AN OVERVIEW OF ANALYTIC COMBINATORICS

KANAD BHATTACHARYA

Abstract. In this expository paper, we give a an overview of Analytic Combinatorics and the various techniques used in the field. We try to include classical and modern ideas in the topic. We cover the symbolic method, Boltzmann sampling, the saddle point method, asymptotics of partitions, the kernel method and the analysis of single and multivariable generating functions.

1. Introduction

Generating functions are ubiquitous in mathematics. They can be used to systematically reduce several enumerative combinatorial problems. They also provide an inroad for (complex) analytic methods into combinatorics. *Analytic Combinatorics* is a field built around doing exactly this. However, the idea of "doing Analytic Combinatorics" is a relatively new notion, although many of the methods in the topic are classical. As we shall see in the next two paragraphs, the history of the field also displays this duality.

The concept of generating functions began with the work of Euler (see page 20 in [FS09]), who in 1751 wrote a letter to Goldbach, mentioning the generating function for Catalan numbers. He worked out the generating function for various other combinatorial objects, such as the one for partitions. However, the next big leap in the history of partitions came with the 1917 landmark work of Ramanujan and Hardy when they found the asymptotic for partitions [HR18] (which we discuss in Section 7). Their approach involved doing some clever analysis on the generating function for partitions. In 1938, Rademacher improved upon their work and found an infinite series that gave the exact expression for P_n [Rad38]. In the meantime, Redfield in 1927 [Red27] and Pólya in 1937 [Pól37] found the Redfield-Pólya enumeration theorem, which used generating functions to capture the contribution of the symmetry group in the enumeration of combinatorial structures (refer to [Har08] for a good exposition on this).

The idea that all the above-mentioned classical results, along with some modern developments, could be brought under a single field came to prominence with the work of Flajolet and Knuth [Pro15], [Pro21]. Knuth in his books [Knu73], [Knu98] developed the topic *Analysis of Algorithms*, which would be hugely influential. For instance, the roots of the kernel method can be traced back to exercise 2.2.1.-4 in [Knu73]. Flajolet would find the solution to many such topical problems proposed by Knuth and others (for example, see [FO82]). His approach often involved often drew inspiration from the previously mentioned traditional mathematics. They also involved more and more analysis, such as in [Fla03]. For instance, Ramanujan's works were a source of inspiration for him. Flajolet coined the term *Analytic*

Date: July 2025.

Combinatorics for his field of work, drawing inspiration from Analytic Number Theory. In 2009, he, along with Sedgewick, wrote a famous textbook on the topic [FS09].

Originally, a large section of the work in Analytic Combinatorics has been in the case of single-variable generating functions. In recent times, there has been an increase in interest in multivariable generating functions. For instance, there has been some work in enumerating paths on lattices [BF02], [FR12] (which we discuss in 9) as well as smooth Analytic Combinatorics in Several Variables (smooth ACSV) [BP11], [BMPS18]. A few books have also been published in the topic such as [Mel21], [PWM24].

In this expository paper we aim to provide an overview of Analytic Combinatorics and reveal how it consists of a wide variety of interesting approaches. We cover the preliminaries in 2. The rest of the paper is structured as follows: we begin with the combinatorial (algebraic) techniques in Sections 3, 4 where we cover symbolic methods and Boltzmann sampling. Then we move onto doing analysis over single variable generating functions in Sections 5, 6, 7. We cover various strategies such as analyzing singularities and the saddle point method. In Section 7, which is the largest section, we cover the asymptotics of partitions. In Sections 8, 9 we cover multivariable generating functions, discussing their properties and using them in the kernel method to enumerate paths on lattices. As analytic combinatorics consists of an assortment of clever arguments, it is important that we look at how the methods mentioned in this paper can be applied to solve various problems.

2. Preliminaries

We begin with the basic definitions and terminology in this section.

Definition 2.1. A *combinatorial class* is a set $\mathcal{A}$ such that

- There is a function $|\bullet|\mathcal{A} \to \mathbb{Z}_{\geq 0}$. This function is called the *size*.
- The sets $\mathcal{A}_n = \{a | a \in \mathcal{A}, |a| = n\}$ are finite. We denote $A_n = |\mathcal{A}_n|$.

Example (Binary Words). Consider the combinatorial class of all binary words $\mathcal{W} = \{e, 0, 1, 00, 01, 10, \dots\}$. Here the size function gives us the length of the word. Observe that $W_n = 2^n$.

Two combinatorial classes $\mathcal{A}$ are isomorphic (denoted $\mathcal{A} \cong \mathcal{B}$) if $\forall n, A_n = B_n$. Combinatorial classes are helpful in defining generating functions.

Definition 2.2. For a combinatorial class $\mathcal{A}$, we define its corresponding *generating function* as

$$A(z) = \sum_{n\geq 0} A_n z^n.$$

We also denote the coefficient of z^n in $A(z)$ (which in this case is just A_n) as

$$[z^n]A(z).$$

Observe that, we could equivalently define the generating function in the following way

$$A(z) = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|}.$$

Example (Binary words). We have

$$W(z) = \sum_{n \geq 0} 2^n z^n \overset{\text{if } |z|<\frac{1}{2}}{=} \frac{1}{1-2z}.$$

We also define a few other combinatorial classes which will be helpful later on.

Definition 2.3. We have the following definitions.

- Let $\mathcal{E}$ be a combinatorial class with a single element with size 0.
- Let $\mathcal{Z}$ be a combinatorial class with a single element of size of 1.

There are two aspects of generating functions: the formal series aspect for algebraic manipulations and the functional series aspect for analysis arguments. In the next two sections we shall focus on the algebraic aspects. However, being able to do analysis on generating functions is a huge advantage. Sometimes the generating functions themselves do not converge over any radius, hence it is useful to work with the *exponential generating* function instead as they can help issues related to convergence.

Definition 2.4. The *exponential generating function* is given by

$$\sum_{n \geq 0} A_n \frac{z^n}{n!}.$$

More analytic properties can be extracted if one works with the *Mellin transform.*

Definition 2.5. A *Mellin transform* $\mathcal{M}$ on a function $f(s)$ is defined as

$$\mathcal{M}f(s) = \int_0^\infty s^{t-1} f(s)\, ds = g(t).$$

For instance Mellin transforms help convert a power series into a Dirichlet series. [ASS25] as well as Appendix B in [FS09] contains further details on this. Now we look into the problem of methodically constructing generating functions which really highlights the combinatorial aspect of generating functions.

3. Constructing Generating Functions

Suppose we can construct a combinatorial class $\mathcal{A}$ out of two other combinatorial classes $\mathcal{B}$ and $\mathcal{C}$. Then how is $A(z)$ related to $B(z)$ and $C(z)$? In this section we provide an overview of systematically constructing generating functions using *symbolic methods.*

We begin things with the following definitions.

Definition 3.1. We define the following.

- Suppose $\mathcal{A} = \mathcal{B} \times \mathcal{C} = \{(\beta, \gamma) : \beta \in \mathcal{B}, \gamma \in \mathcal{C}\}$. Then we define the size of an element $\alpha = (\beta, \gamma) \in \mathcal{A}$ as $|\alpha|_\mathcal{A} = |\beta|_\mathcal{B} + |\gamma|_\mathcal{C}$.
- Suppose $\mathcal{A} = \mathcal{B} + \mathcal{C}$ where $+$ is the symbol for disjoint union (also called a combinatorial sum). Then we define the size for an element $\alpha \in \mathcal{A}$

$$|\alpha|_\mathcal{A} = \begin{cases} |\alpha|_\mathcal{B} & \text{if } \alpha \in \mathcal{B}, \\ |\alpha|_\mathcal{C} & \text{if } \alpha \in \mathcal{C}. \end{cases}$$

This leads naturally to the following lemma.

Lemma 3.2. *We have the following results*

$$\mathcal{A} = \mathcal{B} + \mathcal{C} \quad \Longrightarrow \quad A(z) = B(z) + C(z)$$

$$\mathcal{A} = \mathcal{B} \times \mathcal{C} \quad \Longrightarrow \quad A(z) = \sum_{n \geq 0} z^n \left(\sum_{i=0}^{n} B_i C_{n-i} \right) = B(z) \cdot C(z).$$

Proof. Follows upon expanding the power series. ■

Now we turn our attention to more sophisticated scenarios.

