On O-minimality and the Pila-Wilkie theorem

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Abstract

This paper discusses o-minimal structures and a number of related results and applications. After covering some basic motivations and notions from model theory, three primary theorems will be proven. We will primarily aim to understand the Pila-Wilkie theorem and a few of its generalizations, after covering the more fundamental results of Monotonicity and Cell Decomposition.

1 Introduction

We frequently want to discuss the solutions to polynomials, and possibly understand a property of them. This motivates a number of theorems estimating or bounding the number of rational points that are solutions to those polynomials.

The Pila-Wilkie theorem (Theorem 5.5) is one such result, and it provides a particularly small bound for the number of rational points that are not solutions for any polynomial or polynomial inequality. It provides this bound only given a certain property we call o-minimality, which is applied from model theory. This property has many useful implications, which allow it to provide a helpful setting for theorems such as the one we will prove in this paper.

O-minimal structures are the main concept we will explore in this paper. However, this was not originally developed with results similar to the Pila-Wilkie theorem in mind. A major motivation for its use was [Gro84], written in 1984 , when Alexandre Grothendieck advanced a program of 'tame topology', which he proposed as a topology that would exclude certain pathological spaces and behaviors.

This in turn motivated Lou van den Dries to argue in [vdD98] that o-minimal structures could allow for a tame topology over the reals, and meet Grothendieck's conditions. This relied on progress already made in the theory of o-minimal structures, developed from model theoretic results about the real numbers.

Several major theorems about o-minimal structures had already been proven then, beginning with [PS86] where Pillay and Steinhorn introduced the theory and some initial results.

The Pila-Wilkie theorem was proven more recently in [PW06], and used previous work in [BP89], [Yom87] and [Gro87] to provide an important application of o-minimality.

In this paper, we will begin our discussion of o-minimality with an overview of some preliminary ideas and definitions in model theory. We will then prove one of the early results in the area, the Monotonicity Theorem.

In the next section, we will prove the Cell Decomposition Theorem, another foundational result. We will then turn to an exploration of the Pila-Wilkie theorem, beginning with a discussion of some basic concepts outside of o-minimality, as well as some important lemmas. In the last section, we will prove the Pila-Wilkie theorem and some of its generalizations to definable families.

2 Preliminaries

We begin by covering the necessary knowledge of model theory and o-minimality in general to understand the later theorems and proofs in this paper.

Definition 2.1. A structure A is a set M, which we call its *domain* or *universe*, along with:

1. a collection of relations, or subsets $R \subseteq M^n$,
2. a collection of functions $f : M^n \to M$,
3. and a collection $c \in M$ of constants.

We often call a subset of the domain of a structure A a subset of A.

Definition 2.2. A signature σ is a collection of relation, function, and constant symbols.

When a structure has signature σ we say that it is a σ-structure. We can also discuss the relations and functions on a structure with respect to their arity, which is $n \in \mathbb{N}$ for the number of arguments taken by a relation or function.

For example, the rational numbers, the real numbers and the complex numbers can each be considered as structures of the same signature. We specify the signature

$$\sigma = \{+, \times, -, 0, 1\}$$

with $+$ and $\times$ as binary function symbols, a unary function symbol $-$ for change of sign, and two constants 0 and 1. Then the rational, real and complex numbers are σ-structures over $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, respectively.

Additionally, when discussing functions on structures, we say that $f(x)$ is the image of x, and that for $y = f(x)$, x is the preimage of y. We also say that a set S of points x is the fiber of y if it is the preimage of the singleton set $\{y\}$.

Other fundamental concepts for defining o-minimal structures include those of definable subsets and intervals.

Definition 2.3. A definable subset of a structure A is a subset of its domain M that can be specified with a formula using logical symbols, or the symbols on the signature of A, or both.

Definition 2.4. A subset C of M is convex if for all $c, c' \in C$ and every $m \in M$ with $c \leq m \leq c'$, we have that $m \in C$.

Definition 2.5. An interval of M is a convex subset of M with endpoints in $M \cup \{-\infty, +\infty\}$.

In o-minimal structures, we will largely deal with open intervals, i.e. those that do not include their endpoints. We will use the usual notation for intervals, i.e. (a, b) for an open interval from a to b, and $[a, b]$ for a closed interval.

Definition 2.6. A box is the cartesian product of n finite intervals.

We also define some additional concepts in topology considered relative to the other notions we develop.

Definition 2.7. A homeomorphism is a function $f : A \to B$ that is:

1. bijective, i.e., it maps unique elements in A to unique elements in B and maps at least one element in A to each element in B.

2. continuous.

3. an open map, meaning that it maps open sets on A to open sets on B.

Definition 2.8. The interior of a set S is the largest open set contained in S.

Definition 2.9. A neighborhood of a point p is a set S that contains an open set containing p.

We also say that for some neighborhood S, p is an interior point of S.

Definition 2.10. The boundary $\text{bd}(S)$ of a set S is the set of all points p such that each neighborhood of p contains at least one point in S and one point not in S.

Definition 2.11. A closed set is a set that contains all of the points on its boundary.

Now we define some concepts of ordering, for the convenience of the reader, before defining o-minimal structures.

Definition 2.12. A partial order or strict partial order on a set M is a binary relation $<$ such that, for all $a, b, c \in M$:

1. we never have $a < a$,

2. if $a < b$ then we do not have that $b < a$,

3. and if $a < b$ and $b < c$ then $a < c$.

Alternatively, we say that a strict partial order is irreflexive, asymmetric, and transitive, respectively.

Definition 2.13. A linear or total order is a partial order $<$ on M such that for any $a, b \in M$, if $a \neq b$, then either $a < b$ or $b < a$.

Definition 2.14. A dense order is a partial order such that where $a, b \in M$ and $a < b$, there exists $c \in M$ suh that $a < c < b$.

Definition 2.15. An order on M without endpoints is one such that there is no greatest or lowest element in M.

Definition 2.16. An o-minimal structure is a structure $\mathcal{S}$ such that it has a dense linear order without endpoints $<$ defined on its domain, and where every definable subset of $\mathcal{S}^\setminus$ is a finite union of points and open intervals.

The following are o-minimal structures:

1. the rational numbers $\mathbb{Q}$ with a dense linear ordering without endpoints,

2. the real numbers $\mathbb{R}$ with addition and multiplication and the above ordering,

3. and the real numbers $\mathbb{R}$ with addition, multiplication and exponentiation and the above ordering.

