Isoperimetric Inequalities

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Contents

Abstract

In this paper we will go over the proof of the Isoperimetric Inequality. We will briefly talk about the Fourier Series, Parseval's Theorem, and Wirtinger's Inequality. Later on, we will also look at a simpler version of the problem and its applications.

1 Introduction

The history of the Isoperimetric Inequality goes all the way back to the ancient times in 814 BC when Queen Dido founded Carthage in Northern Africa. Queen Dido came from the City of Tyre where her brother, Pygmalion, ruled. Facing the tyranny, she took a group of inhabitants to leave the city and headed westwards along the coast of the Mediterranean Sea. They eventually came across a patch of land that they decided to settle on. The original inhabitants in the area were not happy and the chief decided to mock Dido by granting them any land that could be encircled by an ox's hide. The Phoenicians decided to cut the ox's hide into thin strips. They then tied the strips together into a long rope and encircled a large piece of land on which Dido later founded the City of Carthage.

The problem now shows up. Dido had to encircle the largest possible piece of land she could using a limited length of ropes. What is the most optimal shape for the largest area? It turns out that the answer is a circle. This is where the Isoperimetric Inequality comes in.

The Isoperimetric Inequality is an inequality that relates the area to the perimeter. The intuitive idea of the inequality is that, given a fixed length L, the shape with the largest area having a perimeter L is a circle. Another way of saying this is given a fixed area A, the shape with the least perimeter is a circle.

The formal statement is as follows. If L is the perimeter of a simple closed curve C, and A is the area it bounds, then

$$L^2 \geq 4\pi A$$

Additionally, equality holds when C is a circle. Before going into the proof, I'd like to introduce some of the prerequisites.

2 Prerequisites

This proof of the Isoperimetric Inequality utilizes 3 important concepts, Fourier Series, Parseval's Theorem, and Wirtinger's Inequality.

2.1 Fourier Series

The Fourier Series is a powerful tool to help approximate any function using only sine and cosine functions. The main idea is that it adds up a collection of waves made from a combination of sines and cosines to estimate the function. As the number of terms in the sum increase, the approximation gets finer and finer. When there are an infinite number of terms in the sequence, the sum is exactly equal to the function.