Definition 3.3. We define the following.

- $\textsc{Seq}(\mathcal{A})$ consists of sequences constructed from elements of $\mathcal{A}$.
- $\textsc{MSet}(\mathcal{A})$ consists of elements of $\textsc{Seq}(\mathcal{A})$ which are distinct up to permutations.
- $\textsc{PSet}(\mathcal{A})$ consists of elements of $\textsc{MSet}(\mathcal{A})$ that are constructed only from distinct elements of $\mathcal{A}$.
- $\textsc{Cyc}(\mathcal{A})$ consists of elements of $\textsc{Seq}(\mathcal{A})$ that are distinct up to cycle permutations.

Theorem 3.4. *We have the following results.*

- *Sequence: If $\mathcal{A} = \textsc{Seq}(\mathcal{B})$, then*

$$A(z) = \frac{1}{1 - B(z)}.$$

- *Powerset: If $\mathcal{A} = \textsc{PSet}(\mathcal{B})$, then*

$$A(z) = \prod_{n \geq 1} (1 + z^n)^{B_n} = \exp\left(\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} B(z^n) \right).$$

- *Multiset: If $\mathcal{A} = \textsc{MSet}(\mathcal{B})$, then*

$$A(z) = \prod_{n \geq 1} (1 - z^n)^{-B_n} = \exp\left(\sum_{n \geq 1} \frac{1}{n} B(z^n) \right).$$

- *Cycle: If $\mathcal{A} = \textsc{Cyc}(\mathcal{B})$, then*

$$A(z) = \sum_{n \geq 1} \frac{\phi(n)}{n} \log\left(\frac{1}{1 - B(z^n)} \right).$$

Proof. For the first part, observe that

$$\mathcal{A} = \textsc{Seq}(\mathcal{B}) = \mathcal{E} + \mathcal{B} + \mathcal{B} \times \mathcal{B} + \mathcal{B} \times \mathcal{B} \times \mathcal{B} + \cdots = \mathcal{E} + \sum_{n \geq 1} \mathcal{B}^n.$$

Hence,

$$A(z) = 1 + B(z) + (B(z))^2 + (B(z))^3 + \cdots = \frac{1}{1 - B(z)}.$$

For the second part, observe the isomorphism

$$\mathcal{A} = \textsc{PSet}(\mathcal{B}) \cong \prod_{\beta \in \mathcal{B}} (\mathcal{E} + \{\alpha\}).$$

Hence,

$$A(z) = \prod_{\beta \in \mathcal{B}} (1 + z^{|\beta|}) = \prod_{n \geq 0} (1 + z^n)^{B_n}.$$

Now we can write

$$\begin{aligned} A(z) &= \exp\left(\sum_{n\geq 0} B_n \log\left(1+z^n\right)\right) \\ &= \exp\left(\sum_{n\geq 0} B_n \left(\sum_{k\geq 1} (-1)^{k-1}\frac{z^{nk}}{k}\right)\right) \\ &= \exp\left(\sum_{k\geq 1} \frac{(-1)^{k-1}}{k} B(z^k)\right). \end{aligned}$$

For the other parts we ask the reader to refer to page 27 in [FS09]. ■

Many more such formulae can be derived from Redfield-Pólya enumeration formula [Red27], [Pól37]. The idea of this theorem is to analyze symmetry group of the thing we wish to count. It provides a generalization of Burnside's formula in the context of multivariable generating functions. We now use Theorem 3.4 to work out a few examples.

Example (Integer Partitions). Observe that the combinatorial class for integer partitions $\mathcal{P}$ is given by $\mathcal{P} = \text{MSET}(\mathcal{I})$ where $\mathcal{I}$ is the combinatorial class defined on the set of positive integers $\mathbb{Z}_{>0}$, where each positive integer n is assigned a size $|n| = n$. Notice that $\mathcal{I} = \mathcal{Z}\text{SEQ}(\mathcal{Z})$. Hence,

$$I(z) = z + z^2 + z^3 + \cdots$$

Hence,

$$P(z) = \prod_{n\geq 1} \frac{1}{1-z^n}.$$

Example (Rooted Plane Trees). A *rooted* tree consists of a tree with one of its vertices specified (the root). A *plane* tree is a tree where the order in which the sub-trees are attached to a node matters. The size of a tree is the number of vertices. Their combinatorial class is denoted $\mathcal{G}$. Observe that we can think of an element of $\mathcal{G}$ as consisting of a root attached to a sequence of other rooted trees. We capture this self-similarity in the following isomorphism

$$\mathcal{G} \cong \mathcal{Z} \times \text{SEQ}(\mathcal{G}).$$

Hence, we have that

$$G(z) = \frac{z}{1-G(z)}$$

which can be solved for as a quadratic equation for $G(z)$. Checking base cases, we arrive at the following expression

$$G(z) = \frac{1}{2}\left(1-\sqrt{1-4z}\right) = \sum_{n\geq 1} \frac{1}{n}\binom{2n-2}{n-1} z^n.$$

See how systematic things have become!

The symbolic method gives an algorithmic flavor to the process of constructing generating functions. This same combinatorial/ algebraic idea of systematically constructing objects carries over to Boltzmann samples, which we discuss in the next section. Additionally in discussing Boltzmann samples we shall find an interesting way to interpret the notion of a combinatorial class (refer to equations (4.1),(4.2) and (4.3)).

4. Combinatorial Classes in Boltzmann Sampling

The idea behind Boltzmann sampling is to assign a probability proportional to $x^{|\alpha|}$ to an element $\alpha \in \mathcal{A}$. Here x is some fixed positive real number. In order to normalize the probabilities, we need to divide by the generating function $A(x)$ as

$$\sum_{\alpha \in \mathcal{A}} x^{|\alpha|} = \sum_{n \geq 0} A_n x^n = A(x).$$

Therefore the probability assigned to the element α is just

$$\mathbb{P}_{\mathcal{A}}(\alpha) = \frac{x^{|\alpha|}}{A(x)}.$$

The Boltzmann sampler associated with the class $\mathcal{A}$, denoted $\Gamma A(x)$, returns elements of $\mathcal{A}$ with above mentioned probability. The motivation for working with Boltzmann samples comes from the Boltzmann distribution from statistical mechanics which assigns the following probability to a state s

$$\mathbb{P}(s) = \frac{e^{-\frac{E(s)}{k_B T}}}{Z(T)}.$$

Here $E(s)$ is the energy of the state, T is the temperature, k_B is the Boltzmann constant and $Z(T) = \sum_s e^{-\frac{E(s)}{k_B T}}$ is the partition function. The connection between Boltzmann sampling and the Boltzmann distribution follow from the following correspondence

$$x \longleftrightarrow e^{-\frac{1}{k_B T}} \tag{4.1}$$

$$|\alpha| \longleftrightarrow E(s) \tag{4.2}$$

$$A(x) \longleftrightarrow Z(T). \tag{4.3}$$

The above relations give us an interesting way to think about a combinatorial class. It turns out that a lot of information about the expected value of various quantities can be calculated in terms of the partitions function $Z(T)$. Similar results can be worked out in the case of the Boltzmann samples, as is shown in the following theorem.

Theorem 4.1. *We have the following results involving expected value*

- $$\mathbb{E}(|\alpha|) = x\frac{A'(x)}{A(x)},$$
- $$\mathbb{E}(|\alpha|^2) = \frac{x^2 A''(x) + xA'(x)}{A(x)},$$

- *for some fixed* $u \in \mathbb{R}$
$$\mathbb{E}(u^{|\alpha|}) = \frac{A(ux)}{A(x)}.$$

Proof. For the first part, we see that

$$x\frac{A'(x)}{A(x)} = x\frac{\frac{d}{dx}\left(\sum_{\alpha\in\mathcal{A}} x^{|\alpha|}\right)}{A(x)} = \sum_{x\in\alpha} \frac{x^{|\alpha|}}{A(x)}|\alpha| = \sum_{x\in\alpha} \mathbb{P}(\alpha)|\alpha| = \mathbb{E}(|\alpha|).$$

The other parts follow in similar fashion after expanding the right hand side of the expressions.