The proof of the o-minimality of these structures is outside the scope of this paper, but proofs can be found in [PS86], some of which rely heavily on quantifier elimination, which in turn is discussed in [Hod97], for example.

However, $\mathbb{R}$ and $<$ with the function $\sin(x)$ is not o-minimal, because we can specify infinitely many points by defining the set where $\sin(x) = 0$.

3 Monotonicity Theorem

The first theorem we cover proves a result about the strict monotonicity and continuity of functions on o-minimal structures. By strictly monotonic we mean that that on each interval on which a function is defined, it is constant, strictly increasing or strictly decreasing.

Theorem 3.1 (Monotonicity Theorem)**.** *Let I be an open interval from a to b on o-minimal structure A and $f : I \to G$ be a definable function, where $M \subseteq A^n$. Then there is a finite sequence*

$$a = a_0 < a_1 < \cdots < a_n = b$$

such that f is continuous on the open interval from a_i to a_{i+1} and strictly monotone for $0 \leq i < n$.

To prove the Monotonicity Theorem, we will first require a series of lemmas about functions on A.

Lemma 3.2. *If $f : I \to G$ is definable, then there is a subinterval $I' \subseteq I$ on which f is strictly monotone.*

To prove this, we require the following additional lemma.

Lemma 3.3. *Let $f : I \to G$ be a definable function such that $f(x) > 0$ for all $x \in I$. Then there is a subinterval $I' \subseteq I$ and some $\varepsilon > 0$ in G such that $f(x') \geq \varepsilon$ for all $x' \in I'$.*

Proof. Define the subset $V \subseteq I$ by:

$$V = \{x \in I : f(y) < f(x) \text{ for all } y < x\}.$$

Since V is definable, it must either be finite or contain an interval by o-minimality. If V contains an interval, then f is strictly increasing on a subinterval of I and we have proven Lemma 3.3.

If V is finite, then we may redefine V on a subinterval of I instead of I, and assume that $V = \emptyset$. By our definition of V, we may choose an infinite decreasing sequence $x_0 > x_1 > \cdots$ in I such that $f(x_0) < f(x_1) < \cdots$. The set of points where $f(x) \geq f(x_0) = \varepsilon$ is infinite and definable, and therefore, it must contain an interval I' that proves Lemma 3.3 because f is everywhere at least $\varepsilon > 0$ there. ∎

We now prove Lemma 3.2.

Proof. For a point $x \in I$, we may consider the three definable sets of points such that $f(y) > f(x)$, $f(y) = f(x)$ or $f(y) < f(x)$, respectively. Thus, we have that, for every y on the open interval from x to x', there is some $x' > x$ such that one of those three conditions is the case.

Since the set of points x for each of these conditions is definable and the union of these three sets completely covers I, one of them will contain an interval. Therefore, if we shrink I to exclude the points on the other sets, we may assume without loss of generality that only one of these circumstances occurs, for example, the first. In the first case we define $g : I \to G$ such that $g(x)$ is the infimum (or greatest lower bound) of the following set:

$$y > x : f(y) \leq f(x).$$

By assuming the first condition above, we have that $g(x) > x$ for all $x \in I$. As g is definable, Lemma 3.3 gives us that there is some subinterval $I' \subseteq I$ and some $\varepsilon > 0$ in G where $g(x) \geq x + \varepsilon$ for every $x \in I'$. If we then shrink I' to have length less than ε, we have from $g \geq x + \varepsilon$ that f is strictly increasing on I', and we have proven Lemma 3.2.

∎

We will require a further lemma for the proof of the Monotonicity Theorem.

Lemma 3.4. *If $f : I \to G$ is definable and strictly monotonic, then there is a subinterval $I' \subseteq I$ on which f is continuous.*

Proof. Assume that f is strictly increasing, so it must be injective, i.e. it maps unique elements from I to unique elements of G. The image $f(I)$ of I is infinite and definable, so there is some interval J that is a subset of $f(I)$. For every $c < d$ in the interval J, the preimage of the open interval from c to d is the open interval in J from the preimage of c to the preimage of d. Therefore because this interval is open in I', we can restrict f to the interval I', which is the preimage of J. Then f is continuous because its preimage is open, and we have proven Lemma 3.4. ∎

We are now ready to prove the Monotonicity Theorem.

Proof. We call f locally continuous and strictly monotone at $x \in I$ when it is continuous and strictly monotone on a subinterval containing x. Thus, by 3.2 and 3.4, the set of points x at which f is locally continuous and strictly monotone is open and dense in I (i.e. every point in I is a point x or arbitrarily close to one such point). As this set is definable, it is cofinite, meaning that the difference of I and the collection of all x is finite.

Shrinking I as necessary, we can then assume that f is continuous and everywhere locally strictly monotone. We now prove that f is strictly monotone.

Consider the set of points $a' < b'$ in the closed interval from a to b, such that f is strictly monotone on the open interval from a to b. As this set is definable, we have that there exists an open interval of I from a' to b' on which f is strictly monotone and is of maximal size with this property. As $a' > a$, then by the local strict monotonicity of f, in this case on a neighborhood of a', it follows that f is strictly monotonic on $(a' - \varepsilon, b')$ for some $\varepsilon > 0$ in R. This contradicts the maximality of (a', b'), and therefore $a' = a$, $b' = b$ and f is strictly monotonic on I. Hence, we have proven Theorem 3.1. ∎

4 Cell Decomposition

In this section, we will prove the Cell Decomposition Theorem, one of the primary early results regarding o-minimal structures. Here we will write $\Gamma(f)$ for the graph of a function f, and $f|S$ for a function f restricted to a set S.

When studying o-minimal structures, we will often want to specify the points and intervals contained in each definable subset. To generalize this across dimensions, we have the notion of a cell.

Definition 4.1. Let $\bar{\imath} = (i_1, \ldots, i_m) \in \{0, 1\}^m$ (i.e. a sequence of zeroes and ones of length m). Then an $\bar{\imath}$-cell $C \subseteq M^m$, where M is the domain of an o-minimal structure, is defined inductively:

1. A 0-cell is a point.

2. A 1-cell is an open interval on M.

3. (We suppose that $\bar{\imath}$-cells have been defined) An $(\bar{\imath}, 0)$-cell is the graph $\Gamma(f)$ of a function $f : C \to M$ that is definable and continuous where C is an $\bar{\imath}$-cell.

