O-Minimality

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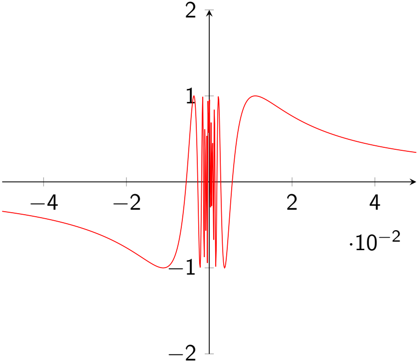
Introduction

Consider the graph of $y = \sin(1/x)$



Structures

Definition 0.1

A *structure* is a set along with a collection of n-ary relations, n-ary functions and constants that are defined on that set.

For example, the real field can be considered as a structure, with the binary functions $+$ and $\times$, the unary function $-$ and the constants 0 and 1.

Definition 0.2

An *o-minimal* structure $\mathcal{A}$ is a structure on $\mathbb{R}^n$ that has a total ordering defined on $\mathbb{R}^n$, and where every definable subset of $\mathcal{A}$ is a finite union of open intervals and points.

Theorems

Theorem 0.3

Continuity Let $f : (a, b) \to R$ be definable in an o-minimal structure $\mathcal{R}$. Then there exist points $a = c_0 < c_1 < \cdots < c_k = b$ such that on each interval (c_i, c_{i+1}), f is continuous.

Theorem 0.4 (Monotonicity Theorem)

Let $f : (a, b) \to R$ be definable in an o-minimal structure $\mathcal{R}$. Then there exist points $a = c_0 < c_1 < \cdots < c_k = b$ such that on each interval (c_i, c_{i+1}), f is either constant, strictly increasing, or strictly decreasing.

Theorem 0.5

Let $f : (a, b) \to R$ be definable in an o-minimal structure $\mathcal{R}$ expanding a real closed field. Then there exist points $a = c_0 < c_1 < \cdots < c_k = b$ such that on each interval (c_i, c_{i+1}) has a continuous derivative if $\mathcal{R}$ expands a real closed field.

Cells

Definition 0.6

A **cell** $C \subseteq \mathbb{R}^n$ is defined as either:

- For $n = 1$: Points or open intervals
- For $n > 1$: Graphs $y = f(x)$ or bands $f(x) < y < g(x)$ where f, g are definable functions

The dimension of a cell is:

- 0 for points
- 1 for intervals
- dim(graph) = dim(domain)
- dim(band) = dim(domain) + 1

Theorem 0.7 (Cell Decomposition)

Any definable set $X \subseteq \mathbb{R}^n$ on an o-minimal structure is a finite union of cells.

Properties of o-minimal structures

Definition 0.8

The *algebraic part* X^{alg} of a subset X of $\mathbb{R}^n$ is the union of every infinite subset of X defined by polynomials or polynomial inequalities.

Definition 0.9

The *transcendental part* X^{tr} of a subset X of $\mathbb{R}^n$ is the difference of that subset and its algebraic part.

Definition 0.10

The Height function is defined, for positive and coprime $a, b \in \mathbb{N}$, by $\mathrm{H}(\frac{a}{b}) \max(|a|, |b|)$.

Pila-Wilkie Theorem

Theorem 0.11 (Pila-Wilkie Theorem)

Let X be a definable subset of $\mathbb{R}^n$ on an o-minimal structure. Then for all ε there exists c such that for all T, the number of rational points on X^{tr} of height at most T is at most $c \cdot T^{\varepsilon}$.

Definitions

Definition 0.12

For $k, n > 1$ and $X \subseteq \mathbb{R}^n$, a *strong k-parametrization* of X is a C^k-map $f : (0,1)^m \to \mathbb{R}^n, m < n$, with image X, such that $|f^{(\alpha)}(a)| \leq 1$ for all $\alpha \in \mathbb{N}^m$ with $|\alpha| < k$ and all $a \in (0,1)^m$.

Definition 0.13

A *hypersurface* in $\mathbb{R}^n$ of degree $\leq e$ is the set of solutions in $\mathbb{R}^n$ of a nonzero polynomial defined on $\mathrm{x} = (x_1, ..., x_n)$ over $\mathbb{R}$ of (total) degree $\leq e$.

Theorems

Theorem 0.14 (Bombieri-Pila)

Let $n \geq 1$ be given. Then for any $e \geq 1$ there are $k = k(n, e) \geq 1$, $\varepsilon = \varepsilon(n, e)$, and $c = c(n, e)$, such that if $X \subseteq \mathbb{R}^n$ has a strong k-parametrization, then for all T at most cT^ε many hypersurfaces in $\mathbb{R}^n$ of degree $\leq e$ are enough to cover the set of rational points of X with height $\leq T$, with $\varepsilon(n, e) \to 0$ as $e \to \infty$.

Theorem 0.15

Given an o-minimal field R, we have that every definable set $X \subseteq [-1, 1]^n_R$ with empty interior and $n \geq 1$ is for every $k \geq 1$ a finite union of definable subsets, each having a definable strong k-parametrization.

Lemmas

Lemma 0.16

If $X = X_1 \cup \cdots \cup X_m$*, then* $X^{\mathrm{alg}} \supseteq X_1^{\mathrm{alg}} \cup \cdots \cup X_m^{\mathrm{alg}}$ *and thus* $X^{\mathrm{tr}} \subseteq X_1^{\mathrm{tr}} \cup \cdots \cup X_m^{\mathrm{tr}}$.

Lemma 0.17

Suppose $S \subseteq \mathbb{R}^n$ *is semialgebraic,* $f : S \to \mathbb{R}^m$ *is semialgebraic and injective, and* f *maps the set* $X \subseteq S$ *homeomorphically onto* $Y = f(X) \subseteq \mathbb{R}^m$*. Then* $f(X^{\mathrm{alg}}) = Y^{\mathrm{alg}}$ *and thus* $f(X^{\mathrm{tr}}) = Y^{\mathrm{tr}}$*.*

Proof Sketch

1. We remove the interior of X in $\mathbb{R}^n$ and arrange that it has empty interior
2. We take some large e and use the second theorem to prove that the number of rational points in the transcendental part of X is a subset of:

$$\bigcup_{i=1}^{M} \bigcup_{j=1}^{J} S_{ij}$$

where S is the number of rational points on the transcendental part of the intersection of X and hypersurfaces H_{ij} in $\mathbb{R}^n$ of degree $\leq e$.
3. We then break the hypersurface into semialgebraic cells and, proceeding by induction on the height, we show that there is an upper bound on the number of rational points it has that does not depend on the hypersurface and the height, so that it obtains a bound that is exactly $c \cdot T^{\varepsilon}$.