# An Overview of Analytic Combinatorics

Kanad Bhattacharya
kanad.bhattacharya.725@gmail.com

Euler Circle

July 2025

# An Introduction

### History

- In 1751, Euler wrote the generating function of Catalan numbers to Goldbach.

# An Introduction

### History

- In 1751, Euler wrote the generating function of Catalan numbers to Goldbach.
- In 1917, Ramanujan and Hardy found the asymptotics of partitions.

# An Introduction

## History

- In 1751, Euler wrote the generating function of Catalan numbers to Goldbach.
- In 1917, Ramanujan and Hardy found the asymptotics of partitions.
- Many of the modern problems in Analytic Combinatorics have been proposed by Knuth.

# An Introduction

## History

- In 1751, Euler wrote the generating function of Catalan numbers to Goldbach.
- In 1917, Ramanujan and Hardy found the asymptotics of partitions.
- Many of the modern problems in Analytic Combinatorics have been proposed by Knuth.
- Flajolet did much of the pioneering work in the field.

# Combinatorial Classes

### Definition

A *combinatorial class* is a set $\mathcal{A}$ such that

- There is a function $|\bullet|\mathcal{A} \to \mathbb{Z}_{\geq 0}$. This function is called the *size*.
- The sets $\mathcal{A}_n = \{a|a \in \mathcal{A}, |a| = n\}$ are finite. We denote $A_n = |\mathcal{A}_n|$.

# Combinatorial Classes

**Definition**

A *combinatorial class* is a set $\mathcal{A}$ such that

- There is a function $|\bullet|\mathcal{A} \to \mathbb{Z}_{\geq 0}$. This function is called the *size*.
- The sets $\mathcal{A}_n = \{a | a \in \mathcal{A}, |a| = n\}$ are finite. We denote $A_n = |\mathcal{A}_n|$.

**Example**

- **Binary words.** $\mathcal{A} = \{e, 0, 1, 00, 01, 10, 11, ...\}$. Here the size function gives us the size of the word. $A_n = 2^n$.
- **Number of Partitions.** What is $A_n$ here?

# Generating Functions

### Definition

For a combinatorial class $\mathcal{A}$, we define its corresponding *generating function* as

$$A(z) = \sum_{n \geq 0} A_n z^n.$$

## Generating Functions

### Definition

For a combinatorial class $\mathcal{A}$, we define its corresponding *generating function* as

$$A(z) = \sum_{n \geq 0} A_n z^n.$$

### Example

For the case of binary words, our generating function is

$$1 + 2z + 4z^2 + \cdots = \frac{1}{1 - 2z}.$$

# Constructing Generating Functions I

**Definition**

- $\textsc{Seq}(\mathcal{A})$ consists of sequences constructed from elements of $\mathcal{A}$.
- $\textsc{MSet}(\mathcal{A})$ consists of elements of $\textsc{Seq}(\mathcal{A})$ which are distinct up to permutations.

# Constructing Generating Functions II

### History

The Theory of Polya (1937) gives us a systematic way to construct the generating functions by considering symmetries.

# Constructing Generating Functions II

### History

The Theory of Polya (1937) gives us a systematic way to construct the generating functions by considering symmetries.

### Theorem

*We can arrive at the corresponding generating functions via the following operations*

- $\mathcal{B} = \textsc{Seq}(\mathcal{A}) \implies B(z) = \frac{1}{1-A(z)}$
- $\mathcal{B} = M\textsc{Set}(\mathcal{A}) \implies B(z) = \prod_{n\geq 1} \frac{1}{(1-z^n)^{A_n}}$.

# Constructing Generating Functions III

Proof.

- We have $\mathcal{B} = \textsc{Seq}(\mathcal{A}) = \mathcal{E} + \mathcal{A} + \mathcal{A} \times \mathcal{A} + \mathcal{A} \times \mathcal{A} \times \mathcal{A} + \cdots$. Hence

$$B(z) = 1 + A(z) + A(z)^2 + A(z)^3 + \cdots = \frac{1}{1 - A(z)}.$$

- Similarly for $\mathcal{B} = \textsc{MSet}(\mathcal{A}) \cong \prod_{a \in \mathcal{A}} (\mathcal{E} + \{a\} + \{a\}^2 + \cdots) \implies$

$$B(z) = \prod_{a \in \mathcal{A}} (1 + z^{|a|} + z^{2|a|} + \cdots) = \prod_{n \geq 1} \frac{1}{(1 - z^n)^{A_n}}.$$