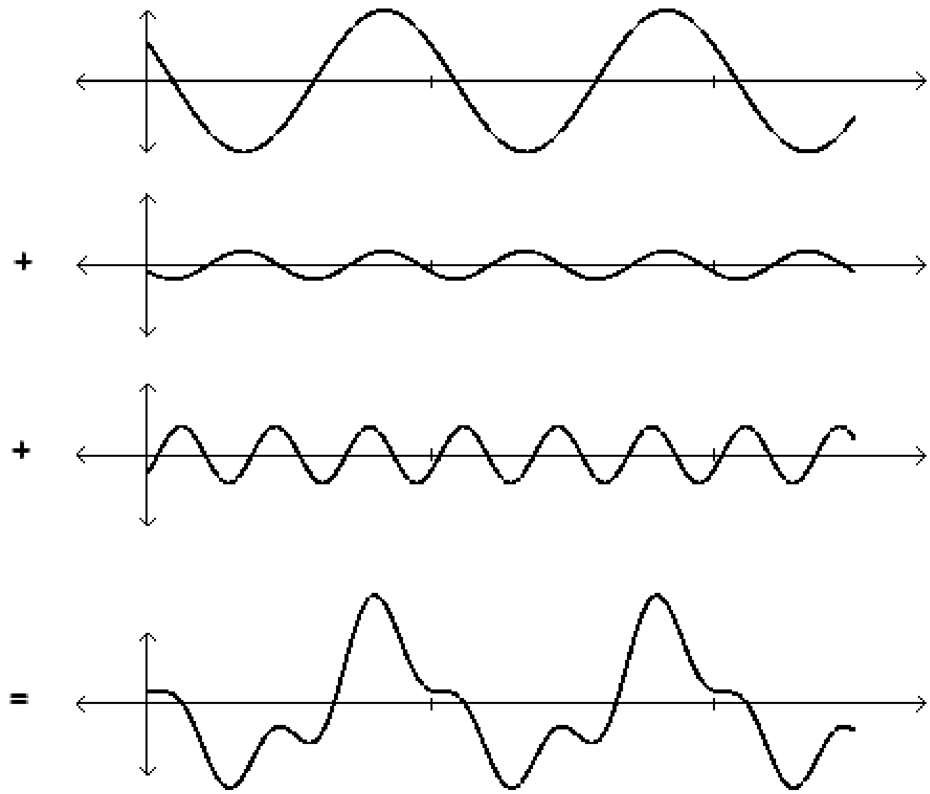

Figure 1: Fourier Series

The figure above shows how the sum of 3 waves made of both sine and cosine functions can sum up to be the new wave. Remember that another way of representing sine and cosine waves are circles on complex planes with vectors rotating inside. Using the circles to lay out the Fourier Series we can see that it becomes a sequence of vectors connected tip to tail, each with a different radius and rotating at different frequencies. This chain of arrows can trace out anything in the 2 dimensional complex plane with enough vectors.

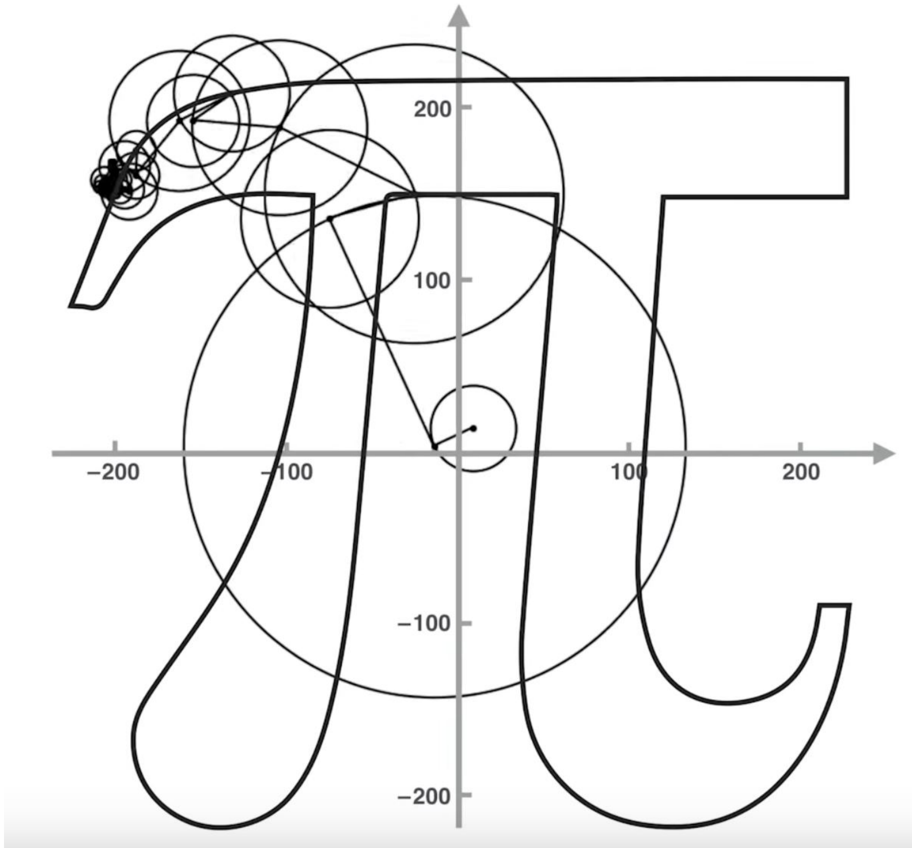

Figure 2: Fourier Series in the Complex Plane

2.2 Parseval's Theorem

Parseval's Theorem [1] states that if a function has a Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx)$$

then

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Now we will prove the theorem. Squaring the Fourier Series of $f(x)$, we get

$$[f(x)]^2 = \frac{1}{4}a_0^2 + a_0 \sum_{n=1}^{\infty}[a_n cos(nx) + b_n sin(nx)] +$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[a_n a_m cos(nx)cos(mx)+$$
$$a_n b_m cos(nx)sin(mx)+$$
$$a_m b_n sin(nx)cos(mx)+$$
$$b_n b_m sin(nx)sin(mx)]$$

