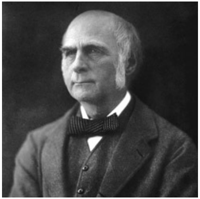
On the Galton–Watson Process and Its Modeling Capabilities

Kevin Zhu

Euler Circle IRPW

July 2025

Origination of the Galton-Watson Process



Francis Galton

Henry William Watson

Defining the Galton-Watson Process

- Generation 0 has 1 individual.
- Each individual within the same branching tree has the same distribution for the number of offspring they produce.
- The offspring of generation n become generation $n + 1$.

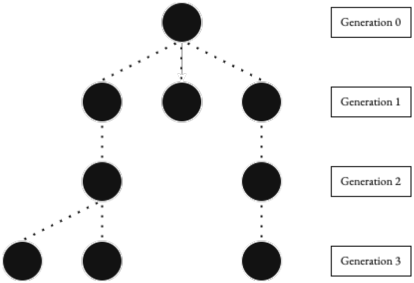


Figure: Possible Branching Tree

The Key Question

What is the probability that a branching tree eventually dies out?

Answer:

The Key Question

What is the probability that a branching tree eventually dies out?

Answer: It depends on the expected number of children an individual has.

Probability Generating Functions

Definition

X is the offspring distribution.

The probability generating function of X is denoted as $g(s)$.

$$g(s) = \sum_{i=0}^{\infty} \mathbb{P}(X = i)s^i$$

Theorem

$\mathbb{E}(X) = g'(1)$.

Preliminary Result

Theorem

The probability that a branching tree eventually dies out is the smallest nonnegative solution to the equation $s = g(s)$.

Proof Sketch

Theorem

The probability that a branching tree eventually dies out is the smallest nonnegative solution to the equation $s = g(s)$.

Definition

q_n is the probability that the branching tree dies out by the nth generation.

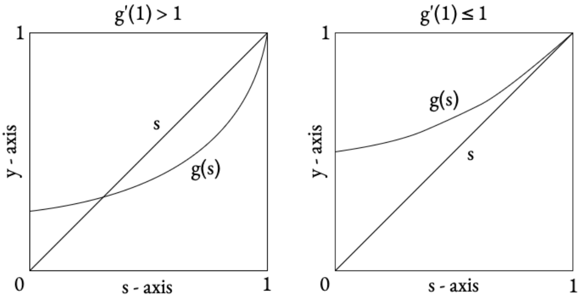
Proof.

$$g(q_{n-1}) = \sum_{i=0}^{\infty} \mathbb{P}(X = i)(q_{n-1})^i = q_n$$

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} g(q_{n-1}) = g\left(\lim_{n\to\infty} q_n\right)$$

□

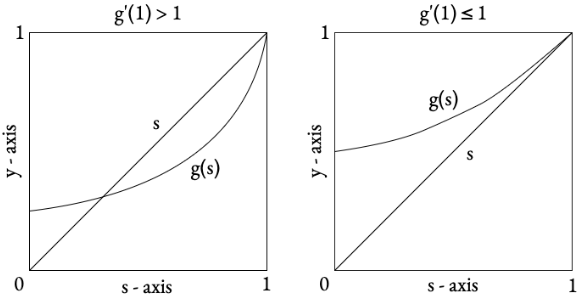
$s = g(s)$



Theorem

- $g(1) = 1$.
- $g''(s) > 0 : s \in (0, 1)$.
- $0 \leq g(0) \leq 1$.

Main Result



Theorem

- *If* $g'(1) = \mathbb{E}(X) \leq 1$, *then* $\lim_{n\to\infty} q_n = 1$.
- *If* $g'(1) = \mathbb{E}(X) > 1$, *then* $\lim_{n\to\infty} q_n < 1$.

When X is geometrically distributed

Definition

$\mathbb{P}(X = i) = (1 - p)p^i$.

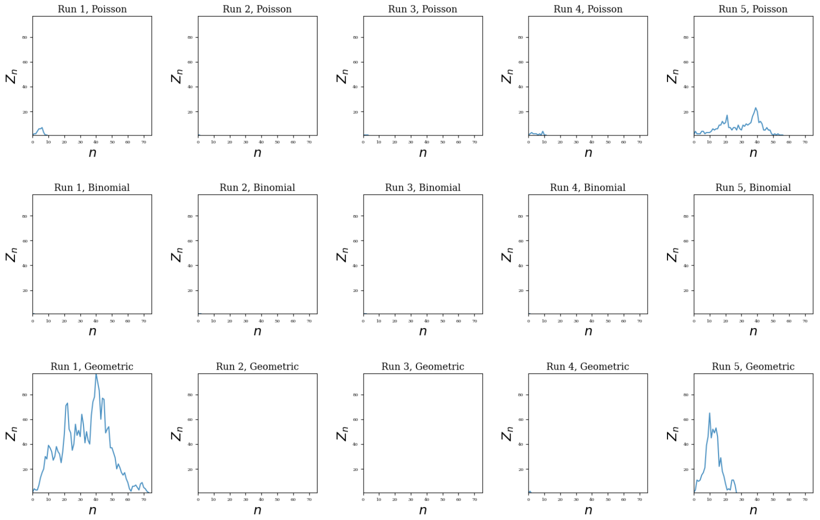
i	0	1	2	3	...
$\mathbb{P}(X = i)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	...