■

# Constructing Generating Functions IV

**Example**

The generating function for partitions can be calculated as follows

$$\mathcal{P} = \textsc{MSet}(\mathbb{Z}_{>0})$$

$$\implies P(z) = \prod_{n \geq 1} \frac{1}{1 - z^n}.$$

# Analysis on one variable generating functions I

**Theorem (Cauchy 1826)**

*Let $\rho$ be the radius of convergence of $A(z)$. Then*

$$A_n = \frac{1}{2\pi i} \oint_{|z|=r} A(z) \frac{dz}{z^{n+1}}$$

*where $r < \rho$.*

# Analysis on one variable generating functions I

**Theorem (Cauchy 1826)**

*Let $\rho$ be the radius of convergence of $A(z)$. Then*

$$A_n = \frac{1}{2\pi i} \oint_{|z|=r} A(z) \, \frac{dz}{z^{n+1}}$$

*where $r < \rho$.*

**Theorem**

*We also can establish the following trivial bound*

$$\left| \int_\gamma f(z) \, dz \right| \leq ||\gamma|| \sup |f(z)|.$$

# Analysis on One Variable Generating Functions II

## Convergence

Hence it makes sense to work with generating functions that converge over some domain.

## Definition

The *exponential generating function* is given by

$$\sum_{n\geq 0} A_n \frac{z^n}{n!}.$$

# Analysis on One Variable Generating Functions II

### Convergence

Hence it makes sense to work with generating functions that converge over some domain.

### Definition

The *exponential generating function* is given by

$$\sum_{n\geq 0} A_n \frac{z^n}{n!}.$$

### Definition

A Mellin transform $\mathcal{M}$ on a function $f(s)$ is defined as

$$\mathcal{M}f(s) = \int_0^\infty s^{t-1} f(s)\, ds = g(t).$$

# Some Advanced Methods

### Methods

The following methods are commonly used

- Singularity Analysis and Tauberian Theorems

# Some Advanced Methods

### Methods

The following methods are commonly used

- Singularity Analysis and Tauberian Theorems
- Saddle Point Method.

# Some Advanced Methods

## Methods

The following methods are commonly used

- Singularity Analysis and Tauberian Theorems
- Saddle Point Method.

We try to approximate integrals of the form

$$\int e^{Nf(z)}\, dz.$$

# Some Advanced Methods

## Methods

The following methods are commonly used

- Singularity Analysis and Tauberian Theorems
- Saddle Point Method.

We try to approximate integrals of the form

$$\int e^{Nf(z)}\, dz.$$

We try to do so using Gaussian integrals.

# Some Results in Analytic Combinatorics

## Interesting Results

We can derive the following results

- **Factorials**

$$n! \sim \sqrt{2\pi n}\frac{n^n}{e^n}$$

# Some Results in Analytic Combinatorics

## Interesting Results

We can derive the following results

- **Factorials**

$$n! \sim \sqrt{2\pi n}\frac{n^n}{e^n}$$

- **Involutions**

$$I_n = n!\frac{e^{-1/4}}{2\sqrt{\pi n}}\, n^{-n/2}e^{n/2+\sqrt{n}}\left(1 + O\left(\frac{1}{n^{1/5}}\right)\right)$$

# Some Results in Analytic Combinatorics

## Interesting Results

We can derive the following results

- **Factorials**

$$n! \sim \sqrt{2\pi n}\frac{n^n}{e^n}$$

- **Involutions**

$$I_n = n!\frac{e^{-1/4}}{2\sqrt{\pi n}}\, n^{-n/2} e^{n/2+\sqrt{n}} \left(1 + O\left(\frac{1}{n^{1/5}}\right)\right)$$

- **Set Partitions (Bell numbers)**

$$S_n = n! \cdot \frac{e^{e^r - 1}}{r^n\sqrt{2\pi r(r+1)e^r}} \left(1 + O\left(e^{-r/5}\right)\right)$$

where $r$ satisfies $re^r = n + 1$ (i.e. $r = W_0(n+1)$).