4. An $(\bar{\imath}, 1)$-cell is the image of a pair of functions (f, g) on C where C is an $\bar{\imath}$-cell, f, g are definable functions either $C \to M$ or a constant function on C to either $-\infty$ or $+\infty$, and for all $x \in C$, $f(x) < g(x)$.

In particular, our theorem concerns cell decompositions.

Definition 4.2. A partition of a set A is a family of sets S such that S does not contain the empty set, S covers A (or the union of the sets in S is A), and the sets in S are pairwise disjoint (or their intersection is the empty set).

Definition 4.3. A decomposition of M^m the domain of some o-minimal structure is a partition defined by induction on m:

1. A decomposition of $M^1 = M$ is a collection of points and open intervals (where $a_1 < \cdots < a_k$) are points in M:

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\}.$$

2. A decomposition of M^{m+1} is a finite partition of M^{m+1} into cells A such that the set of $\pi(A)$ is a decomposition of M^m. (Where π is the map $M^{m+1} \to M^m$)

We say that a decomposition $\mathcal{D}$ of M^m partitions a set $B \subseteq M^m$ if each cell in $\mathcal{D}$ is either part of B or disjoint from B, i.e. of B is a union of cells of B.

Lastly, we have the following notion of definable connectedness, which will be useful for the proofs of some initial lemmas:

Definition 4.4. A set is connected if it is not the union of two nonempty disjoint open sets.

Definition 4.5. We say that a nonempty definable set S is definably connected when S is not the disjoint union of two nonempty definable open subsets.

Theorem 4.6 (Cell Decomposition)**.** *Given any definable sets $A_1, \ldots, A_k \subseteq M^m$, there is a decomposition of M^m partitioning each of $A_1, \ldots, A_k$.*

To prove this theorem, we proceed by an induction on m. For this induction, we use the following result from the monotonicity theorem.

Theorem 4.7. *For each definable function $f : A \to M, A \subseteq M^m$, there is a decomposition $\mathcal{D}$ of M that partitions A such that the restriction $f|B : B \to M$ to each cell $B \in \mathcal{D}$ with $B \subseteq A$ is continuous.*

By o-minimality, Theorem 4.6 holds for $m = 1$. We assume that Theorem 4.6 holds for $m = 1 \ldots n$, so it remains to prove the theorem for the $m + 1$ case. Additionally, Theorem 4.7 follows directly from monotonicity (as we proved that each such f is continuous), and so we also assume that it holds for $m = 1 \ldots n$.

We begin this induction with a finiteness lemma.

A set $Y \subseteq M^m$ is finite over M^m if for each $x \in M^m$ the fiber $Y_x := \{w \in M : (x, w) \in Y\}$ is finite, and Y is uniformly finite over M^m if there is $N \in \mathbb{N}$ such that $|Y_x| \leq N$ for all $x \in M^m$.

Lemma 4.8. *Suppose the definable subset Y of $M^m + 1$ is finite over M^m. Then Y is uniformly finite over M^m.*

To prove this lemma, we will make and justify a number of claims about boxes.

Definition 4.9. A box $B \subseteq M^m$ is Y-good if for each point $(x, w) \in Y$ with $x \in B$ there is an interval I around w where $Y \cap (B \times I)$ is the graph of f for some continuous function $f : B \to M$.

Claim 4.10. *If the box $B \subseteq M^m$ is Y-good, then there are continuous definable functions $f_1 < \cdots < f_k : B \to M$ such that $Y \cap (B \times M)$ is the union of the graphs of $f_1, \ldots, f_k$.*

To prove this claim, we will require two further subclaims.

We fix $x \in B$ and we write that $Y_x = \{r_1, \ldots, r_k\}$ with $r_1 < \cdots < r_k$. We take intervals $I_1, \ldots, I_k$ around $r_1 < \cdots < r_k$ respectively, and continuous functions $f_1, \ldots, f_k : B \to M$ such that $Y \cap (B \times I_j) = \Gamma(f_j)$ where $j = 1, \ldots, k$.

Claim 4.11. *$f_1 < \cdots < f_k$.*

We will prove the following definable connectedness theorem, and then proceed to prove the above claim.

Theorem 4.12. *Every cell is definably connected.*

Proof. For a 0-cell, it is a point and thus cannot be the disjoint union of two subsets. We suppose that a 1-cell U is not definably connected. We then have that $U = A \cup B$ for nonempty, open and disjoint $A, B \subseteq U$. We can express each subset as a union of intervals $A = A_1 \cup \cdots \cup A_i$ and $B = B_1 \cup \cdots \cup B_i$. We can express these unions as $(a_{11}, a_{12}), \ldots, (a_{i1}, a_{i2})$ and $(b_{11}, b_{12}), \ldots, (b_{i1}, b_{i2})$ respectively. Each such interval has limit points, and therefore we can consider a_{k2} such that it bounds (a_{k1}, a_{k2}) and is not in any other interval. It cannot be in B by its disjointness with A, so this contradicts $U = A \cup B$ and U must be definably connected. Now if K is a cell in M^{m+1}, then we assume inductively that the cell $\pi(K)$ where $\pi : M^{m+1} \to M^m$ is definably connected, and use the fact that each fiber $\pi^{-1}(x) \cap K$ is definably connected to complete the proof. ∎

Now we turn to a justification of Claim 4.11.

Proof. We can prove that $f_1 < f_2$ and the other inequalities will follow by the same method. Assume that there is a point $p \in B$ with $f_1(p) = f_2(p)$. Thus $f_2(p) \in I_1$, and because f_2 is continuous there is a neighborhood $U \subseteq B$ of p such that $f_2(U) \subseteq I_1$. Since $Y \cap (U \times I_1) = \Gamma(f_1|U)$, and $\Gamma(f_2|U) \subseteq (Y \cap (U \times I_1)$, we have that $f_1|U = f_2|U$. It follows that the set $\{p \in B : f_1(p) = f_2(p)\}$ is open. Since $p \in B : f_1(p) < f_2(p)$ and $p \in B : f_1(p) > f_2(p)$ are also open, and B is definably connected, by Theorem 4.12 and with $f_1(x) = r_1 < r_2 = f_2(x)$, we have that $f_1(x) < f_2(x)$. Similarly, we obtain that $f_1 < \cdots < f_k$ across each $f_i < f_{i+1}$. ∎

Claim 4.13. *$Y \cap (B \times M) = \Gamma(f_1) \cup \cdots \cup \Gamma(f_k)$.*

Proof. Consider some point $(a, s) \in Y \cap (B \times M)$ and let $f : B \to M$ be a continuous definable function such that $f(a) = s$ and $\Gamma(f) \subseteq Y$. As $(x, f(x)) \in Y$, it follows that $f(x) = r_i = f_i(x)$ for some $i \in \{1, \ldots, k\}$. As above, this gives that $f = f_i$, and therefore we have justified Claim 4.10. ∎

We now require a further two claims, to prove Lemma 4.8. We will say that some point x is Y-good if x belongs to a Y-good box.