Table: $p = \frac{1}{2}$

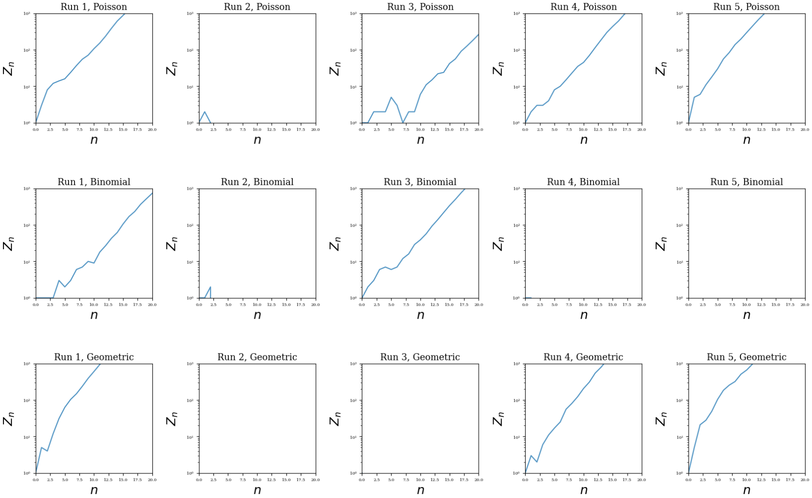
Theorem

- *If* $p \leq \frac{1}{2}$, *then* $\lim_{n\to\infty} q_n = 1$.
- *If* $p > \frac{1}{2}$, *then* $\lim_{n\to\infty} q_n = \frac{1-p}{p}$.

Poisson V. Binomial V. Geometric ($\mathbb{E}(X) = 1$)



Poisson V. Binomial V. Geometric ($\mathbb{E}(X) = 1.5$)



Bisexual Galton-Watson Process

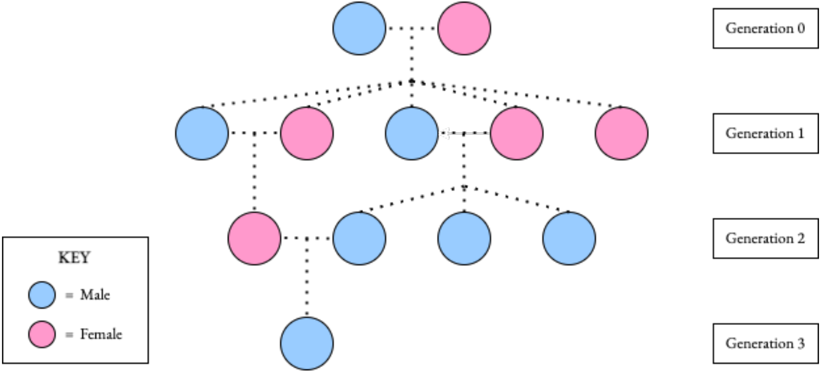


Figure: Possible Bisexual Branching Tree

Age-dependent Galton-Watson Process

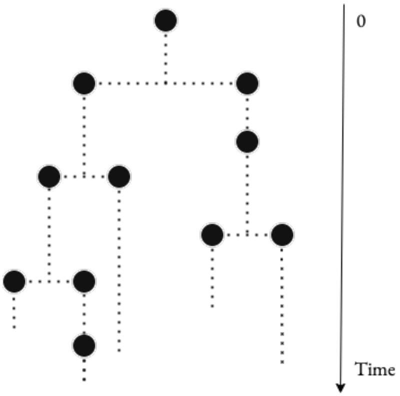
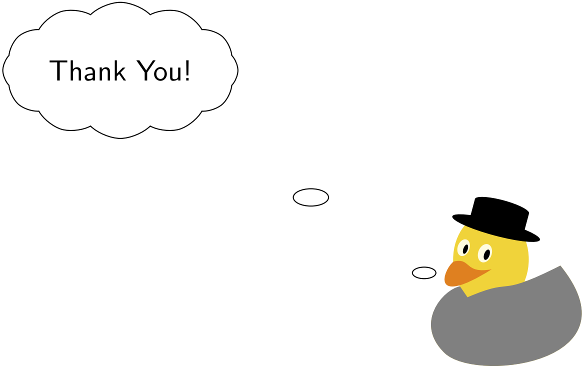


Figure: Possible Age-dependent Branching Tree


Thank You!