Isoperimetric Inequality

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Queen Dido

The Isoperimetric Inequality originated from a story about Queen Dido. Dido originally lived in the city of Tyre where she faced the tyranny of her brother. Leaving the city with some inhabitants, they found a piece of land to settle on. The local chief was not happy and told Dido that she could have any land that could be encircled by an oxen's hide. Dido cut the oxen's hide into strips and tied them into a long rope, encircling a large piece of land on which she later founded Carthage.

The problem now shows up. What is the shape that will result in the largest area?

Isoperimetric Inequality

It turns out that the shape that gives the largest area is a circle. In fact, the Isoperimetric Inequality describes the relation between the area and perimeter of an encircled area. It states

$$L^2 \geq 4\pi A$$

where equality holds when the encircled area is a circle.

Fourier Series

- The Fourier Series is a sum that can estimate any function.
- Sums sine and cosine waves to make any function
- Generally, the formula for the Fourier Series of a function is

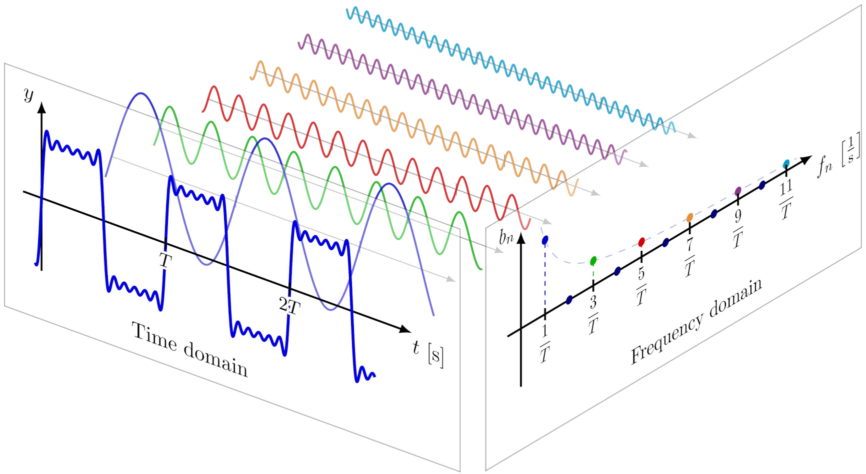
$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(n x) + \sum_{n=1}^{\infty} b_n \sin(n x),$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, d x$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(n x)\, d x$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(n x)\, d x$$



Parseval's Theorem

Parseval's Theorem states that if a function has Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx)$$

then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

The full proof of this theorem is in the paper.

Wirtinger's Inequality

Wirtinger's Inequality states that let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function. It is continuous and also has a continuous derivative. If

$$\int_0^{2\pi} f(x) = 0$$

then

$$\int_0^{2\pi} f'(x)^2 dx \geq \int_0^{2\pi} f(x)^2 dx$$

The equality case holds when $f(x) = acosx + bsinx$ for some a and b. Proving this requires the Fourier Series and Parseval's Theorem. The full proof is in the paper.

Isoperimetric Inequality on a Triangle

Before we tackle the actual problem, let's try out a simpler version. The Isoperimetric Inequality on a Triangle states that if L is the perimeter and A is the area,

$$L^2 \geq 12\sqrt{3} \cdot A$$

where equality holds when it is an equilateral triangle. The proof of this requires the AM-GM Inequality. AM-GM states for all non-negative real numbers $a_1, ..., a_n$

$$\frac{a_1 + ... + a_n}{n} \geq \sqrt[n]{a_1 \cdot ... \cdot a_n}$$

Equality holds when $a_i = a_j$ for all i and j. The proof of this inequality utilizes Cauchy Induction and is covered in the paper.

Isoperimetric Inequality on a Triangle

Cauchy Induction:

- Prove base case works (most of the time its n = 0, 1, 2)
- Prove that n = 2^k for all k works
- Prove that n - 1 given n works

We don’t have time for proving every part, but I just wanted to share an interesting proof of the base case of n = 2.