Claim 4.14. *If $A \subseteq M^m$ is a definably connected set and all points of A are Y-good, then there are continuous functions $f_1 < \cdots < f_k : A \to M$ such that $Y \cap (A \times M) = \Gamma(f_1) \cup \cdots \cup \Gamma(f_k)$.*

Proof. We can choose a point $x \in A$, if A is nonempty, and let $k = |Y_x|$. By our first claim the set $\{a \in A : |Y_a| = k\}$ is open and closed in A, hence $|Y_a| = k$ for all $a \in A$. ∎

Claim 4.15. *Every open cell in M^m contains a Y-good point.*

Proof. It suffices to show that each box B in M^m contains a Y-good point. We write that $B = B' \times (a, b)$ for B' a box in M^{m-1}. For each point $p \in B$ consider that $Y(p) := \{(w, s) \in M^2 : a < w < b \text{ and } (p, r, s) \in Y\}$ which is finite over M. We can thus apply the following theorem to $A = Y(p)$ and conclude that the set $\{w \in M : w \text{ is not } Y(p)\text{-good}\}$ is finite.

Theorem 4.16. *Let $A \subseteq \mathbb{R}^2$ be definable such that A_x is finite for each $x \in \mathbb{R}$. Then there are points $a_1 < \cdots < a_k$ in $\mathbb{R}$ such that the intersection of A with each vertical strip $(a_i, a_{i+1}) \times \mathbb{R}$ has the form $\Gamma(f_{i1}) \cup \cdots \cup \Gamma(f_{in(i)})$ for certain definable continuous functions $f_{ij} : (a_i, a_{i+1}) \to \mathbb{R}$ with $f_{i1}(x) < \cdots < f_{in(i)}(x)$ for $x \in (a_i, a_{i+1})$. (Here we have set $a_0 := -\infty$, $a_{k+1} := +\infty$.)*

A proof of this theorem can be found in [vdD98], Chapter 3, Section 1.7.

Therefore, the definable set $\mathrm{Bad}(Y) := \{(p, w) \in B : w \text{ is not } Y(p)\text{-good}\}$ has no interior point. By our inductive assumption on m there is a decomposition which partitions B and $\mathrm{Bad}(Y)$. Take an open cell C of this partition such that $C \subseteq B$. Then $C \cap \mathrm{Bad}(Y) = \emptyset$, hence if we replace B by a box contained in C we have then reduced to the case that $\mathrm{Bad}(Y) = \emptyset$, i.e. for each $p \in B'$ we can apply Claim 4.14 above (with $Y(p) \subseteq M^2$ instead of Y) to find a number $k(p) \in \mathbb{N}$ such that $|Y_x| = k(p)$ for each point $x = (p, w) \in B$. We now must show that the numbers $k(p), p \in B$ have a finite bound.

Thus, we choose a $w \in (a, b)$ and we consider the set:

$$Y^w := \{(p, s) : (p, w, s) \in Y\} \subseteq M^m.$$

As Y is finite over M^m, the set Y^w is finite over R^{m-1}, so by the inductive assumption Y^w is uniformly finite over M^{m-1}, i.e. there is $N \in \mathbb{N}$ such that for each $p \in B' : |\{s \in M : (p, s) \in Y^w\}| \leq N$, or $|Y_{(p,w)}| \leq N$ for all $p \in B'$. Therefore $k(p) \leq N$ for all $p \in B'$ and $|Y_x| \leq N$ for all $x \in B$.

We let $B_i := \{x \in B : |Y_x| = i\}$ for each $i \in \{0, \ldots, N\}$ and we define the functions $f_{i1}, \ldots, f_{ij}$ on each B_i by $f_{i1} < \cdots < f_{ij}$, and $Y_x = \{f_{i1}(x), \ldots, f_{ij}(x)\}$. Applying the inductive assumption for Theorem 4.7 on dimension m to each f_{ij} separately, and then using the inductive hypothesis for Theorem 4.6 to find a common refinement of the decompositions obtained via the induction by Theorem 4.7, we get a decomposition $\mathcal{D}$ of M^m partitioning each of the sets B_i, such that for each $A \in \mathcal{D}$, if $A \subseteq B_i$, then $f_{ij}|A$ is continuous for $j = 0, \ldots, i$. Since B is open and partitioned by $\mathcal{D}$, we have that there exists an open cell $A \in \mathcal{D}$ with $A \subseteq B$. With $B = \bigcup_i B_i$, it follows that $A \subseteq B_i$ for some i, and therefore the functions $f_{i1}, \ldots, f_{ij}$ are continuous on A. Hence each part of A is Y-good and we have established Claim 4.15, because $A \subseteq B$. ∎

We can now prove Lemma 4.8.

Proof. Consider a decomposition $\mathcal{D}$ of M^m that partitions the set of Y-good points, and let $A \in \mathcal{D}$. If A is open, then by Claim 4.15 there is a number $N_A \in \mathbb{N}$ such that $|Y_x| \leq N_A$ for all $x \in A$.

We then rely on the proof in [vdD98], Chapter 3, Section 2.7 using the definable homeomorphism p_A, to show that such a number N_A also exists for the non-open cells $A \in \mathcal{D}$. Now take $N := \max\{N_A : A \in \mathcal{D}\}$. Then $|Y_x| \leq N$ for all x in M^m, and Lemma 4.8 is proven. ∎

We note that a definable set $S \subseteq M$ has finite boundary $\mathrm{bd}(S)$, and that the interval between two successive boundary points is either part of S or disjoint from S.

For a definable set $A \subseteq M^{m+1}$, we write

$$\mathrm{bd}_m(A) := \{(x, w) \in M^{m+1} : w \in \mathrm{bd}(A_x)\}$$

and we can note that $\mathrm{bd}_m(A)$ is a definable set that is finite over M^m, so the uniform finiteness property is therefore applicable, which we employ in the proof below.