Taking the integral on both sides we get

$$\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{1}{4}a_0^2\int_{-\pi}^{\pi} dx+$$
$$a_0\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}[a_n cos(nx) + b_n sin(nx)]dx+$$
$$\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[a_n a_m cos(nx)cos(mx)+$$
$$a_n b_m cos(nx)sin(mx)+$$
$$a_m b_n sin(nx)cos(mx)+$$
$$b_n b_m sin(nx)sin(mx)]dx$$

We will calculate each term separately. For the first term, we get

$$\frac{1}{4}a_0^2\int_{-\pi}^{\pi} dx = \frac{1}{4}a_0^2(2\pi)$$

For the second term, we notice that because sine and cosine are 2π periodic functions

$$a_0\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}[a_n cos(nx) + b_n sin(nx)]dx = 0$$

For the third term, we will use the Orthogonality Relations [7]. By the Orthogonality Relations, we can see that the following 3 equations hold.

$$\int_{-\pi}^{\pi} cos(nx)cos(mx)dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$
$$\int_{-\pi}^{\pi} sin(nx)sin(mx)dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$
$$\int_{-\pi}^{\pi} cos(nx)sin(mx)dx = 0$$

Plugging it into the third term, we get

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[a_n a_m \pi \delta_{nm} + 0 + 0 + b_n b_m \pi \delta_{nm}]$$

$$\pi \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Summing the 3 terms together, we get

$$\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{1}{4}a_0^2(2\pi) + 0 + \pi \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

So,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Note that Parseval's Theorem is the equality case of Bessel's Inequality [3].

2.3 Wirtinger's Inequality

Wirtinger's Inequality [2] says that let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of period 2π. The function is continuous and also has a continuous derivative throughout $\mathbb{R}$. If

$$\int_0^{2\pi} f(x) = 0$$

then

$$\int_0^{2\pi} f'(x)^2 dx \geq \int_0^{2\pi} f(x)^2 dx$$

The equality case only holds when $f(x) = acosx + bsinx$ for some a and b.

To prove it, we first see that because $f(x)$ is continuous, its derivative is continuous, and its period is 2π, Dirichlet's conditions [4] are met and we can write the Fourier Series for $f(x)$ to be

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n sinnx + b_n cosnx)$$

Note that $a_0 = 0$ because

$$\int_0^{2\pi} f(x) = 0$$

Therefore the Parseval's Theorem now becomes

$$\frac{1}{\pi}\int_0^{2\pi} f(x)^2 dx = \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{1}$$

Note that we changed the parameters of the integral from $-\pi$ and π to 0 and 2π. This doesn't change anything because $f(x)$ is a 2π periodic function. We can first take the derivative of $f(x)$ to get

$$f'(x) = \sum_{n=1}^{\infty}(a_n ncos(nx) - b_n nsin(nx))$$

Plugging this into Parseval's Theorem gives us

$$\frac{1}{\pi}\int_0^{2\pi} f'(x)^2 dx = \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2) \tag{2}$$

Note that the minus sign in the equation for $f'(x)$ is canceled out due to squaring on the right hand side of the Parseval's Theorem. Because $n \geq 1$, (2) $\geq$ (1), which proves the Wirtinger's Inequality.

3 Isoperimetric Inequality on a Triangle

Let's start things off with a simpler version of the problem. We will be proving the Isoperimetric Inequality on a triangle.

The Isoperimetric Inequality for Triangles states that if L is the perimeter and A is the area

$$L^2 \geq 12\sqrt{3} \cdot A$$

To prove this, we first need to prove the AM-GM Inequality.

3.1 Proof of the AM-GM Inequality

The AM-GM Inequality states for all nonnegative reals $a_1, ..., a_n$

$$\frac{a_1 + ... + a_n}{n} \geq \sqrt[n]{a_1 \cdot ... \cdot a_n}$$

The equality case holds when $a_i = a_j$ for all i, j.

We will prove this via Cauchy Induction. Cauchy Induction is a form of the regular induction. In Cauchy Induction, we first prove the base case, which is with 2 elements. Then we prove all the cases with 2^n elements. Finally, we prove that given n elements work, $n-1$ elements work.