■

Now, we turn to the question of constructing Boltzmann samples. Suppose we know how to construct a combinatorial class $\mathcal{A}$ from two classes $\mathcal{B}$ and $\mathcal{C}$. The question (again) is how $\Gamma A(x)$ related to $\Gamma B(x)$ and $\Gamma C(x)$. We shall work this out in three cases, and the answer shall be provided in the form of an algorithm.

4.1. **Disjoint Unions.** Suppose we have $\mathcal{A} = \mathcal{B} + \mathcal{C}$. Then $A(x) = B(x) + C(x)$. Observe that

$$\mathbb{P}_{\mathcal{A}}(\alpha) = \begin{cases} \frac{x^{|\alpha|}}{B(x)}\frac{B(x)}{A(x)} = \mathbb{P}_{\mathcal{B}}(\alpha)\frac{B(x)}{A(x)} \text{ if } \alpha \in \mathcal{B}, \\ \frac{x^{|\alpha|}}{C(x)}\frac{C(x)}{A(x)} = \mathbb{P}_{\mathcal{C}}(\alpha)\frac{C(x)}{A(x)} \text{ if } \alpha \in \mathcal{C}. \end{cases}$$

How do we use this to express $\Gamma A(x)$ in terms of $\Gamma B(x)$ and $\Gamma C(x)$? The first step is to define the Bernoulli switch Bern(p).

Definition 4.2. For a given real number $p \in [0, 1]$ (the probability), Bern(p) returns 1 with probability p and returns 0 with probability $1 - p$.

Now it is quite straightforward to write the algorithm for $\Gamma A(x)$, as shown in Algorithm 1.

Algorithm 1 Sampler for $\mathcal{A} = \mathcal{B} + \mathcal{C}$

```
function ΓA(x)
    Let p_A = B(x)/A(x)
    if Bern(p_A) = 1 then
        return ΓB(x)
    else
        return ΓC(x)
    end if
end function
```

4.2. **Cartesian Products.** Suppose we have $\mathcal{A} = \mathcal{B}\times\mathcal{C}$. Then $A(x) = B(x)\cdot C(x)$. Observe that for an element $\alpha = (\beta, \gamma)$ where $\alpha \in \mathcal{A}, \beta \in \mathcal{B}, \gamma \in \mathcal{C}$, we have that

$$\mathbb{P}_{\mathcal{A}}(\alpha) = \frac{x^{|\beta|}}{B(x)} \cdot \frac{x^{|\gamma|}}{C(x)} = \mathbb{P}_{\mathcal{B}}(\beta)\mathbb{P}_{\mathcal{C}}(\gamma).$$

In this case the algorithm can be constructed as in Algorithm 2.

Algorithm 2 Sampler for $\mathcal{A} = \mathcal{B} \times \mathcal{C}$

function $\Gamma\mathcal{A}(x)$
  return Ordered pair $(\Gamma\mathcal{B}(x), \Gamma\mathcal{C}(x))$ ▷ independent calls
end function

4.3. **Sequences.** Suppose we have $\mathcal{A} = \textsc{Seq}(\mathcal{B})$. In this case we provide two different algorithms to construct $\Gamma A(x)$. The first one is recurrence based and uses the fact that $A(x)(1 - B(x)) = 1 \iff A(x) = 1 + B(x)A(x)$, or equivalently, $\mathcal{A} \cong \mathcal{E} + B \times \mathcal{A}$. Using the probabilities calculated in the previous two subsections on disjoint unions and cartesian products, we arrive at

$$\mathbb{P}_{\mathcal{A}}(\alpha) = \begin{cases} \mathbb{P}_{\mathcal{B}\times\mathcal{A}}(\alpha)\frac{B(x)A(x)}{A(x)} = \mathbb{P}_{\mathcal{B}\times\mathcal{A}}(\alpha)B(x) = \mathbb{P}_{\mathcal{B}}(\beta)\mathbb{P}_{\mathcal{A}}(\alpha')B(x) \text{ if } \alpha = (\beta, \alpha) \in \mathcal{B} \times \mathcal{A}, \\ \mathbb{P}_{\mathcal{E}}(\alpha)\frac{1}{A(x)} = \frac{1}{A(x)} \text{ if } \alpha \in \mathcal{E}. \end{cases}$$

Observe the recursion built into the above expression. Thus combining Algorithms 1 and 2 to arrive at Algorithm 3.

Algorithm 3 Recursive Sampler for $\mathcal{A} = \textsc{Seq}(\mathcal{B})$

function $\Gamma\mathcal{A}(x)$
  if $\text{Bern}(B(x)) = 1$ **then**
    return $(\Gamma\mathcal{B}(x), \Gamma\mathcal{A}(x))$ ▷ recursive call
  else
    return 1
  end if
end function

The other approach is not recursive. Picking an element $\alpha \in \mathcal{A}$ can be described as follows: we first pick the $\mathcal{B}^k$ in $\mathcal{A} = \sum_{n\geq 0} \mathcal{B}^n$ and then picking the α from this $\mathcal{B}^k$. For this we need the following definition.

Definition 4.3. For a real number $\lambda < 1$, $\text{Geom}(\lambda)$ returns an integer $n \geq 0$ with probability $(1 - \lambda)\lambda^n$.

The desired result is shown in 4.

Algorithm 4 Geometric Sampler for $\mathcal{A} = \textsc{Seq}(\mathcal{B})$

function $\Gamma\mathcal{A}(x)$
  Draw n according to $\text{Geom}(B(x))$
  return the n-tuple $(\Gamma\mathcal{B}(x), \ldots, \Gamma\mathcal{B}(x))$ ▷ n independent calls
end function

Algorithms for various other Boltzmann sampler have been worked out in [FFP07]. There are also some interesting variants of this problem where one analyzes *exponential Boltzmann generators* [BF02].

5. Using Analysis on Generating Functions

Till now we have been treating Generating Functions as purely algebraic/ combinatorial objects. However, deriving asymptotics falls into the field of analysis. Hence it is only natural that we try to do analysis on generating functions to understand their coefficients!

We work in the complex plane as that provides a clearer picture of the generating function $A(z)$. It is helpful at times to look at the analytic continuation of $A(z)$. The poles of $A(z)$ turn out to be critical. For instance, the distance from the origin to the nearest pole of the gives us the radius of convergence ρ. This in turn tells us that:

$$a_n \propto \rho^{-n}$$

Additionally, working in the complex plane enables us to use Cauchy's formula

$$A_n = \frac{1}{2\pi i} \oint_{|z|=r} A(z) \, \frac{dz}{z^{n+1}}$$

where $r < \rho$.

Another topical idea is to use the following classical bound:

$$\left| \int_\gamma f(z) dz \right| \leq ||\gamma|| \sup |f(z)|.$$

This can be useful in the following sense: suppose $\rho \leq 1$. Then we can find A_n by integrating over a radius $R > 1$ and subtracting the residues

$$A_n = -\sum \text{Res} + \frac{1}{2\pi i} \int_{|z|=R} A(z) \, \frac{dz}{z^{n+1}} = -\sum \text{Res} + \mathcal{O}(R^{-n})$$

where the residues are due to the singularities in the region $r \leq |z| \leq R$. If the singularities are poles and are finite, it can be proven that (see Chapter 2 in [Mel21])

$$A_n = \sum_k P_k(n) \sigma_k^n + \mathcal{O}(R^{-n})$$

where $P_k(n)$ are polynomials and σ_k are the poles. If one of the σ_k has modulus greater than 1, then we can ignore the contribution of $\mathcal{O}(R^{-n})$ while analyzing the asymptotic behavior of a_n. This highlights the power of considering residues.

An instance of this method is to classify solutions of linear recurrence relations with constant coefficients as shown in the following theorem. The idea of the proof is to realize that the generating function must be a rational function and then analyzing residues due to the poles of the generating function.