We now directly prove Theorem 4.6.

Proof. Define $A_1, \ldots, A_k$ as subsets of M^{m+1}. Define

$$Y := \mathrm{bd}_m(A_1) \cup \cdots \cup \mathrm{bd}_m(A_k).$$

Then $Y \subseteq M^{m+1}$ is definable and finite over M^m, so there is $P \in \mathbb{N}$ such that $|Y_x| \leq P$ for all x in M^m. For each $i \in \{0, \ldots, M\}$, let $B_i := \{x \in M^m : |Y_x| = i\}$, and define functions $f_{i1}, f_{i2}, \ldots, f_{ii}$ on B_i by

$$Y_x = \{f_{i1}(x) \ldots f_{ii}(x)\}, f_{i1}(x) < \cdots < f_{ii}(x).$$

Further, we let $f_{i0} = -\infty$ and $f_{ii+1} = +\infty$ be functions on B_i. Finally we define for each $\lambda \in \{1, \ldots, k\}, i \in \{0, \ldots, M\}$ and $1 \leq j \leq i$:

$$C_{\lambda ij} = \{x \in B_i : f_{ij}(x) \in (A_\lambda)_x\},$$

and for each $\lambda \in \{1, \ldots, k\}, i \in \{0, \ldots, M\}$ and $0 \leq j \leq i$,

$$D_{\lambda ij} = \{x \in B_i : (f_{ij}(x), f_{ij+1}(x)) \subseteq (A_\lambda)_x\}.$$

We then take a decomposition $\mathcal{D}$ of M^m which partitions each set B_i; each set $C_{\lambda ij}$ and each set $D_{\lambda ij}$, and furthermore, which has the property that if $E \in \mathcal{D}$ is contained in B_i, then $f_{i1}|E, \ldots, f_{ii}|E$ are continuous functions. By our inductive assumptions for Theorems 4.6 and 4.7, this decomposition must exist.

For each cell $E \in \mathcal{D}$ we let $\mathcal{D}_E$ be the following partition of $E \times M$:

$$\mathcal{D}_E := \{(f_{i0}|E, f_{i1}|E), \ldots, (f_{ii}|E, f_{ii+1}|E), \Gamma(f_{i1}|E), \ldots, \Gamma(f_{ii}|E)\}$$

where $i \in \{0, \ldots, M\}$ is such that $E \subseteq B_i$. Then $\mathcal{D}^* := \bigcup\{\mathcal{D}_E : E \in \mathcal{D}\}$ is a decomposition of M^{m+1} which partitions each set $A_1, \ldots, A_k$, and we have proven Theorem 4.6. ∎

5 Preliminaries for the Pila-Wilkie Theorem

In this section, we will provide a preliminary understanding of the notions and lemmas from number theory that are required to understand the Pila-Wilkie theorem.

Note that we will work in a structure over $\mathbb{R}$ for this section and the proof of the Pila-Wilkie theorem, and we will use the following notation:

Throughout, we will define $d, e, k, l, m, n \in \mathbb{N}$, and $\varepsilon, c, K \in \mathbb{R}^{>}$, or are positive real numbers. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ we set $|\alpha| := \alpha_1 + \cdots + \alpha_m$.

For $a_1, \ldots, a_n \in \mathbb{R}^{>0}$ the number $\max\{a_1, \ldots, a_n\} \in \mathbb{R}^{>0}$ will equal 0 by convention if $n = 0$. For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ we write $w(a) := \max\{|a_1|, \ldots, |a_n|\} \in \mathbb{R}^{>}$.

The Pila-Wilkie theorem relies on a number of definitions about polynomials and subsets of structures, which we now define.

Definition 5.1. We say that a set $S \subseteq \mathbb{R}^n$ is semialgebraic if it is a finite union of sets defined by some polynomials P where $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : P(x_1, \ldots, x_n) = 0\}$ or $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : P(x_1, \ldots, x_n) > 0\}$ for each P.

Definition 5.2. The algebraic part X^{alg} of definable $X \subseteq \mathbb{R}^n$ is the union of the connected infinite semialgebraic subsets of X. So for $n > 1$, the interior of X is part of X^{alg}.

Definition 5.3. The transcendental part X^{tr} of X is the difference of X and X^{alg}.

Over these sets, we will be interested in counting rational points, which will require the following definitions.

Definition 5.4. The multiplicative height function $H : \mathbb{Q} \to \mathbb{R}$ is defined as $H\left(\frac{a}{b}\right) := \max(|a|, |b|) \in \mathbb{N}^{>1}$ for coprime $a, b \in \mathbb{Z}$, $b \neq 0$.

For this, we also use the following further notation:

1. $X(\mathbb{Q}) := X \cap \mathbb{Q}^n$.
2. $X(\mathbb{Q}, T) := \{a \in X(\mathbb{Q}) : H(a) \leq T\}$ is the set of rational points of X of height $\leq T$ where $T \in \mathbb{R}$.
3. $N(X, T) :=$ the number of points in $X(\mathbb{Q}, T)$.

We can now introduce the Pila-Wilkie theorem:

Theorem 5.5. *Let $X \subseteq \mathbb{R}^n$ with $n > 1$ be definable in some o-minimal structure on the real field. Then for all ε there is a c such that for all T,*

$$N(X^{tr}, T) \leq cT^{\varepsilon}.$$

To prove this theorem, we will rely on a series of major theorems and lemmas. We first define some foundational concepts for presenting those theorems and lemmas, and then state the results themselves.

We first define the necessary notions of parametrization.

Definition 5.6. A C^k-map f is a function that has k continuous derivatives.

Definition 5.7. For definable $X \subseteq \mathbb{R}^m$, X is strongly bounded if $X \subseteq [-N, N]^m$ for some N in $\mathbb{N}$, and a definable function $f : X \to \mathbb{R}^n$ is strongly bounded if its graph $\Gamma(f) \subseteq \mathbb{R}^{m+n}$ is strongly bounded; i.e., $X \subseteq \mathbb{R}^m$ and $f(X) \subseteq \mathbb{R}^n$ are strongly bounded.

Definition 5.8. A partial k-parametrization of X is defined as a definable C^k-map $f : (0,1)^l \to \mathbb{R}^m$ such that $l = \dim X$ (so $X \neq \emptyset$), the image of f is contained in X, and $f^{(\beta)}$ is strongly bounded for all $\beta \in \mathbb{N}^l$ with $|\beta| \leq k$.