3.1.1 The Base Case

The base case is the AM-GM Inequality with 2 elements.

$$\frac{x+y}{2} \geq \sqrt{xy}$$

The more obvious and boring way to prove this is by doing some simple algebra.

$$(x-y)^2 \geq 0$$
$$x^2 - 2xy - y^2 \geq 0$$
$$x^2 + 2xy - y^2 \geq 4xy$$
$$\frac{(x+y)^2}{4} \geq xy$$
$$\frac{x+y}{2} \geq \sqrt{xy}$$

But we are here to have fun, so here is a much more interesting proof.

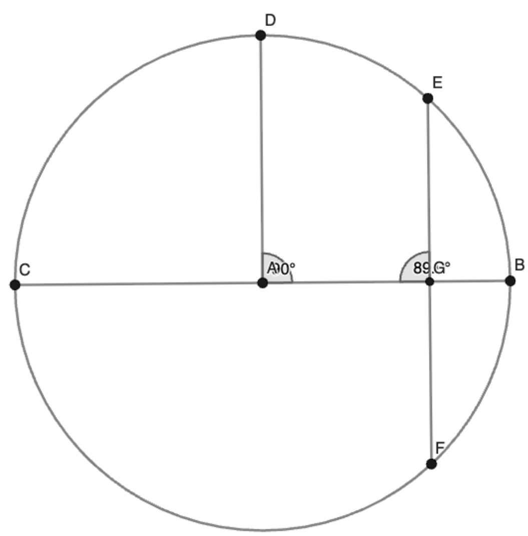

Figure 3: AM-GM Base Case

We set the circle to have diameter $x + y$, where $\overline{CG}$ has length x and $\overline{GB}$ has length y. $\overline{AD}$ is perpendicular to $\overline{CB}$ and is the radius, so it has length $\frac{x+y}{2}$. $\overline{EF}$ is perpendicular to $\overline{CB}$ which makes $\overline{EG} = \overline{GF}$.

By using Power of the Point on the chords $\overline{CB}$ and $\overline{EF}$, we can see that $xy = \overline{EG} \cdot \overline{GF}$. Thus, $\overline{EG} = \overline{GF} = \sqrt{xy}$. We can see that by changing x and y, the position of the point G on the line $\overline{CB}$ changes. However, no matter where G is, $\overline{EG}$ will always be smaller than $\overline{AD}$, which means $\frac{x+y}{2} \geq \sqrt{xy}$.

3.1.2 Powers of Two

Suppose the AM-GM Inequality works for n elements. We want to prove that it works for $2n$ elements. Suppose that we have a list of $2n$ elements $a_1, ..., a_{2n}$. We can split this into two sequences of n elements $a1, ..., a_n$ and $a_{n+1}, ..., a_{2n}$. From this we can write two inequalities.

$$\frac{x_1 + ... + x_n}{n} \geq \sqrt[n]{x_1 ... x_n}$$

and

$$\frac{x_{n+1} + ... + x_{2n}}{n} \geq \sqrt[n]{x_{n+1} ... x_{2n}}$$

Now we add them and divide by 2.

$$\frac{x_1 + ... + x_{2n}}{2n} \geq \frac{\sqrt[n]{x_1 ... x_n} + \sqrt[n]{x_{n+1} ... x_{2n}}}{2}$$

Notice that now we can apply the AM-GM on the 2 elements $\sqrt[n]{x_1 ... x_n}$ and $\sqrt[n]{x_{n+1} ... x_{2n}}$ to get

$$\frac{\sqrt[n]{x_1 ... x_n} + \sqrt[n]{x_{n+1} ... x_{2n}}}{2} \geq \sqrt[2n]{x_1 ... x_{2n}}$$

Which means that

$$\frac{x_1 + ... + x_{2n}}{2n} \geq \sqrt[2n]{x_1 ... x_{2n}}$$

Since the base case is 2 elements, this proves the AM-GM Inequality for all powers of 2.