Theorem 5.1. *Consider a recurrence relation of the form*

$$\sum_{i=0}^{k} k_i a_{n+i} = 0.$$

All solutions of this equation are of the form

$$a_n = \sum_{i=0}^{k'} P_i(n)\sigma_i^n$$

for some constants σ *and polynomials* $P_i(n)$*.*

This idea of doing analysis can be used to arrive at much more rich and interesting results. One can refer to [FO90], [Fla03], [FS09] for more interesting results on uni-variate generating functions. Till now we have focused on analyzing the singularities of a generating function to arrive at asymptotics. In the next section we turn our attention to a different, yet extremely powerful, approach to extract asymptotics when singularity analysis fails us. It is called the *saddle point method* and can be used to find a good approximation for a large class of integrals.

6. The Saddle Point Method

The *saddle point method* has its roots in some classical ideas such as Laplace's method (see Chapter 8 in [FS09]). The idea of this method can be described as follows: suppose we are interested in approximating an integral of the form

$$\int_\gamma e^{Nf(z)} dz$$

where N is varied over large numbers and $f(z)$ is analytic. Furthermore assume we have a point z_0 on γ where $f'(z_0) = 0$ (this is the saddle point!) and $f''(z_0) = -|f''(z_0)|$. (Often, we "chose" γ so that a saddle point lies on it.) Hence, points near z_0 can be approximated as follows

$$f(z) = f(z_0) + f''(z_0)\frac{(z-z_0)^2}{2} + \mathcal{O}((z-z_0)^3).$$

We write

$$\int_\gamma e^{Nf(z)} dz = \int_{\gamma_1} e^{Nf(z)} dz + \int_{\gamma_2} e^{Nf(z)} dz$$

where γ_1 is a small region around z_0 where it makes sense to discard the $O((z-z_0)^3)$ term in the power series of $f(z)$. If set up correctly, we can ensure that $\int_{\gamma_2} e^{Nf(z)} dz << \int_{\gamma_1} e^{Nf(z)} dz$. Then, under the right circumstances and using the Gaussian integral, we can arrive at the following asymptotic

$$\int_{\gamma_1} e^{Nf(z)} dz \sim e^{Nf(z_0)} \int_{\gamma_1} e^{Nf''(z_0)\frac{(z-z_0)^2}{2}} dz \sim e^{Nf(z_0)}\sqrt{\frac{2\pi}{N|f''(z_0)|}}.$$

This method has a variety of applications. For instance, we find the asymptotic for set partitions (Bell numbers) as [FS09]

$$S_n = n! \cdot \frac{e^{e^r - 1}}{r^n\sqrt{2\pi r(r+1)e^r}}\left(1 + O\left(e^{-\frac{r}{5}}\right)\right)$$

where r satisfies $re^r = n+1$ (i.e. $r = W_0(n+1)$). Recently, stronger bounds have been placed on the Bell numbers [GS25]. Furthermore, this method can be used to find the asymptotics of permutation involutions and fragmented permutation. Due to the importance of this topic in one variable generating functions, we work out the following example where we derive the asymptotic for $n!$ using this method.

Example (Stirling's approximation). Observe that $[z^n]e^z = \frac{1}{n!}$. Hence, we are interested in the asymptotic of the following expression

$$I_n = \frac{1}{2\pi i} \oint e^z \frac{dz}{z^{n+1}}$$

where the contour is taken over a circle of arbitrary radius (as e^z is entire). The idea is to chose the specific radius R which shall give us a saddle point to exploit. First we express the integral in polar coordinates

$$I_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{e^{Re^{i\theta}}}{R^n e^{in\theta}}\, d\theta.$$

Observe that if we set $R = n$, then the function $e^{ne^{i\theta} - n\theta}$ has a saddle point at $\theta = 0$. We now try to use this fact. We write

$$I_n = \frac{e^n}{n^n} \cdot \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{n(e^{i\theta} - 1 - i\theta)}\, d\theta.$$

The spirit of the saddle point method is to use the approximation $e^{n(e^{i\theta}-1-i\theta)} = e^{n\left(-\frac{\theta^2}{2} + \mathcal{O}(\theta^3)\right)}$ over a suitable range of θ to extract the main asymptotic behavior. Clearly in this range, we want $n\theta^3 \to 0$ while $n\theta^2 \to \infty$. Observe that $\theta_0 = n^{-\frac{2}{5}}$ fits this criteria. Hence, we write

$$\begin{aligned} J_n &= \int_{-\theta_0}^{\theta_0} e^{n(e^{i\theta}-1-i\theta)} d\theta \\ K_n &= \int_{\theta_0}^{2\pi-\theta_0} e^{n(e^{i\theta}-1-i\theta)} d\theta \\ I_n &= \frac{e^n}{2\pi n^n}(J_n + K_n). \end{aligned}$$

We see that K_n decays exponentially

$$K_n \le \left| \int_{\theta_0}^{2\pi - \theta_0} d\theta \right| \sup_{\theta \in [\theta_0, 2\pi - \theta_0]} |e^{n(e^{i\theta}-1-i\theta)}| = (2\pi - 2\theta_0) e^{n\cos(\theta_0) - n} = \mathcal{O}\left(\exp\left(-\frac{1}{2} n^{\frac{1}{5}}\right)\right)$$

On the other hand, to evaluate J_n, we use the substitution $\theta = \phi\sqrt{n}$ to arrive at

$$\begin{aligned} J_n &= \int_{-\theta_0}^{\theta_0} e^{n(e^{i\theta}-1-i\theta)} d\theta \\ &= \int_{-\theta_0}^{\theta_0} e^{-n\frac{\theta^2}{2}} \exp(\mathcal{O}(n\theta^3))\, d\theta = \int_{-\theta_0}^{\theta_0} e^{-n\frac{\theta^2}{2}}\, d\theta \left(1 + \mathcal{O}(n^{-\frac{1}{5}})\right) \\ &\sim \frac{1}{\sqrt{n}} \int_{-n^{\frac{1}{10}}}^{n^{\frac{1}{10}}} e^{-\frac{\phi^2}{2}} d\phi = \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} e^{-\frac{\phi^2}{2}} + \mathcal{O}\left(\exp\left(-\frac{n^{\frac{1}{5}}}{2}\right)\right) \\ &\sim \sqrt{\frac{2\pi}{n}}. \end{aligned}$$

Putting everything together, we see that

$$\frac{1}{n!} \sim \frac{1}{\sqrt{2\pi n}} \frac{e^n}{n^n}.$$

To recapitulate, in the the saddle point method (when applied to generating functions), we convert $A_n = \oint A(z)\frac{dz}{z^{n+1}}$ into the form $\oint e^{f(z)} dz$, choose a radius so that a saddle point lies on the contour and then use a Gaussian integral to arrive at the required asymptotic. In general, the radius over which we wish to evaluate $\oint A(z)\frac{dz}{z^{n+1}}$ is given by

$$\frac{d}{dR} \log\left(\frac{A(R)}{R^n}\right) = 0.$$

Conditions in which the saddle point method can be applied have been worked out in [Hay56], [FS09]. In fact we shall use the saddle point radius to prove 7.8 while deriving the asymptotic for partitions in the next section.

7. Asymptotics for the Partition Function

In this section we aim to find the asymptotics for partitions. The original 1918 work of Ramanujan and Hardy used Farey sequences, Ford Circles and Modular forms [HR18]. Here, we do not use such methods and present a slight variation of Newman's proof of this result [New62]. However, this approach shall give rise to a bigger error term, although we arrive at the correct asymptotic (see Theorem 7.9).

A lot of the arguments will be lengthy and technical; hence we advise the reader to hold on tight! There will be three important theorems in the buildup to the main result. In this section, when we say $\int_{z_1}^{z_2} f(z)\, dz$, it is assumed that the integral is taken along the straight line from z_1 to z_2. We start things with two lemmas that shall lead to our first big theorem.