Definition 5.9. A k-parametrization of X is a finite set of partial k-parametrizations of X whose images cover X; i.e. X is strongly bounded.

Definition 5.10. For $k, n > 1$ and $X \subseteq \mathbb{R}^n$ a strong k-parametrization of X is a C^k-map $f : (0,1)^m \to \mathbb{R}^n, m < n$, with image X, such that $w(f^{(\alpha)}(a)) \leq 1$ for all $\alpha \in \mathbb{N}^m$ with $w(\alpha) \leq k$ and all $a \in (0,1)^m$.

Definition 5.11. A hypersurface in $\mathbb{R}^n$ of degree $\leq e$ is the set of solutions in $\mathbb{R}^n$ of a nonzero polynomial in $x = (x_1, \ldots, x_n)$ over $\mathbb{R}$ of degree less than e.

The following theorems rely are results of the proofs found in [BP89], [Yom87] and [Gro87].

Theorem 5.12. *Let $n > 1$ be given. Then for any $e > 1$ there are $k = k(n,e) > 1$, $\varepsilon = \varepsilon(n,e)$, and $c = c(n,e)$, such that if $X \subseteq \mathbb{R}^n$ has a strong k-parametrization, then for all T at most cT^ε many hypersurfaces in $\mathbb{R}^n$ of degree less than e are enough to cover $X(\mathbb{Q}, T)$, with $\varepsilon(n,e) \to 0$ as $e \to \infty$.*

Theorem 5.13. *Given an o-minimal structure on $\mathbb{R}$, every definable set $X \subseteq [-1,1]^n$ with empty interior and $n > 1$ is for every $k > 1$ a finite union of definable subsets, each having a definable strong k-parametrization.*

The proofs of both theorems can be found in [BvdD22], for example.

We depend also on a number of lemmas, which we now state.

Lemma 5.14. *If $X = X_1 \cup \cdots \cup X_m$, then $X^{alg} \supseteq X_1^{alg} \cup \cdots \cup X_m^{alg}$, and thus $X^{tr} \subseteq X_1^{tr} \cup \cdots \cup X_m^{tr}$.*

Proof. X^{alg} includes the finite unions of semialgebraic sets in different X_k^{alg} for $k \in \mathbb{N}$, so it may be larger than $X_1^{\text{alg}} \cup \cdots \cup X_m^{\text{alg}}$, and hence $X^{\text{tr}} \subseteq X_1^{\text{tr}} \cup \cdots \cup X_m^{\text{tr}}$. ■

Lemma 5.15. *We assume that $S \subseteq \mathbb{R}^n$ is semialgebraic, $f : S \to \mathbb{R}^m$ is semialgebraic and injective, and f maps the set $X \subseteq S$ homeomorphically onto $Y = f(X) \subseteq \mathbb{R}^m$. Therefore $f(X^{alg}) = Y^{alg}$ and thus $f(X^{tr}) = Y^{tr}$. (We allow $m = 0$ for later inductions.)*

Proof. We have that $f(X^{\text{alg}}) \subseteq Y^{\text{alg}}$ and for any connected infinite semialgebraic set $C \subseteq Y$, the set $f^{-1}(C) \subseteq S$ is semialgebraic because C and f are, contained in X because f is injective. Hence $f^{-1}(C) \subseteq S$ is connected and infinite, and $f^{-1}(C) \subseteq X^{\text{alg}}$. This gives us that $f^{-1}(Y^{\text{alg}}) \subseteq X^{\text{alg}}$, and thus $f(X^{\text{alg}}) = Y^{\text{alg}}$. ■

An initial step we make towards proving Theorem 5.5 is to reduce to the case of subsets of $[-1,1]^n$, so that Theorem 5.13 can be applied, which we do below

For $X \subseteq \mathbb{R}^n$ and $I \subseteq \{1,\dots,n\}$, we define

$$X_I := \{a \in X : w(a_i) > 1 \text{ for all } i \in I, w(a_i) \leq 1 \text{ for all } i \notin I\}.$$

We also define the semialgebraic map $f_I : \mathbb{R}^n \to \mathbb{R}^n$ by $f_I(a) = b$ where $b_i := a_i^{-1}$ for $i \in I$ and $b_i := a_i$ for $i \notin I$. Thus f_I maps $\mathbb{R}^n$ homeomorphically onto its image, a subset of $[-1,1]^n$. If $I = \emptyset$, then f_I is the inclusion map $\mathbb{R}^n = [-1,1]^n \to \mathbb{R}^n$. Note that for $a \in \mathbb{Q}^n$ we have $f_I(a) \in \mathbb{Q}^n$ and $H(a) = H_{f_I(a)}$. Moreover, X is the disjoint union of the sets X_I, and for $Y_I = f_I(X_I)$ we have $Y_I \subseteq [-1,1]^n$, $Y_I^{\text{tr}} = f_I(X_I^{\text{tr}})$ by Lemma 2.2, so $N(Y_I^{\text{tr}}, T) = N(X^{\text{tr},I}, T)$ for all T.

6 Proof of the Pila-Wilkie Theorem

In this section, we will complete the proof of Theorem 5.5.

We first prove the theorem as it is stated above. However, we will see that it relies on a final assumption, and then briefly discuss definable families, which allows for two generalizations of the proof.

Proof. We let $X \subseteq \mathbb{R}^n$ be definable, and then proceed by induction on n. By 5.14 we have that if X is open in $\mathbb{R}^n$, then $X^{\text{tr}} = \emptyset$. Then we can remove the interior of X in $\mathbb{R}^n$ from X and arrange that X has empty interior. Additionally, we arrange that $X \subseteq [-1,1]^n$ as above.

We let ε be given and we take some $e \geq 1$ large enough that $\varepsilon(n,e) \leq \varepsilon/2$ in Theorem 5.12. Also taking $k = k(n,e)$, we have from Theorem 5.13 for the structure over $\mathbb{R}$ that $M \in \mathbb{N}$ where $X = X_1, \dots, X_M$ where X_i is definable and admits a strong k-parametrization.