3.1.3 Stepping Backwards

Assuming that AM-GM works for n elements, we must prove that it works for $n-1$ elements. First, we substitute x_n with $\frac{x_1+...+x_{n-1}}{n-1}$. Plugging it in, we get

$$\frac{x_1 + ... + x_{n-1} + \frac{x_1+...+x_{n-1}}{n-1}}{n} \geq \sqrt[n]{x_1 ... x_{n-1} \left(\frac{x_1 + ... + x_{n-1}}{n-1} \right)}$$

Note that because we assumed the inequality to be true with n elements, equality must hold if and only if $x_1 = x_2 = ... = x_{n-1} = \frac{x_1+...+x_{n-1}}{n-1}$. However notice that if $x_1 = x_2 = ... = x_{n-1}$, $\frac{x_1+...+x_{n-1}}{n-1}$ must be equal as well. So, the equality case holds if and only if $x_1 = x_2 = ... = x_{n-1}$.

We can now continue by multiplying $n-1$ to the numerator and the denominator of the left hand side.

$$\frac{x_1 + ... + x_{n-1} + \frac{x_1+...+x_{n-1}}{n-1}}{n} = \frac{nx_1 + ... + nx_{n-1}}{n(n-1)} = \frac{x_1 + ... + x_{n-1}}{n-1}$$

Plugging it back in, we get

$$\frac{x_1 + ... + x_{n-1}}{n-1} \geq \sqrt[n]{x_1 ... x_{n-1} \left(\frac{x_1 + ... + x_{n-1}}{n-1}\right)}$$
$$\left(\frac{x_1 + ... + x_{n-1}}{n-1}\right)^n \geq x_1 ... x_{n-1} \left(\frac{x_1 + ... + x_{n-1}}{n-1}\right)$$
$$\left(\frac{x_1 + ... + x_{n-1}}{n-1}\right)^{n-1} \geq x_1 ... x_{n-1}$$
$$\frac{x_1 + ... + x_{n-1}}{n-1} \geq \sqrt[n-1]{x_1 ... x_{n-1}}$$

Thus, by Cauchy Induction, we have proved the AM-GM Inequality.

3.2 Applying the AM-GM Inequality

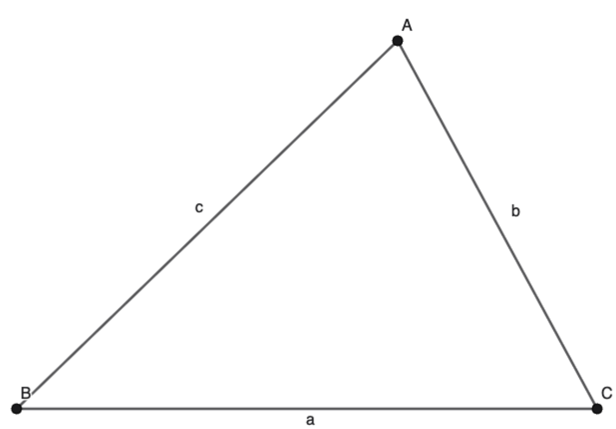

Figure 4: Triangle

First, we will define $s = \frac{a+b+c}{2}$. Now, we will apply AM-GM on the 3 terms $(s-a)$, $(s-b)$, and $(s-c)$ to get

$$\frac{(s-a)+(s-b)+(s-c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}$$
$$\frac{s}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}$$
$$\frac{s^3}{27} \geq (s-a)(s-b)(s-c)$$

Using Heron's Formula, we get

$$\frac{s^4}{27} \geq A^2$$
$$s^2 \geq 3\sqrt{3} \cdot A$$
$$L^2 \geq 12\sqrt{3} \cdot A$$

4 A General Idea

When it comes to intuition, with a slight bit of reasoning, one can come to the conclusion that if the perimeter is fixed, the largest possible area is from a circle. The intuitive logic will most likely go like this. The shape will most likely be uniform throughout, and the only shape that is absolutely uniform is a circle. However, the truth is that the proof is more complicated than this.

Some interesting parts I noticed about the problem included my observation that the shapes cannot be concave, since one could just invert the part caving inward to get a larger area with the same perimeter.