Lemma 7.1. *For an analytic function g we have*

$$\left|\sum_{n\geq 1} g(n) - \int_0^{\infty} g(s)\, ds\right| \leq \int_0^{\infty} |g'(s)| ds.$$

Proof. We can write

$$\begin{aligned}\left|\sum_{n\geq 1} g(n) - \int_0^\infty g(s)\,ds\right| &= \left|\int_0^\infty g(s) - g([s])\,ds\right| \\ &\leq \int_0^\infty \left(|\Re(g(s)) - \Re(g([s])) + |\Im(g(s)) - \Im(g([s]))|\right) ds \\ &\leq \int_0^\infty \left(|\Re(g'(s))| + |\Im(g'(s))|\right) ds \quad \text{(Riemann Sums)} \\ &\leq \int_0^\infty |g'(s)|\,ds \quad \text{(Cauchy)}.\end{aligned}$$

.

■

Lemma 7.2. *When $|z| < 1$, we have:*

$$\log(P(z)) = \sum_{n\geq 1} \frac{z^n}{1-z^n}.$$

Proof. Follows trivially upon expanding power series. ■

We arrive at our first big theorem which gives us the asymptote of $P(z)$ as $z \to 1$.

Theorem 7.3. *When $|z| < 1$ and $|1-z| \leq 2(1-|z|)$, we have that*

$$\log(P(z)) = \frac{\pi^2}{6(1-z)} + \frac{1}{2}\log(1-z) - \frac{1}{2}\log(2\pi) - \frac{\pi^2}{12} + \mathcal{O}(1-z).$$

Proof. Let $z = e^{-w}$ where $|\arg(w)| \leq \pi$. As $|1-z| \leq 2(1-|z|)$, we have that $|\arg(w)| < \frac{\pi}{2}$. Observe that from Lemma 7.2

$$\begin{aligned}\log P(z) &= \sum_{n\geq 1} \frac{1}{n(e^{nw}-1)} \\ &= \frac{\pi^2}{6w} + \frac{1}{2}\log(1-e^{-w}) + w\left(\sum_{n\geq 1} \frac{1}{nw(e^{nw}-1)} - \frac{1}{n^2w^2} + \frac{1}{2nw}e^{-nw}\right).\end{aligned}$$

Our aim is to approximate the last term in the above equation. We shall employ Lemma 7.1 to do so. Let us define

$$g(s) = w\left(\frac{1}{sw(e^{sw}-1)} - \frac{1}{s^2w^2} + \frac{1}{2sw}e^{-sw}\right).$$

Let $u = sw$. Observe that

$$\begin{aligned}\int_0^\infty g(s)\,ds &= \int_0^{w\cdot\infty} \frac{g(\frac{u}{w})}{w}\,du \\ &= \int_0^\infty \left(\frac{1}{u(e^u-1)} - \frac{1}{u^2} + \frac{1}{2u}e^{-u}\right) du \quad \text{(Cauchy)} \\ &= -\frac{\log(2\pi)}{2} \quad \text{(Standard Integral)}.\end{aligned}$$

Next, we look at

$$\int_0^\infty |g'(s)|\,ds = w\int_0^{w\cdot\infty}\left|\frac{d}{du}\left(\frac{1}{u(e^u-1)} - \frac{1}{u^2} + \frac{1}{2u}e^{-u}\right)\right|\,du = w\mathcal{O}(1).$$

(The above integral converges when $|\arg(u)| = |\arg(w)| < \frac{\pi}{2}$, a criteria which has already been imposed as result of the condition $|1-z| \leq 2(1-|z|)$ in the theorem statement.)

Hence, we conclude using Lemma 7.1 that

$$w\left(\sum_{n\geq 1}\frac{1}{nw(e^{nw}-1)} - \frac{1}{n^2w^2} + \frac{1}{2nw}e^{-nw}\right) = \mathcal{O}(w).$$

Finally we observe $(1-z) \sim w$ as $z \to 1$ and

$$\frac{\pi^2}{6(1-z)} = \frac{\pi^2}{6w} + \frac{\pi^2}{12} + \mathcal{O}(w).$$

Putting everything together, we arrive at

$$\log(P(z)) = \frac{\pi^2}{6(1-z)} + \frac{1}{2}\log(1-z) - \frac{1}{2}\log(2\pi) - \frac{\pi^2}{12} + \mathcal{O}(1-z),$$

which was desired. ∎

In the above proof, the substitution of $z = e^{-w}$ might seem unmotivated. The reason behind looking at this substitution is that

$$\log P(e^{-w}) \overset{\mathcal{M}}{\to} \int_0^\infty w^{v-1}P(e^{-w})\,dw = \zeta(v)\zeta(v+1)\Gamma(v)$$

where $\mathcal{M}$ is the Mellin Transform. There is a proof of the above theorem using this result but it involves using Modular Forms, which we do not include here.

Before we proceed with the rest of the proof of Theorem7.9, we first make use of the following definition.

Definition 7.4. Let $Q(z)$ denote the asymptotic of $P(z)$ as $z \to 1$

$$Q(z) := \left(\frac{1-z}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{\pi^2}{12}} e^{\frac{\pi^2}{6(1-z)}}.$$

Also let $Q_n := [z^n]Q(z)$.

The rest of the proof to Theorem 7.9 can be divided into two parts: finding the asymptotic for Q_n and relating P_n to Q_n. (Intuitively, it should make sense that this approach is easier than directly trying to extract P_n from $P(z)$.) Both of these parts can be done using the saddle point method. However using the saddle point method to approximate Q_n is lengthy (and the exact saddle point radius of $Q(z)$ is not fun to work with.) Hence, we take a different route where we use Lemma 7.5 to arrive at the asymptotic in Theorem 7.6. As for bound $P_n - Q_n$ we use an approximated version of the saddle point radius $R = 1 - \frac{\pi}{\sqrt{6n}}$.

Now that our strategy is clear, we turn to approximating Q_n. We have the following wonderful lemma which helps us greatly.

Lemma 7.5. *When $|z| < 1$, we have*

$$\int_{-\infty}^{\infty} e^{\pi t\sqrt{\frac{2}{3}}-(1-z)t^2}\,dt = \frac{\pi\sqrt{2}}{(1-z)} e^{\frac{\pi^2}{12}} Q(z).$$

Proof. We have

$$\begin{aligned}\int_{-\infty}^{\infty} e^{\pi t\sqrt{\frac{2}{3}}-(1-z)t^2}\,dt &= e^{\frac{2\pi^2}{4\cdot 3(1-z)}}\sqrt{\frac{\pi}{(1-z)}} \quad \text{(Gaussian Integral)}\\ &= \frac{\pi\sqrt{2}}{(1-z)} e^{\frac{\pi^2}{12}} Q(z).\end{aligned}$$

■

Hence, we do not need a loop integral to compute Q_n: we can just power series expand the integral! We do so and arrive at the asymptotic of Q_n in the following important theorem.

Theorem 7.6. *We have that*

$$Q_n \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}.$$

Proof. Let $k = t + \sqrt{n}$. We have

$$\begin{aligned}Q_n &= [z^n]\frac{e^{-\frac{\pi^2}{12}}}{\pi\sqrt{2}}(1-z)\int_{-\infty}^{\infty} e^{\pi t\sqrt{\frac{2}{3}}-(1-z)t^2}\,dt \quad \text{(Lemma 7.5)}\\ &= \frac{e^{-\frac{\pi^2}{12}}}{\pi\sqrt{2}}\int_{-\infty}^{\infty} e^{\pi t\sqrt{\frac{2}{3}}-t^2}\left(\frac{t^{2n}}{n!}-\frac{t^{2n-2}}{(n-1)!}\right)dt\\ &= \frac{e^{-\frac{\pi^2}{12}}}{\pi\sqrt{2}}\int_{-\infty}^{\infty} e^{\pi k\sqrt{\frac{2}{3}}-k^2-2\sqrt{n}k} e^{\pi\sqrt{\frac{2n}{3}}-n}\frac{(k+\sqrt{n})^{2n-2}}{n!}((k+\sqrt{n})^2-n)\,dk\\ &\sim \frac{e^{-\frac{\pi^2}{12}}}{\pi\sqrt{2}}\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{\sqrt{2\pi n}}\int_{-\infty}^{\infty} k e^{\pi k\sqrt{\frac{2}{3}}-k^2-2\sqrt{n}k}\left(1+\frac{k}{\sqrt{n}}\right)^{2n-2}\left(2+\frac{k}{\sqrt{n}}\right)dk \quad \text{(Stirling)}.\end{aligned}$$