Then by Theorem 5.12 we have that $X(\mathbb{Q},T) \subseteq \bigcup_{i=1}^M \bigcup_{j=1}^J H_{ij}$, where H_{ij} is a hypersurface in $\mathbb{R}^n$ of degree $\leq e$, $J \in \mathbb{N}$, $J \leq cT^{\varepsilon/2}$ and $c = c(n,e)$. If $a \in X^{\text{tr}}(\mathbb{Q},T)$ and $a \in H_ij$, then we have $a \in (X \cap H_{ij})^{\text{tr}}$ and thus $X^{\text{tr}}(\mathbb{Q},T) \subseteq \bigcup_{i=1}^M \bigcup_{j=1}^J (X \cap H_{ij})^{\text{tr}}(\mathbb{Q},T)$.

We let H be any hypersurface in $\mathbb{R}^n$ of degree $\leq e$. We will now prove an upper bound on $N((X \cap H)^{\text{tr}}, T)$ of the form $c_1 T^{\varepsilon/2}$ with $c_1 \in \mathbb{R}^>$ independent of H and T. By applying this to the hypersurfaces H_{ij}, we have $N(X^{\text{tr}},T) \leq MJc_1T^{\varepsilon/2} \leq Mc_1T^{\varepsilon/2}c_1T^{\varepsilon/2} = Mc_1cT^{\varepsilon}$ and complete the proof.

We take semialgebraic cells $C_1,\dots,C_L$ in $\mathbb{R}^n$ and $L \in \mathbb{N}$, such that $H = C_1 \cup \dots \cup C_L$. Then assume that $C = C_l$ is one such cell. By the result from [vdD98], Chapter III, Section 2.7 we have a semialgebraic homeomorphism $p = p_C : C \to p(C) = p(C_l)$ onto an open cell $p(C_l)$ in $\mathbb{R}^{n_l}$ with $n_l < n$. Thus p maps homeomorphically onto its image $Y_l \subseteq p(C_l) \subseteq \mathbb{R}^{n_l}$. p is now given by omitting $n - n_l$ of the coordinates, so for $a \in C_l(\mathbb{Q})$ we have $p(a) \in \mathbb{Q}^{n_l}$ and $H(p(a)) \leq H(a)$. The hypersurfaces of degree $\leq e$ in $\mathbb{R}^n$ belong to one semialgebraic family. Now by [vdD98], Chapter III, Section 3.6, we take $L \leq L(e,n)$, with $L(e,n) \in \mathbb{N}^{\geq 1}$ depending only on e, n. By Lemma 5.14, we then have that $(X \cap H_{ij}) \subseteq (X \cap C_1)^{\text{tr}}, \dots, (X \cap C_1)^{\text{tr}}$. As $n_l < n$ we can assume inductively that for all T, $N(Y_l^{\text{tr}}, T) \leq B_l T^{\varepsilon/2}$ where $l = 1,\dots,L$ with $B_l \in \mathbb{R}^>$ independent of T. Then for all T, $N(((X \cap C_l)^{\text{tr}}), T) \leq B_l T^{\varepsilon/2}, l = 1,\dots,L$ when Lemma 5.15 is applied to the maps $p = p_{C_l}$, and then $N(((X \cap C_l)^{\text{tr}}), T) \leq (B_1 + \dots + B_L)T^{\varepsilon/2}$.

We assume that we can take $B_1, \ldots, B_L \leq B$ with $B \in \mathbb{R}^>$ dependent only on X, ε and not on $H, Y_1, \ldots, Y_L$. Therefore $c_1 := L(e,n)B$ is a positive real number and we have Theorem 5.5. ■

We now provide a brief discussion of definable families, before proving two generalized versions of the Pila-Wilkie theorem.

Definition 6.1. For $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$, and $s \in E$, we define a section $X(s)$ as $\{a \in \mathbb{R}^n : (s,a) \in X\}$.

For the family $\{X(s)\}_{s \in E}$ of sections $X(s) \subseteq \mathbb{R}^n$; and we call these sets $X(s)$ the members of the family described by E, X. If E and X are definable, we write that it is a definable family, and it follows that its members are definable subsets of $\mathbb{R}^n$.

Definition 6.2.

We will usually divide the family given by E, X into the subfamilies that we have from some covering $E = E_1 \cup \cdots \cup E_N$, where E_ν is the set of $s \in E$ for which $X(s)$ satisfies some specified condition e_ν. Then $X = X_1 \cup \cdots \cup X_N$ with $X_\nu := X \cap (E_\nu \times \mathbb{R}^n)$, such that $X_\nu(s)$ satisfies e_ν for all $s \in E_\nu$.

For the next lemma, (which follows from the proof in [vdD98], Chapter III, Section 3), for $i = (i_1, \ldots, i_n) \in \{0,1\}^n$ we have from [vdD98], Chapter III, Section 2 that there is

$$p_i : \mathbb{R}^n \to \mathbb{R}^d, \quad d := i_1 + \cdots + i_n,$$

which maps every i-cell homeomorphically onto its image, an open cell in $\mathbb{R}^d$.

Lemma 6.3. *Let $e > 1$ and define that $D := \binom{e+n}{n}$, the dimension of the set of polynomials over $\mathbb{R}$ in n variables and of degree $\leq e$. Then there are $L \in \mathbb{N}_{>1}$ and semialgebraic sets $H, C_1, \ldots, C_L \subseteq F \times \mathbb{R}^n$, $F := \mathbb{R}^D \setminus \{0\}$, such that*

$$\{H(t) : t \in F\} = \textit{set of hypersurfaces in } \mathbb{R}^n \textit{ of degree } \leq e,$$

$H(t) = C_1(t) \cup \cdots \cup C_L(t)$ for all $t \in F$, and for each $l \in \{1, \ldots, L\}$ there is an $i = (i_1, \ldots, i_n) \in \{0,1\}^n$, $i \neq (1, \ldots, 1)$, with the property that every $C_l(t)$ with $t \in F$ is a semialgebraic i-cell in $\mathbb{R}^n$ or empty.

In the following proofs we assume that $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$ are definable.