Isoperimetric Inequality on a Triangle

The base case of the AM-GM Inequality is with 2 variables. We need to prove that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

There is a simple way to prove this by expanding and rearranging the inequality

$$(x-y)^2 \geq 0$$

But we are here to have fun, so we will prove it geometrically.

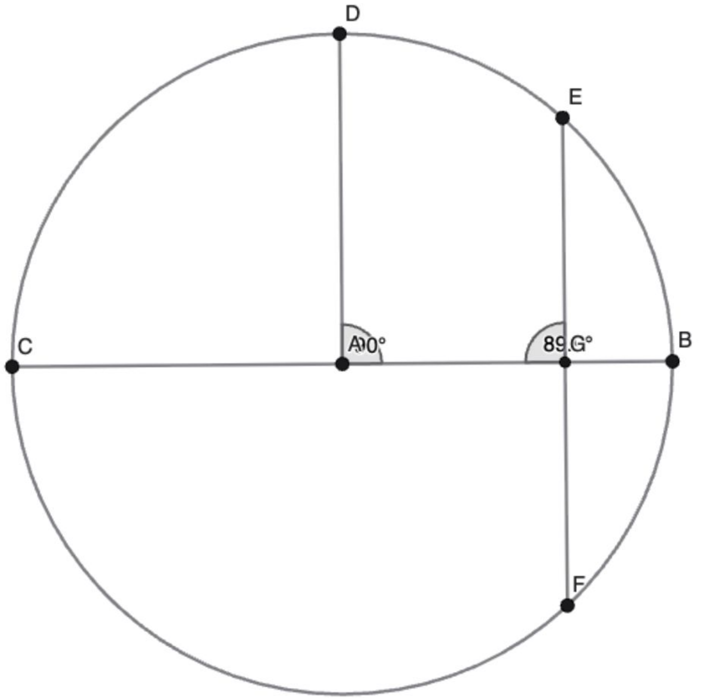
Isoperimetric Inequality on a Triangle

We start off by drawing a circle with diameter x + y with CG being equal to x and GB being equal to y.

Notice that AD is the radius, so it has length $\frac{x+y}{2}$

Using Power of a Point on chords CB and EF, we get that EG = GF = $\sqrt{xy}$. We can see that no matter where G is on CB, the length of EG will always be shorter than AD, which means

$$\frac{x+y}{2} \geq \sqrt{xy}$$



Isoperimetric Inequality on a Triangle

Now we will prove the Isoperimetric Inequality on a Triangle. Define the semiperimeter to be s and the 3 sides to be a, b, and c. We will use the AM-GM Inequality on the 3 terms (s - a), (s - b), and (s - c). We get

$$\frac{(s-a)+(s-b)+(s-c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}$$

$$\frac{s}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}$$

$$\frac{s^3}{27} \geq (s-a)(s-b)(s-c)$$

Isoperimetric Inequality on a Triangle

Applying Heron's Formula, we get

$$\frac{s^4}{27} \geq A^2$$

$$s^2 \geq 3\sqrt{3} \cdot A$$

$$L^2 \geq 12\sqrt{3} \cdot A$$

thus proving the inequality.

Adolf Hurwitz's Proof

Now we will prove the regular Isoperimetric Inequality. We can start by splitting the closed curve into 2 piecewise functions x(s) and y(s) where s goes from 0 to L. It satisfies

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$$

Now we will make these 2 functions 2π periodic.

$$f(\theta) = x\left(\frac{L\theta}{2\pi}\right)$$

$$g(\theta) = y\left(\frac{L\theta}{2\pi}\right)$$

and we will define

$$s = \frac{L\theta}{2\pi}$$

Adolf Hurwitz's Proof

By taking the derivative of f and g, we can piece it together to form

$$(f'(\theta))^2 + (g'(\theta))^2 = \frac{L^2}{4\pi^2}$$

This is based off of the condition from last slide. Now we will calculate the area integral of the shape. Remember that the formula for the area enclosed by a curve is

$$A = \frac{1}{2}\int_0^L (x(s)y'(s) - y(s)x'(s))ds$$

Adolf Hurwitz's Proof

Now we need to state a few conditions. Because $s = \frac{L\theta}{2\pi}$, the following must be true.

