leq m_p - m_{p-1} + m_{p-2} - \ldots + (-1)^p m_0.$$

Applications in Homology

Figure 5: Visualizing the Discrete Morse Inequalities



Applications in Homology

Proof of the Weak Discrete Morse Inequalities)

1. We proceed by induction on the number of simplices, checking if the highest value simplex is regular (in which case we can collapse it by) or critical (in which case removing it removes a hole).
2. Since each regular pair of simplices consists of two codimension-1 simplices, their Euler characteristic is 0. Canceling out all of the regular simplices, we are left with just the critical simplices.

Thank you for attending my presentation!

Thank you very much!