# Asymptotics of Partitions I

## Generating Function

We cover the asymptotics for the number of integer partitions. The generating function for partitions is given by

$$P(z) = \prod_{n \geq 1} \frac{1}{1 - z^n}.$$

# Asymptotics of Partitions I

### Generating Function

We cover the asymptotics for the number of integer partitions. The generating function for partitions is given by

$$P(z) = \prod_{n \geq 1} \frac{1}{1 - z^n}.$$

### Theorem

*Taking the Mellin transform, we arrive at the following result*

$$\log P(e^{-w}) \overset{\mathcal{M}}{\to} \int_0^\infty w^{v-1} P(e^{-w})\, dw = \zeta(v)\zeta(v+1)\Gamma(v).$$

# Asymptotics of Partitions II

**Theorem**

*When* $|z| < 1$ *and* $|1 - z| \leq 2(1 - |z|)$, *we have that*

$$\log(P(z)) = \frac{\pi^2}{6(1-z)} + \frac{1}{2}\log(1-z) - \frac{1}{2}\log(2\pi) - \frac{\pi^2}{12} + \mathcal{O}(1-z).$$

# Asymptotics of Partitions II

**Theorem**

*When* $|z| < 1$ *and* $|1 - z| \leq 2(1 - |z|)$, *we have that*

$$\log(P(z)) = \frac{\pi^2}{6(1-z)} + \frac{1}{2}\log(1-z) - \frac{1}{2}\log(2\pi) - \frac{\pi^2}{12} + \mathcal{O}(1-z).$$

**Definition**

Let $Q(z)$ denote the asymptotic of $P(z)$ as $z \to 1$

$$Q(z) := \left(\frac{1-z}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{\pi^2}{12}} e^{\frac{\pi^2}{6(1-z)}}.$$

Also let $Q_n := [z^n]Q(z)$.

# Asymptotics of Partitions III

**Lemma**

*When* $|z| < 1$, *we have*

$$|\log P(z)| \leq \left( \frac{1}{1 - |z|} + \frac{1}{|1 - z|} \right).$$

# Asymptotics of Partitions III

**Lemma**

*When* $|z| < 1$, *we have*

$$|\log P(z)| \leq \left( \frac{1}{1 - |z|} + \frac{1}{|1 - z|} \right).$$

**Theorem**

*Using the saddle point radius* $|z| = 1 - \frac{\pi}{6\sqrt{n}}$, *we arrive at*

$$P_n = Q_n + O\left(n^{-5/4} \exp\left(\pi\sqrt{2n/3}\right)\right).$$

# Asymptotics of Partitions IV

**Lemma**

*When* $|z| < 1$

$$\int_{-\infty}^{\infty} e^{\pi t\sqrt{\frac{2}{3}}-(1-z)t^2}\,dt = \frac{\pi\sqrt{2}}{(1-z)} e^{\frac{\pi^2}{12}} Q(z).$$

# Asymptotics of Partitions IV

**Lemma**

*When* $|z| < 1$

$$\int_{-\infty}^{\infty} e^{\pi t\sqrt{\frac{2}{3}}-(1-z)t^2}\, dt = \frac{\pi\sqrt{2}}{(1-z)} e^{\frac{\pi^2}{12}} Q(z).$$

**Theorem (Ramanujan-Hardy 1917)**

*Using a Power Series expansion of the integral, we arrive at*

$$P_n \sim Q_n \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}.$$

# Multivariable Generating Functions I

## Definition

- For a dimension $n$, we define

$$\mathbf{z} = (z_1, z_2, \ldots, z_n)$$

$$d\mathbf{z} = dz_1 dz_2 \cdots dz_n.$$

For $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n$, we define

$$\mathbf{z}^{\mathbf{i}} = z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}.$$

# Multivariable Generating Functions I

## Definition

- For a dimension $n$, we define

$$\mathbf{z} = (z_1, z_2, \dots, z_n)$$

$$d\mathbf{z} = dz_1 dz_2 \cdots dz_n.$$

For $\mathbf{i} = (i_1, i_2, \dots, i_n) \in \mathbb{Z}^n$, we define

$$\mathbf{z}^{\mathbf{i}} = z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}.$$

## Definition

We define a multivariable power series as

$$f(\mathbf{z}) = \sum_{\mathbf{i}} f_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}.$$

# Multivariable Generating Functions II

**Definition**

We define an *open polydisk* $D$ as a set of the form

$$D = \{\mathbf{z} \in \mathbb{C}^n : r_i > |z_i - a_i|, r_i \in \mathbb{R}^+, a_i \in \mathbb{C}\}.$$

# Multivariable Generating Functions II

**Definition**

We define an *open polydisk* $D$ as a set of the form

$$D = \{\mathbf{z} \in \mathbb{C}^n : r_i > |z_i - a_i|, r_i \in \mathbb{R}^+, a_i \in \mathbb{C}\}.$$

