Linear Forms in Logarithms

Krish Agrawal

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Outline

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Algebraic Numbers

Definition

A complex number α is **algebraic** if it satisfies some non-zero polynomial equation $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ where $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ and $a_n \neq 0$.

- The *degree* of α is the smallest degree of such a polynomial.
- Examples: $\sqrt{2}$ has degree 2, $\sqrt[3]{2}$ has degree 3, $\frac{1+\sqrt{5}}{2}$ has degree 2.
- Every rational number is algebraic of degree 1.

Transcendental Numbers

Definition

A complex number is **transcendental** if it is not algebraic—that is, it satisfies no polynomial equation with integer coefficients.

- Classical examples: e (Hermite, 1873) and π (Lindemann, 1882).
- The Gelfond-Schneider theorem (1934) established transcendence of numbers like $2^{\sqrt{2}}$ and e^{π}.
- Transcendental numbers are abundant: they form an uncountable set, while algebraic numbers are countable.

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Basic Definition

Definition

A **linear form in logarithms** is an expression of the form

$$\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + \cdots + b_n \log \alpha_n$$

where

- $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers;
- $b_1, \ldots, b_n \in \mathbb{Z}$ with at least one $b_i \neq 0$;
- log denotes a fixed branch of the complex logarithm.

Central Questions

1. When does $\Lambda = 0$?
2. If $\Lambda \neq 0$, how close to zero can $|\Lambda|$ be?

Motivation from Diophantine Equations

Linear forms in logarithms arise naturally when studying exponential Diophantine equations:

- Consider the equation $\alpha_1^{x_1}\alpha_2^{x_2}\cdots\alpha_n^{x_n} = 1$ where $x_i \in \mathbb{Z}$.
- Taking logarithms: $x_1 \log \alpha_1 + x_2 \log \alpha_2 + \cdots + x_n \log \alpha_n = 2\pi i k$ for some $k \in \mathbb{Z}$.
- Lower bounds on $|\Lambda|$ translate to upper bounds on $|x_i|$.

Concrete Example

To solve $2^x - 3^y = 1$ in positive integers:

- Rearrange to $2^x = 3^y + 1$.
- This leads to studying $|x \log 2 - y \log 3|$.
- Bounds on this linear form constrain possible values of (x, y).

Historical Timeline

- **1844**: Liouville proves the first approximation theorem for algebraic numbers.
- **1873**: Hermite proves e is transcendental.
- **1882**: Lindemann proves π is transcendental.
- **1934**: Gelfond and Schneider independently resolve Hilbert's 7th problem.
- **1966**: Baker proves lower bounds for linear forms in logarithms.

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The Gelfond-Schneider Theorem

Theorem (Gelfond-Schneider, 1934)

If α is algebraic with $\alpha \neq 0, 1$ and β is algebraic and irrational, then α^β is transcendental.

- This resolved Hilbert's 7th problem about the transcendence of a^b for algebraic a, b.
- One of the key examples of the same is: $2^{\sqrt{2}}$
- More about the proof in the full paper..

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Baker's Theorem: The Statement

Theorem (Baker, 1966)

Let $\alpha_1, \ldots, \alpha_n$ *be non-zero algebraic numbers, not all equal to 1, and let* $b_1, \ldots, b_n$ *be integers, not all zero. If*

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n \neq 0,$$

then

$$|\Lambda| > \exp\left(-C(n) \cdot D^2 \cdot H \cdot \log B\right)$$

where $D = \prod_{i=1}^n d_i$, $H = \prod_{i=1}^n h(\alpha_i)$, $B = \max\{|b_1|, \ldots, |b_n|\}$, *and* $C(n)$ *depends only on* n.

Baker's Theorem: The Details

- $d_i = [\mathbb{Q}(\alpha_i) : \mathbb{Q}]$ is the degree of α_i.
- $h(\alpha_i)$ is the absolute logarithmic height of α_i.
- For α with minimal polynomial $a_d x^d + \cdots + a_0$ (with $\gcd(a_0, \ldots, a_d) = 1$):

$$h(\alpha) = \frac{1}{d}\left(\log|a_d| + \sum_{\sigma} \log^{+}|\sigma(\alpha)|\right)$$

where the sum is over all conjugates $\sigma(\alpha)$ of α.

Significance

- Generalizes Gelfond-Schneider from single logarithms to arbitrary linear combinations.
- Completely effective: all constants are explicit and computable.
- Provides the foundation for solving exponential Diophantine equations.

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The Computational Challenge

- Baker's bounds, while explicit, are often impractically large.
- Example: A bound like $|\Lambda| > \exp(-10^{100})$ can translate to $|x|, |y| < 10^{480}$.
- Direct computer search becomes impossible.
- **Baker-Davenport (1969)**: Combine Baker bounds with continued fraction theory to dramatically reduce search spaces.

Typical Improvement

Initial bound: $|x| < 10^{480} \rightarrow$ Refined bound: $|x| < 500$.

The Baker-Davenport Method

Consider a linear form $\Lambda = x \log \alpha - y \log \beta + \gamma$.
If we have $|\Lambda| < C_1 e^{-C_2|x|}$ for large $|x|$, then:

$$\left| \frac{\log \alpha}{\log \beta} - \frac{y}{x} \right| < \frac{C_3}{|x|^2}$$

This means $\frac{y}{x}$ must be a convergent in the continued fraction expansion of $\frac{\log \alpha}{\log \beta}$.

Key Insight

Since continued fractions have good approximation properties, there are only finitely many "exceptional" values of x to check, typically reducing millions of cases to dozens.

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The Simultaneous Pell Equation Problem

Problem

Find all positive integers x such that both $3x^2 - 2$ and $8x^2 - 7$ are perfect squares.

- Parametrize solutions to $3x^2 - 2 = y^2$ using the fundamental solution.
- Parametrize solutions to $8x^2 - 7 = z^2$ similarly.
- Equating parametrizations leads to a linear form in logarithms.
- Baker's theorem gives an initial bound like $m < 10^{480}$.
- Baker-Davenport reduction brings this down to $m < 500$.
- Direct verification finds solutions: $x = 1$ and $x = 11$.

Other Notable Applications

- **Ramanujan-Nagell equation**: $x^2 + 7 = 2^n$ (solved completely).
- **Fibonacci powers**: When is F_n a perfect power? (Luca, 2000s).
- **Catalan-Mersenne conjecture**: $2^p - 1 = x^q$ for primes p, q.
- **Generalized Fermat equations**: $x^p + y^q = z^r$ with constraints.
- **S-unit equations**: $x + y = 1$ where x, y have prime factors from a finite set S.

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Modern Developments

- **Sharper effective constants**: Work of Matveev, Waldschmidt, and others has made bounds more practical.
- ***p*-adic linear forms**: Analogous theory for p-adic logarithms (Waldschmidt, Gel'fond).
- **Elliptic logarithms**: Linear forms involving logarithms on elliptic curves (David, Hirata-Kohno).

Selected References

- A. Baker, *Transcendental Number Theory*, Cambridge University Press, 1975.
- A. Baker & H. Davenport, "The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$", *Q. J. Math.*, 20 (1969), 129-137.
- E. M. Matveev, "An explicit lower bound for a linear form in logarithms of algebraic numbers", *Math. Notes*, 64 (1998), 566-573.
- M. Waldschmidt, *Diophantine Approximation on Linear Algebraic Groups*, Springer, 2000.
- Y. Bugeaud & F. Luca, "On the equation $x^2 + 2^a \cdot 3^b = y^n$", *Pacific J. Math.*, 193 (2000), 257-268.