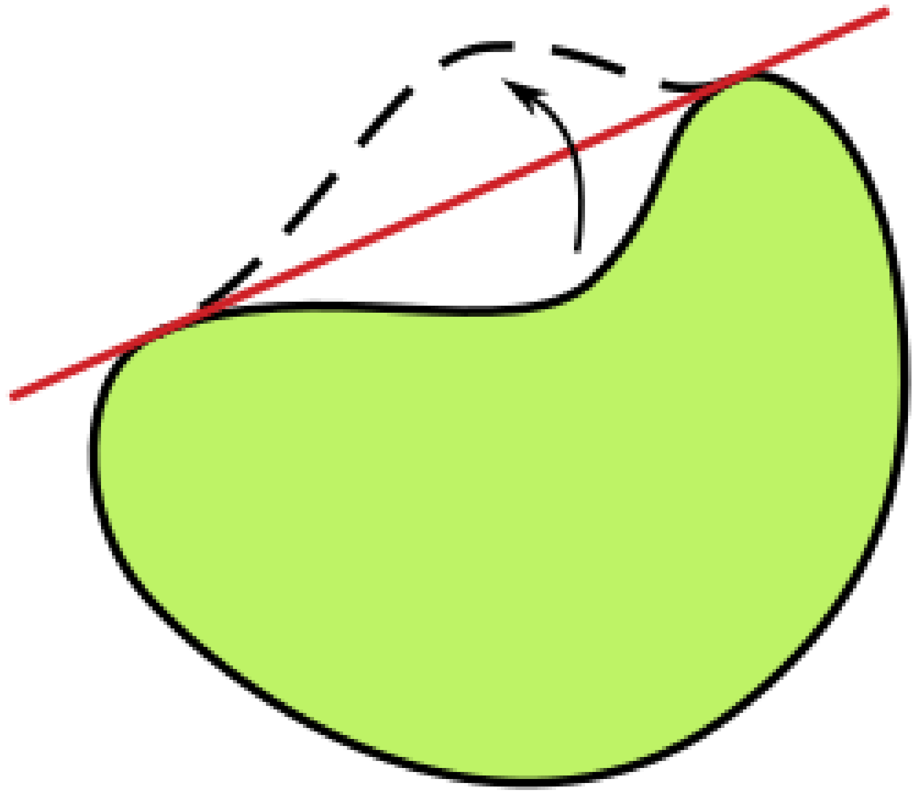

Figure 5: Cannot be concave

Alas, I wish that I could say that I have a good geometric proof, but we all must face the truth that most geometric questions are answered via algebra. The main idea for this proof is that it calculates the area of the shape, and compares it to another quantity using the Wirtinger's Inequality. The other quantity is cleverly set up so that the result directly gives us the Isoperimetric Inequality.

5 Adolf Hurwitz's Proof of the Isoperimetric Inequality [5]

5.1 Parameterizing by Arc Length

We start off by noticing that we can represent the closed curve using 2 piecewise functions $x(s)$ and $y(s)$ where $s \in [0, L]$ that satisfy

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1 \tag{3}$$

We can make this assumption because this is what it means to parameterize by arc length. It essentially limits how much you go around the perimeter in each step to 1. We can think of this as a "speed" of 1.

5.2 Rescaling to 2π Periodicity

Because the functions $x(s)$ and $y(s)$ are L periodic (the perimeter is a closed curve of length L), we can make new functions $f(\theta)$ and $g(\theta)$ to be functions of $x(s)$ and $y(s)$ so that they are 2π periodic, where $\theta \in [0, 2\pi]$.

$$f(\theta) = x\left(\frac{L\theta}{2\pi}\right)$$

$$g(\theta) = y\left(\frac{L\theta}{2\pi}\right)$$

We will also define

$$s = \frac{L\theta}{2\pi}$$

Notice that we have the ability to shift the shape around the plane since it does not affect the inequality in any way. So for later convenience, we will shift the graph so that $\int_0^{2\pi} f(\theta)d\theta = 0$. Now, we can take the derivative $\frac{d}{d\theta}$ and plug it back into (3). Using the chain rule, we get

$$f'(\theta) = x'\left(\frac{L\theta}{2\pi}\right) \cdot \frac{L}{2\pi}$$

$$g'(\theta) = y'\left(\frac{L\theta}{2\pi}\right) \cdot \frac{L}{2\pi}$$

So,

$$(f'(\theta))^2 + (g'(\theta))^2 = \left[\left(x'\left(\frac{L\theta}{2\pi}\right)\right)^2 + \left(y'\left(\frac{L\theta}{2\pi}\right)\right)^2\right] \cdot \left(\frac{L}{2\pi}\right)^2$$