We simplify our calculations by making the observation

$$\lim_{n\to\infty} e^{-2\sqrt{n}k}\left(1+\frac{k}{\sqrt{n}}\right)^{2n-2}\left(2+\frac{k}{\sqrt{n}}\right) = 2e^{-k^2}.$$

However, care must be taken as we simply cannot take a limit inside an integral without any justification. Fortunately, Lebesgue's dominated convergence theorem comes to help! We notice that the integrand is dominated by

$$F(k) = \begin{cases} k e^{\pi\sqrt{\frac{2}{3}}\,k-k^2}(2+k) & \text{if } k \geq 0,\\ |k| e^{\pi\sqrt{\frac{2}{3}}\,k-k^2}(2-k)e^{k^2+1} & \text{if } k \leq 0\end{cases}$$

and that $\int_{-\infty}^{\infty} F(k)dk$ converges. Hence, we take the limit within the integral and arrive at

$$
\begin{aligned}
Q_n &\sim \frac{e^{-\frac{\pi^2}{12}}}{\pi\sqrt{2}} \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{\sqrt{2\pi}\, n} \int_{-\infty}^{\infty} 2ke^{\pi\sqrt{\frac{2}{3}}\,k-2k^2}\, dk \\
&= \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}.
\end{aligned}
$$

Now the theorem follows immediately upon simplifying the algebraic expressions. ■

All that is left for us is to approximate $P_n - Q_n$. We shall do so by bounding $\oint \frac{P(z)-Q(z)}{z^{n+1}} dz$. The radius of contour in the integral will be the (approximate) saddle point radius of $Q(z)$

$$R = 1 - \frac{\pi}{\sqrt{6n}}.$$

However, we cannot use Theorem 7.3 on the entire loop of integration as the condition $|1-z| \leq 2|1-|z||$ may not hold. Hence, we break the loop integral into two integrals. In one of those two integrals, we use Theorem 7.3, while in the other integral we use the following very crude approximation.

Lemma 7.7. *When $|z| < 1$, we have*

$$|\log P(z)| \leq \left(\frac{1}{1-|z|} + \frac{1}{|1-z|}\right).$$

Proof. We have that

$$
\begin{aligned}
|\log P(z)| &= \left|\sum_{n\geq 1} \frac{1}{n} \frac{z^n}{1-z^n}\right| \quad \text{(Lemma 7.2)} \\
&\leq \left|\frac{z}{1-z}\right| + \sum_{n\geq 2} \frac{1}{n^2} \frac{n|z^n|}{1-|z|^n} \\
&< \left|\frac{1}{1-z}\right| + \frac{1}{1-|z|} \sum_{n\geq 2} \frac{1}{n^2} \frac{n|z^n|}{1+|z|+|z|^2+|z|^{n-1}} \\
&\leq \left|\frac{1}{1-z}\right| + \frac{1}{1-|z|} \sum_{n\geq 2} \frac{1}{n^2} \\
&\leq \left|\frac{1}{1-z}\right| + \frac{1}{1-|z|}.
\end{aligned}
$$

■

Now we finally bound $P_n - Q_n$.

Theorem 7.8. *We have*

$$P_n - Q_n = o\left(\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{n}\right).$$

Proof. We shall be approximating $\oint (P(z) - Q(z)) \frac{dz}{z^{n+1}}$. Our integral will be around the contour Γ consisting of points $|z| = 1 - \frac{\pi}{\sqrt{6n}}$. Furthermore the contour will be broken into the following regions

$$\Theta = \left\{ z \in \mathbb{C} : |1 - z| < \pi\sqrt{\frac{2}{3n}} \right\},$$

$$\Omega = \Gamma \backslash \Theta.$$

Observe that we can successfully use Theorem 7.3 while bounding the integral over Θ. However, we use Lemma 7.7 over Ω. The benefit of working with $|z| = 1 - \frac{\pi}{\sqrt{6n}}$ is that $1 - |z| = \mathcal{O}(n^{-\frac{1}{2}})$ and that $|z|^n = e^{-\pi\sqrt{\frac{n}{6}}}$. Also notice that $|1 - z| = \mathcal{O}(n^{-\frac{1}{2}})$ over Θ and $\frac{1}{|1-z|} = \mathcal{O}(n^{\frac{1}{2}})$ over Ω. Using these observations, we now prove the theorem

$$\begin{aligned}
\oint_{\Theta} (P(z) - Q(z)) \frac{dz}{z^{n+1}} &= \mathcal{O}\left(\int_{\Theta} |z|^{-n-1} |1-z|^{3/2} \exp\left(\frac{\pi}{6} \frac{1}{1 - |z|} \right) |dz| \right) \\
&= \mathcal{O}\left(\frac{e^{\pi\sqrt{\frac{n}{6}}}}{e^{-\pi\sqrt{\frac{n}{6}}}} \cdot n^{-\frac{3}{4}} \right) \cdot \mathcal{O}\left(\int_{\Theta} |dz| \right) \\
&= \mathcal{O}\left(n^{-\frac{3}{4}} e^{\pi\sqrt{\frac{2}{3n}}} \right) \cdot \mathcal{O}\left(n^{-\frac{1}{2}} \right) \\
&= o\left(\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{n} \right)
\end{aligned}$$

and

$$\begin{aligned}
\oint_{\Omega} (P(z) - Q(z)) \frac{dz}{z^{n+1}} &= \mathcal{O}\Bigg(\int_{\Omega} |z|^{-n-1} \left[\exp\left(\frac{\pi z}{6(1-z)} \right) + \exp\left(|1-z|^{-1} + (1 - |z|)^{-1} \right) \right] |dz| \Bigg) \\
&= \mathcal{O}\Bigg(\int_{\Omega} e^{\pi\sqrt{\frac{n}{6}}} \left[\exp\left(\frac{\pi}{6}\sqrt{\frac{3n}{2}} \right) + \exp\left(\frac{1}{\pi} \left(\sqrt{\frac{3n}{2}} + \sqrt{6n} \right) \right) \right] |dz| \Bigg) \\
&= o\left(\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{n} \right).
\end{aligned}$$

■

Hence, from Theorems 7.6 and 7.8 we find that $P_n \sim Q_n$ as well as the asymptotic of P_n.

Theorem 7.9. *We have that*

$$P_n \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}.$$

Deriving the asymptotics for P_n has truly highlighted the power of doing analysis on single variable generating functions. In recent years there has been a growing interest in multivariable generating functions, which is the topic of the next section.

8. Working with Multivariable Generating Functions

We turn our attention to multivariable generating functions. They are incredibly helpful when there are more than one parameter that can be varied. At times they can encapsulate information of a large class of problems. For example in Section9 we shall see how we can systematically enumerate lattice paths. However, in this section, we shall cover the some standard properties of multivariable generating functions. Let us first get the preliminary definitions out of the way.

Definition 8.1. For a dimension n, we define

$$\mathbf{z} = (z_1, z_2, \dots, z_N)$$

$$d\mathbf{z} = dz_1 dz_2 \cdots dz_N.$$

For $\mathbf{i} = (i_1, i_2, \dots, i_N) \in \mathbb{Z}^N$, we define

$$\mathbf{z}^{\mathbf{i}} = z_1^{i_1} z_2^{i_2} \dots z_n^{i_N}.$$

Definition 8.2. We define a multivariable power series (centered at $\mathbf{a}$) as

$$f(\mathbf{z}) = \sum_{\mathbf{i}} f_{\mathbf{i}} (\mathbf{z} - \mathbf{a})^{\mathbf{i}}$$

where $\mathbf{i} \in \mathbb{Z}^N_{\geq 0}$.