Theorem 6.4. *Let any ε be given. Then there is a constant $c = c(X, \varepsilon)$ such that for all $s \in E$ and all T we have $N(X(s)^{tr}, T) \leq cT^\varepsilon$.*

Proof. As in the proof above, we proceed by induction on n and reduce to the case where $X(s)$ is for every $s \in E$ a subset of $[-1,1]^n$ with empty interior. We take $e > 1$ sufficiently large that $\varepsilon(n,e) \leq \varepsilon/2$ in Theorem 5.12, and set $k = k(n,e)$. Then for every $Z \subseteq \mathbb{R}^n$ with a strong k-parameterization we can cover $Z(Q,T)$ with at most $cT^{\varepsilon/2}$ hypersurfaces of degree $\leq e$ such that $c = c(n,e)$ is as in Theorem 5.12. From Theorem 5.13 we have that there are definable sets $X_1, \ldots, X_M \subseteq E \times \mathbb{R}^n$, $M \in \mathbb{N}$, such that for all $s \in E$, $X(s) = X_1(s) \cup \cdots \cup X_M(s)$ and each $X_i(s)$ is empty or has a strong k-parametrization. We let $s \in E$, and let H be a hypersurface of degree $\leq e$. As in the previous proof we see that by our choice of k, e it suffices to prove:

$$N((X(s) \cap H)^{\text{tr}}, T) \leq c_1 T^{\varepsilon/2}, \text{ for all } T,$$

where $c_1 \in \mathbb{R}_>$ depends only on X, ε, not on s, H, T. Now we provide some such c_1. With the above values of e and n, we define $D := \binom{e+n}{n}$, $F := \mathbb{R}^D \setminus \{0\}$, and let $H, C_1, \ldots, C_L \subseteq F \times \mathbb{R}^n$ be as in Lemma 6.3. For $l = 1, \ldots, L$, take $i_l = (i_l^1, \ldots, i_l^n)$ in $\{0,1\}^n$, not equal to $(1, \ldots, 1)$, such that for all $t \in F$ the subset $C_l(t)$ of $\mathbb{R}^n$ is a semialgebraic i_l-cell or empty, so

$$p_{i_l} : \mathbb{R}^n \to \mathbb{R}^{n_l}, \quad n_l := i_l^1 + \cdots + i_l^n < n,$$

maps $C_l(t)$ homeomorphically onto its image. Then we have for $l = 1, \ldots, L$ a definable set $Y_l \subseteq (E \times F) \times \mathbb{R}^{n_l}$ such that for all $(s,t) \in E \times F$,

$$Y_l(s,t) = p_{i_l}\left(X(s) \cap C_l(t)\right).$$

Since all $n_l < n$ we can assume inductively that for all $(s,t) \in E \times F$ and all T,

$$N(Y_l(s,t)^{\text{tr}}, T) \leq B_l T^{\varepsilon/2}, \quad l = 1, \ldots, L$$

with $B_l = B_l(Y_l, \varepsilon) \in \mathbb{R}_>$ independent of s, t, T. Since $H = H(t)$ for some $t \in F$,

$$N\left((X(s) \cap H)^{\text{tr}}, T\right) \leq (B_1 + \cdots + B_L) T^{\varepsilon/2},$$

as in the sketch. Thus $c_1 := B_1 + \cdots + B_L$ is as promised. ∎

Lastly, we prove a variant of Theorem 6.4 where from the sets $X(s)$ only a definable part $V(s)$ of $X(s)$ alg is removed instead of all of it.

Theorem 6.5. *Theorem 2.5. Let any ε be given. Then there is a definable set $V = V(X, \varepsilon) \subseteq X$ and a constant $c = c(X, \varepsilon)$ such that for all $s \in E$ and all T, $V(s) \subseteq X(s)$ alg and N*
$X(s) \setminus V(s), T \leq cT^{\varepsilon}$.

Proof. The proof follows similarly to that of Theorem 6.4. We let $V_0 \subseteq X$ be given by $V_0(s) =$ the interior of $X(s)$ in $\mathbb{R}^n$ for $s \in E$. This definable set V_0 will be part of some V as necessary. By replacing X by $X \setminus V_0$ we arrange that $X(s)$ has empty interior for all $s \in E$. Then we also arrange that $X(s) \subseteq [-1,1]^n$ for all $s \in E$. Now take e and $k = k(n,e)$ as in the proof of Theorem 6.4. It now suffices to find a definable $V \subseteq X$ and a constant $c_1 \in \mathbb{R}$ $>$ such that for all $s \in E$, every hypersurface H of degree $\leq e$ in $\mathbb{R}^n$, and all T we have $V(s) \subseteq X(s)$ alg , N
$(X(s) \cap H) \setminus V(s), T \leq c_1 T^{\varepsilon/2}$. We now take the semialgebraic sets $H, C_1, \ldots, C_L \subseteq F \times \mathbb{R}^n$ and the definable sets $Y_l \subseteq E \times F \times \mathbb{R}^{n_l}$ for $l = 1, \ldots, L$ as in the proof of Theorem 6.4. For such l we have $n_l < n$, so we can assume inductively that there is also a definable set $W_l \subseteq Y_l$ and a number $B_l = B_l(Y_l, \varepsilon) \in \mathbb{R}$ $>$ such that for all $s \in E$, $t \in F$, and T we have $W_l(s,t) \subseteq Y_l(s,t)$ alg and $N(Y_l(s,t) \setminus W_l(s,t), T) \leq B_l T^{\varepsilon/2}$. Now the definable set $V \subseteq X$ such that for all $s \in E$, $V(s) = \bigcup_{l=1}^{L} \bigcup_{t \in F} C_l(t) \cap p_i^{-1} W_l(s,t)$ has the specified property and the proof is complete. ∎

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References

[BP89] E. Bombieri and J. Pila. The number of integral points on arcs and ovals. *Duke Math. J.*, 59(2):337–357, 1989.

[BvdD22] Neer Bhardwaj and Lou van den Dries. On the Pila-Wilkie theorem. *Expo. Math.*, 40(3):495–542, 2022.

[Gro84] Alexandre Grothendieck. *Esquisse d'un programme*. Chez l'auteur, 1984.

[Gro87] Mikhail Gromov. Entropy, homology and semialgebraic geometry. *Ast\'{e} risque*, (145-146):5, 1987.

[Hod97] Wilfrid Hodges. *A shorter model theory*. Cambridge University Press, Cambridge, 1997.

[PS86] Anand Pillay and Charles Steinhorn. Definable sets in ordered structures. i. *Transactions of the American Mathematical Society*, 295(2):565–592, 1986.

[PW06] J. Pila and A. J. Wilkie. The rational points of a definable set. *Duke Math. J.*, 133(3):591–616, 2006.

[vdD98] Lou van den Dries. *Tame topology and o-minimal structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[Yom87] Y. Yomdin. C^k-resolution of semialgebraic mappings. Addendum to: "Volume growth and entropy". *Israel J. Math.*, 57(3):301–317, 1987.