$$ds = \frac{L}{2\pi} d\theta$$

$$x(s) = f(\theta)$$

$$y(s) = g(\theta)$$

$$x'(s) = \frac{2\pi}{L} \cdot f'(\theta)$$

$$y'(s) = \frac{2\pi}{L} \cdot g'(\theta)$$

Putting all the conditions here into the area integral and solving, we get

$$A = \frac{1}{2} \int_0^{2\pi} \left(\frac{2\pi}{L} f(\theta) g'(\theta) - \frac{2\pi}{L} g(\theta) f'(\theta) \right) \frac{L}{2\pi} d\theta$$

$$2A = \int_0^{2\pi} (f(\theta) g'(\theta) - g(\theta) f'(\theta)) d\theta$$

Adolf Hurwitz's Proof

Remember that the formula for Integration by parts is

$$\int_0^{2\pi} f(\theta)g'(\theta)d\theta = [f(\theta)g(\theta)]_0^{2\pi} - \int_0^{2\pi} g(\theta)f'(\theta)d\theta$$

However, since f and g are 2π periodic functions,

$$[f(\theta)g(\theta)]_0^{2\pi} = f(2\pi)g(2\pi) - f(0)g(0) = 0$$

Which means

$$\int_0^{2\pi} f(\theta)g'(\theta)d\theta = -\int_0^{2\pi} g(\theta)f'(\theta)d\theta$$ and it leads to $$2A = 2\int_0^{2\pi} f(\theta)g'(\theta)d\theta$$

Adolf Hurwitz's Proof

Completing the square, we get

$$2A = \int_0^{2\pi} (f(\theta)^2 + g'(\theta)^2 - (f(\theta) - g'(\theta))^2)d\theta$$

Because the last term is non-negative,

$$2A \leq \int_0^{2\pi} (f(\theta)^2 + g'(\theta)^2)d\theta$$

Applying Wirtinger's Inequality, we get

$$2A \leq \int_0^{2\pi} (f'(\theta)^2 + g'(\theta)^2)d\theta$$

Because $(f'(\theta))^2 + (g'(\theta))^2 = \frac{L^2}{4\pi^2}$, we get $2A \leq \int_0^{2\pi} \frac{L^2}{4\pi^2} d\theta = \frac{L^2}{4\pi^2} \cdot 2\pi = \frac{L^2}{2\pi}$

So,

$$L^2 \geq 4\pi A$$

Different Ways of Generalization

There are many different ways of generalization, the most obvious being generalization to higher dimensions. For example, in a 3d, the inequality compares the surface area to the volume. The most optimal shape in 3d is a sphere.

We can also generalize to different spaces instead of the 2d Euclidean Space. For example we could generalize to the plane on a sphere. The inequality becomes

$$L^2 \geq 4\pi A - \frac{A^2}{R^2}$$

when it is on a sphere.

Applications of the Inequality

- Bound mixing times of Markov Chains
- Analyzing random walks
- Explain things in physics such as why bubbles always eventually solidify into spheres
- Most liquids on surfaces obey the isoperimetric properties

References

[1] Weisstein, Eric W. "Parseval's Theorem." From MathWorld–A Wolfram Resource. https://mathworld.wolfram.com/ParsevalsTheorem.html

[2] rspuzio (6075). "Wirtinger's Inequality." 2013-3-22. https://planetmath.org/wirtingersinequality

[3] Weisstein, Eric W. "Bessel's Inequality." From MathWorld–A Wolfram Resource. https://mathworld.wolfram.com/BesselsInequality.html

[4] Singh, Pushpendra. 2021. "Proper Definitions and Demonstration of Dirichlet Conditions." https://www.techrxiv.org/users/45748/articles/677069-proper-definitions-and-demonstration-of-dirichlet-conditions.

[5] Treibergs, Andrejs. 2009. "Mixed Area and the Isoperimetric Inequality." https://www.math.utah.edu/ treiberg/MixedAreaSlides.pdf.

Questions?