Definition 8.3. We define an *open polydisk* $D_{\mathbf{r}}(\mathbf{a})$ of radius $\mathbf{r} \in \mathbb{R}^n$ around a point $\mathbf{a} \in \mathbb{C}^n$ as a set of the form

$$D_{\mathbf{r}}(\mathbf{a}) = \{\mathbf{z} \in \mathbb{C}^N : r_i > |z_i - a_i|\}.$$

We also define a *polytorus* as a set of the form

$$T_{\mathbf{r}}(\mathbf{a}) = \{\mathbf{z} \in \mathbb{C}^N : r_i = |z_i - a_i|\}.$$

Observe that $T_{\mathbf{r}}(\mathbf{a}) \subset \partial D_{\mathbf{r}}(\mathbf{a})$. In the case of multivariable functions, we define a function $f(\mathbf{z})$to be *analytic* at a point $\mathbf{a}$ if there exists an open polydisk around $\mathbf{a}$ in which $f(\mathbf{z})$ can be expressed as a power series centered at $\mathbf{a}$. As it turns out, a lot of properties of multivariable analytic functions are similar to those of single variable analytic functions (refer to Chapter 3 in [Mel21]).

Theorem 8.4. *We have the following generalization of Cauchy's Theorem*

$$f_{\mathbf{i}} = \frac{1}{(2\pi i)^n} \oint_{T_{\mathbf{r}}(\mathbf{0})} f(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}^{\mathbf{i}+\mathbf{1}}}$$

where $\mathbf{1} = (1, 1, \dots)$ *and the integral is taken to be a suitable polytorus around* $\mathbf{0}$.

Theorem 8.5. *Suppose a power series of the form*

$$f(\mathbf{z}) = \sum_{\mathbf{i}} f_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$$

converges at some $\mathbf{z} = \mathbf{w}$. *Then the power series converges absolutely in the open ball* $O_{\mathbf{w}}(\mathbf{0})$.

It is common notation to write $\bar{\mathbf{z}} = \left(\frac{1}{z_1}, \frac{1}{z_2}, ..., \frac{1}{z_N}\right)$. Elements of $\mathbb{C}[z, \bar{z}]$ are called *Laurent polynomials* and elements of

$$\mathbb{C}((z)) = \Big\{ \sum_{n \geq N} a_n z^n, N \in \mathbb{Z}, a_n \in \mathbb{C} \Big\}$$

are called *Laurent series*. All these definitions lead us to the *Puiseux series*.

Definition 8.6. We define a *Puiseux series* centered at a as an expression of the form

$$\sum_{n \geq N} a_n (z-a)^{n/R}$$

where $R \in \mathbb{Z}_{>0}$, $N \in \mathbb{Z}$ and $a_n \in \mathbb{C}$. Together they form a field denoted by $\mathbb{C}^{\text{fra}}((z))$.

The reason why Puiseux series is so helpful is due to the following theorem [Mel21].

Theorem 8.7. $\mathbb{C}^{\text{fra}}((z))$ *is the algebraic closure of* $\mathbb{C}((z))$.

Multivariable generating functions are incredibly useful. One of their most powerful applications is enumerating paths on lattices. However, they do not seem to be as well studied as single variable generating functions. [Mel21], [PWM24] provide an overview of various methods employed in analytic combinatorics.

9. Enumerating paths on Lattices

Multivariable generating functions give us a single, robust way to deal with many problems on enumerating lattice path. Let us make things concrete with the following definition.

Definition 9.1. A *lattice path model* consists of

- a finite set of *steps* $\mathcal{S} \subset \mathbb{Z}^N$
- a *region* $\mathcal{R} \subset \mathbb{R}^N$
- a *starting point* $\mathbf{p} \in \mathcal{R}$
- a *terminal set* $\mathcal{T} \subset \mathcal{R}$
- the combinatorial class of all finite tuples called *paths* $(\mathbf{s}_1, \ldots, \mathbf{s}_r) \in \mathcal{S}^r$ such that $\mathbf{p} + \mathbf{s}_1 + \cdots + \mathbf{s}_r \in \mathcal{T}$ and $\mathbf{p} + \mathbf{s}_1 + \cdots + \mathbf{s}_k \in \mathcal{R}$ for all $1 \leq k \leq r$.

Sometimes it is helpful to assign weights to the steps. For instance, we might wish to understand a random walk where every step is assigned a probability.

Definition 9.2. A *weighted path model* assigns a *weight* $w_{\mathbf{i}}$ for each $i \in \mathcal{S}$. The *weight* of a path $(\mathbf{s}_1, \ldots, \mathbf{s}_r) \in \mathcal{S}^r$ is the product of the weights $w_{\mathbf{s}_1} \cdots w_{\mathbf{s}_r}$. *Counting* the number of paths of length n refers to adding weights of all valid paths of length n.

Observe that the following generating functions provide us a comprehensive understanding of lattice path models.

Definition 9.3. We define the *characteristic polynomial* as

$$S(\mathbf{z}) = \sum_{\mathbf{i} \in \mathcal{S}} w_{\mathbf{i}} \mathbf{z}^i.$$

We similarly define

$$W_n(\mathbf{z}) = \sum_{\mathbf{i}\in\mathbb{Z}^d} f_{\mathbf{i},n}\mathbf{z}^{\mathbf{i}}$$

where $f_{\mathbf{i},n}$ is the number of paths of length n beginning at $\mathbf{p}$ and ending at $\mathbf{i}$. Finally, we define

$$W(\mathbf{z},t) = \sum_{n\geq 0} W_n(\mathbf{z})t^n.$$

We first look at the case when $\mathcal{R} = \mathbb{R}^N$ as this will help highlight the *kernel method*. The idea is to inductively construct W_{n+1} from W_n.

Lemma 9.4. *When $\mathcal{R} = \mathbb{R}^N$, we have*

$$W_{n+1}(\mathbf{z}) = S(\mathbf{z})W_n(\mathbf{z}).$$

Proof. Follows upon expanding the right hand side of the equation. ■

This lemma, in a sense, allows us to build $W(\mathbf{z})$ recursively using W_n. This is captured in the following result

Theorem 9.5. *When $\mathcal{R} = \mathbb{R}^N$, we find the explicit relation*

$$W(\mathbf{z},t) = \frac{\mathbf{z}^{\mathbf{p}}}{1 - tS(\mathbf{z})}.$$

Proof. From Lemma 9.4, it is easy to see $W_n = W_0 S^n = \mathbf{z}^{\mathbf{p}} S^n$. Upon using the formula for geometric progression, we arrive at the theorem. ■

Hence, we come upon the following definition which is the reason why it is called the *kernel method*.

Definition 9.6. The *kernel* is the following polynomial

$$K(\mathbf{z},t) = 1 - tS(\mathbf{z}).$$

Analyzing the roots of the kernel is one of the key ideas. Suppose we are interested in one dimensional paths that begin and end at origin i.e. $\mathcal{R} = \mathbb{R}$, $\mathcal{S} \subset \mathbb{Z}$, $\mathbf{p} = \mathbf{0}$ and $\mathcal{T} = \{\mathbf{0}\}$. Such paths are called *bridges*. Hence, we are interested in the following generating function which enumerates bridges

$$B(t) = [z^0]W(z,t) = [z^0]\frac{1}{1 - tS(z)} = \frac{1}{2\pi i}\oint \frac{1}{1 - tS(z)}\frac{dz}{z}. \tag{9.1}$$

Here the loop integral is taken over a sufficiently small radius around the origin. Hence, we can find the expression for $B(t)$ by computing residues in the loop integral. We do so by analyzing the roots of the kernel. Suppose the smallest element and largest elements in $\mathcal{S}$ are $-m$ and l where $m, l > 0$. We are interested in the solutions of

$$z^m K(z,t) = 0$$

which are of the form

$$z = r(t).$$

We are only interested in those solutions which are bounded as $t \to 0$. The following lemma tells us that there are m such roots.

Lemma 9.7. *The equation $z^m K(z,t) = 0$ has $m + M$ roots which can be represented using a Puiseux series in t. Of these $m + M$ roots, there are m "small" roots*

$$z = r_n(t), 1 \leq n \leq m$$

that go to 0 as t goes to 0. The rest of the M solutions blow up as t goes to 0.