**Theorem**

*We have the following generalization of Cauchy's Theorem*

$$f_{\mathbf{i}} = \frac{1}{(2\pi i)^n} \oint f(\mathbf{z}) \, \frac{d\mathbf{z}}{\mathbf{z}^{\mathbf{i}+\mathbf{1}}}.$$

# Enumerating Paths on Lattices I

**Definition**

A *lattice path model* consists of

- a finite set of *steps* $\mathcal{S} \subset \mathbb{Z}^d$
- a *region* $\mathcal{R} \subset \mathbb{R}^d$
- a *starting point* $\mathbf{p} \in \mathcal{R}$
- a *terminal set* $\mathcal{T} \subset \mathcal{R}$
- the combinatorial class of all finite tuples called *paths* $(\mathbf{s}_1, \ldots, \mathbf{s}_r) \in \mathcal{S}^r$ such that $\mathbf{p} + \mathbf{s}_1 + \cdots + \mathbf{s}_r \in \mathcal{T}$ and $\mathbf{p} + \mathbf{s}_1 + \cdots + \mathbf{s}_k \in \mathcal{R}$ for all $1 \leq k \leq r$.

# Enumerating Paths on Lattices II

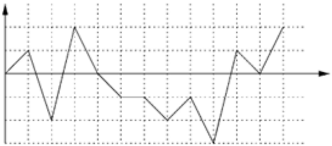

Figure: Path on $\mathcal{R} = \mathbb{R}$

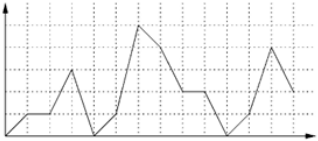

Figure: Path on $\mathcal{R} = \mathbb{R}_{\geq 0}$

# Enumerating Paths on Lattices II

## Definition

- A *weighted path model* assigns a *weight* $w_{\mathbf{i}}$ for each $i \in \mathcal{S}$.
- The *weight* of a path $(\mathbf{s}_1, \ldots, \mathbf{s}_r) \in \mathcal{S}^r$ is the product of the weights $w_{\mathbf{s}_1} \cdots w_{\mathbf{s}_r}$.

# Enumerating Paths on Lattices II

## Definition

- A *weighted path model* assigns a *weight* $w_{\mathbf{i}}$ for each $i \in \mathcal{S}$.
- The *weight* of a path $(\mathbf{s}_1, \ldots, \mathbf{s}_r) \in \mathcal{S}^r$ is the product of the weights $w_{\mathbf{s}_1} \cdots w_{\mathbf{s}_r}$.

## Definition

We look at the following generating functions

- $$S(\mathbf{z}) = \sum_{\mathbf{i} \in \mathcal{S}} w_{\mathbf{i}} \mathbf{z}^{i}$$
- $$W_n(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{Z}^d} f_{\mathbf{i},n} \mathbf{z}^{\mathbf{i}}$$

  where $f_{\mathbf{i},n}$ is the number of paths of length $n$ from $\mathbf{p}$ to $\mathbf{i}$.

# The Kernel Method

**Definition**

We define
$$W(\mathbf{z}, t) = \sum_{n \geq 0} W_n(\mathbf{z}) t^n.$$

# The Kernel Method

## Definition

We define

$$W(\mathbf{z}, t) = \sum_{n \geq 0} W_n(\mathbf{z}) t^n.$$

## Lemma

*When* $\mathcal{R} = \mathbb{R}^d$*, we have*

$$W_{n+1}(\mathbf{z}) = S(\mathbf{z}) W_n(\mathbf{z}).$$

# The Kernel Method

## Definition

We define
$$W(\mathbf{z}, t) = \sum_{n \geq 0} W_n(\mathbf{z}) t^n.$$

## Lemma

*When* $\mathcal{R} = \mathbb{R}^d$*, we have*
$$W_{n+1}(\mathbf{z}) = S(\mathbf{z}) W_n(\mathbf{z}).$$

## Theorem

*When* $\mathcal{R} = \mathbb{R}^d$*, we have*
$$W(\mathbf{z}, t) = \frac{\mathbf{z}^p}{1 - tS(\mathbf{z})}.$$

# Acknowledgments

I would like to thank the following people.

- Dr. Simon Rubinstein Salzedo
- Rachana Madhukara
- My fellow students at Euler Circle.