But because of (3), everything inside the square brackets equals to 1.

$$(f'(\theta))^2 + (g'(\theta))^2 = \frac{L^2}{4\pi^2} \tag{4}$$

5.3 Writing the Area Integral in Terms of f and g

Recall that the formula for the area enclosed by a curve is [6]

$$A = \frac{1}{2}\int_0^L (x(s)y'(s) - y(s)x'(s))ds \tag{5}$$

This is similar to how the determinant of a matrix calculates the area of the shape it represents. In fact, the area of a triangle can be calculated by multiplying $\frac{1}{2}$ to the determinant of the matrix, similar to the equation above. Now we need to define and calculate a few conditions that we will use later. Because

$$s = \frac{L\theta}{2\pi}$$

the following conditions hold.

$$ds = \frac{L}{2\pi}d\theta$$
$$x(s) = f(\theta)$$
$$y(s) = g(\theta)$$

Now we need to calculate what $f'(\theta)$ and $g'(\theta)$ are.

$$f(\theta) = x\left(\frac{L\theta}{2\pi}\right)$$

By the chain rule,

$$f'(\theta) = x'\left(\frac{L\theta}{2\pi}\right) \cdot \frac{L}{2\pi}$$
$$x'(s) = \frac{2\pi}{L} \cdot f'(\theta)$$

Similarly,

$$y'(s) = \frac{2\pi}{L} \cdot g'(\theta)$$

Now, substituting everything back into (5) we get

$$A = \frac{1}{2}\int_0^{2\pi} \left(\frac{2\pi}{L}f(\theta)g'(\theta) - \frac{2\pi}{L}g(\theta)f'(\theta)\right)\frac{L}{2\pi}d\theta$$
$$2A = \int_0^{2\pi} (f(\theta)g'(\theta) - g(\theta)f'(\theta))d\theta \qquad (6)$$

The formula for integration by parts says

$$\int_0^{2\pi} f(\theta)g'(\theta)d\theta = [f(\theta)g(\theta)]_0^{2\pi} - \int_0^{2\pi} g(\theta)f'(\theta)d\theta$$

Since $f(\theta)$ and $g(\theta)$ are 2π periodic functions,

$$f(0) = f(2\pi)$$

and

$$g(0) = g(2\pi)$$

Then,

$$[f(\theta)g(\theta)]_0^{2\pi} = f(2\pi)g(2\pi) - f(0)g(0) = 0$$

So,

$$\int_0^{2\pi} f(\theta)g'(\theta)d\theta = -\int_0^{2\pi} g(\theta)f'(\theta)d\theta$$

Returning back to (6) we can rewrite it into this.

$$2A = 2\int_0^{2\pi} f(\theta)g'(\theta)d\theta \tag{7}$$

5.4 Reorganizing the Equation

We can complete the square so that (7) becomes

$$2A = \int_0^{2\pi} (f(\theta)^2 + g'(\theta)^2 - (f(\theta) - g'(\theta))^2)d\theta$$

Because the last term is non-negative,

$$2A \leq \int_0^{2\pi} (f(\theta)^2 + g'(\theta)^2)d\theta$$

Wirtinger's Inequality states that

$$\int_0^{2\pi} f(\theta)^2 d\theta \leq \int_0^{2\pi} f'(\theta)^2 d\theta$$

We are able to use the inequality because we shifted the graph earlier so that $\int_0^{2\pi} f(\theta)d\theta = 0$. It follows that

$$\int_0^{2\pi} (f(\theta)^2 + g'(\theta)^2)d\theta \leq \int_0^{2\pi} (f'(\theta)^2 + g'(\theta)^2)d\theta$$

So,

$$2A \leq \int_0^{2\pi} (f'(\theta)^2 + g'(\theta)^2)d\theta$$

Substituting the right hand side with (4) gives

$$2A \leq \int_0^{2\pi} \frac{L^2}{4\pi^2}d\theta = \frac{L^2}{4\pi^2} \cdot 2\pi = \frac{L^2}{2\pi}$$

Rearanging the equation gives us the Isoperimetric Inequality.

$$L^2 \geq 4\pi A$$

6 Generalization of the Problem

The first step one should take after proving something is to attempt to generalize. I will not be able to prove these generalizations in this paper, but I will give out some ideas of how they work. The usual Isoperimetric Inequality works on a 2 dimensional plane. The natural move is to generalize to more dimensions. For example in 3 dimensional space, the inequality would relate the surface area to the volume of an object. It turns out that it is possible to generalize to any higher dimension, but it is extremely complex.

Another way of generalizing the inequality is to do it in different spaces.

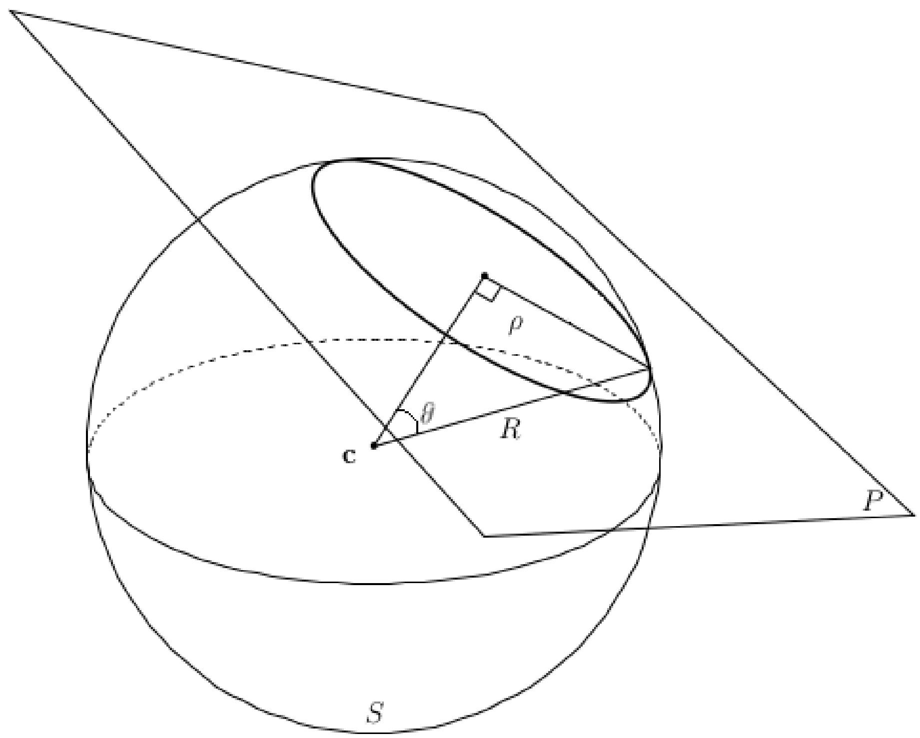

Figure 6: Isoperimetric Inequality on a Sphere

For example, on a sphere, the Isoperimetric Inequality becomes

$$L^2 \geq 4\pi A - \frac{A^2}{R^2}$$

where R is the radius of the sphere. Other generalizations include the Isoperimetric Inequality in Hadamard Manifolds and in metric measure spaces.

7 Applications of the Isoperimetric Inequality

The Isoperimetric Inequality is useful in many situations. For example in geometry and calculus, it can be used to solve problems involving identifying optimal configurations of shapes with constraints. In probability and statistics, it can be used to bound mixing times of Markov Chains and analyzing random walks.

However, if we take a closer look, pure mathematics isn't the only thing it is useful in. In physics and engineering it is used to explain that a bubble always forms in a sphere because it minimizes surface area for a given enclosed volume and it also explains how to use the least material for the greatest strength in buildings. Most liquid droplets on surfaces often obey the isoperimetric principles as well.

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