We ask the reader to refer to page 147 in [Mel21] for the proof of this result. Hence, we find the expression for $B(t)$ by computing residues due to the above mentioned small roots.

Theorem 9.8. *The generating function for the bridges of a given length is given by*

$$B(t) = t \sum_{n=1}^{m} \frac{r_n'(t)}{r_n(t)}.$$

Proof. From equation (9.1) we have

$$B(t) = \frac{1}{2\pi i} \oint \frac{1}{1 - tS(z)} \frac{dz}{z}.$$

Observe that $z = 0$ is not a source of residue as $m > 0$. Also observe that the singularities are simple poles as

$$S'(r_n(t)) = \frac{\frac{d}{dt}(-1 + tS(r_n(t))) - S(r_n(t))}{tr_n'(t)} = -\frac{1}{t^2 r_n'(t)} \neq 0.$$

Hence, we have

$$\begin{aligned} B(t) &= \sum_n \mathrm{Res}_{z=r_n(t)} \left(\frac{1}{z(1 - tS(z))} \right) = \sum_n \lim_{z \to r_n(t)} \left(\frac{z - r_n(t)}{z(1 - tS(z))} \right) \\ &= \sum_n \left(\frac{1}{r_n(t) t S'(r_n(t))} \right) = t \sum_{n=1}^{m} \frac{r_n'(t)}{r_n(t)}. \end{aligned}$$

■

Now we turn our attention to the half plane i.e. $\mathcal{R} = \mathbb{R}_{\geq 0}$ and $\mathcal{S} \subset \mathbb{Z}$. Paths in this case are called *meanders*. If they begin and end at the origin, they are called *excursions*. In this case we need to subtract out all the paths that cross into $\mathbb{Z}_{<0}$. In order to do so, we arrive at the following definition.

Definition 9.9. We define

$$S_{<-j} := \sum_{n < -j, n \in \mathcal{S}} w_n z^n.$$

The corresponding theorems from the case when $\mathcal{R} = \mathbb{R}$ are as follows.

Theorem 9.10. *When $\mathcal{R} = \mathbb{R}_{\geq 0}$ and $\mathcal{S} \subset \mathbb{Z}$, we have*

$$W_{n+1}(z) = S(z) W_n(z) - \sum_{j=0}^{m-1} S_{<-j}(z)\, z^j [z^j] W_n(z).$$

Furthermore, if $\mathbf{p} = 0$,

$$K(z,t) W(z,t) = 1 - t \sum_{n=0}^{m-1} S_{<-n}(z)\, z^n [z^n] W(z,t). \tag{9.2}$$

Proof. Follows upon expanding the expressions as in Theorem 9.5 and Lemma 9.4. ■

We can once again determine a closed form expression for $W(z,t)$ by analyzing the roots of the kernel! We highlight this in the following theorem.

Theorem 9.11. *When $\mathcal{R} = \mathbb{R}_{\geq 0}$, $\mathcal{S} \subset \mathbb{Z}$ and $\mathbf{p} = 0$ we have*

$$W(z,t) = \frac{\prod_{n=1}^{m} (1 - \bar{z} r_n(t))}{1 - tS(z)}.$$

Additionally we have the following expressions for the generating functions for the number of meanders and excursions

$$M(t) = W(1,t) = \frac{\prod_{n=1}^{m} (1 - r_n(t))}{1 - tS(1)},$$

$$E(t) = [z^0]W(z,t) = W(0,t) = \frac{(-1)^{m-1}}{w_{-m} t} \prod_{n=1}^{m} r_n(t).$$

Proof. Observe that the right hand side of 9.2 $1 - t\sum_{j=0}^{m-1} S_{<-j}(z)\, z^j [z^j] W(z,t)$ is of degree m in $\bar{z}$ with constant term 1. Also observe that $\prod_j (z - r_j(t))$ divides this expression. Hence, the first part of the theorem trivially follows. The rest of the results are immediate from the first part of the theorem. ■

The previous proof highlights the power of analyzing roots of the kernel. This idea is taken to the extreme in the *algebraic kernel method*. Typically, this method is applied to when $\mathcal{R} = \mathbb{R}^2_{\geq 0}$ and $\mathcal{S} \subset \mathbb{Z}^2$. We highlight this method in the following example taken from [BM05].

We take the case when $\mathcal{S} = \{(-1,0),(0,-1),(1,1)\}$ to demonstrate the method. Also let $\mathbf{z} = (x,y)$. It is not too hard to arrive at the functional equation for $W(x,y,t)$.

Theorem 9.12. *When $\mathcal{R} = \mathbb{R}^2_{\geq 0}$ and $\mathcal{S} = \{(-1,0),(0,-1),(1,1)\}$, we have*

$$xy(1 - t(\bar{x} + \bar{y} + xy))W(x,y,t) = xy - xt\, W(x,0,t) - yt\, W(0,y,t).$$

Proof. As in Theorems 9.5, 9.10 this follows from expanding the expressions and doing mild casework. ■

In this case we take the kernel to be $K(x,y,t) = 1 - t(\bar{x} + \bar{y} + xy)$. Also, we denote $R(x,t) = xtW(x,0,t) = xtW(0,x,t)$. Now our task is to squeeze out as much is possible from the right hand side of Theorem 9.12. The key is to analyze the symmetries of the kernel. We observe

$$K(x,y,t) = K(\bar{x}\bar{y},y,t) = K(x,\bar{x}\bar{y},t).$$

This leads us to define the following two involutions which preserve the kernel

$$\Phi\colon (x,y) \mapsto (\bar{x}\bar{y}, y),$$
$$\Psi\colon (x,y) \mapsto (x, \bar{x}\bar{y}).$$

We look at the group $\mathcal{G}$ generated by Φ and Ψ. We see that

$$\mathcal{G} = \{(x,y),(\bar{x}\bar{y},y),(x,\bar{x}\bar{y}),(\bar{x}\bar{y},x),(y,\bar{x}\bar{y}),(y,x)\}.$$

Unfortunately we are able to arrive only at the following relations

$$xy\,K(x,y,t)\,W(x,y,t) = xy - R(x,t) - R(y,t),$$
$$\bar{x}\,K(x,y,t)\,W(\bar{x}\bar{y},y,t) = \bar{x} - R(\bar{x}\bar{y},t) - R(y,t),$$
$$\bar{y}\,K(x,y,t)\,W(x,\bar{x}\bar{y},t) = \bar{y} - R(x,t) - R(\bar{x}\bar{y},t)$$

(ideally $\mathcal{G}$ would give us more relations). Algebraically manipulating, we arrive at

$$xyW(x,y,t) - \bar{x}W(\bar{x}\bar{y},y,t) + \bar{y}W(x,\bar{x}\bar{y},t) + \frac{1}{t} = \frac{1}{K(x,y,t)}\left(\frac{1}{t} - 2\bar{x} - 2R(x)\right). \tag{9.3}$$

Now the idea is to expand $\frac{1}{K(x,y,t)}$ by analyzing the roots of the kernel. For a given x, let $y = Y_0, Y_1$ be the solutions of the equation $xyK(x,y,t) = 0$. From the quadratic formula, we have

$$Y_0(x,t) = \frac{1 - t\bar{x} - \sqrt{\Delta(x,t)}}{2tx} = t + \bar{x}t^2 + O(t^3),$$
$$Y_1(x,t) = \frac{1 - t\bar{x} + \sqrt{\Delta(x,t)}}{2tx} = \frac{\bar{x}}{t} - \bar{x}^2 - t - \bar{x}t^2 + O(t^3).$$

where $\Delta(x,t) = (1-t\bar{x})^2 - 4t^2x$ is the discriminant. Hence, we can write

$$\frac{1}{K(x,y,t)} = \frac{1}{\sqrt{\Delta(x,t)}}\left(\frac{1}{1-\bar{y}Y_0} + \frac{1}{1-y/Y_1} - 1\right) \tag{9.4}$$
$$= \frac{1}{\sqrt{\Delta(x,t)}}\left(\sum_{n\geq 0}\bar{y}^nY_0^n + \sum_{n\geq 1}y^nY_1^{-n}\right). \tag{9.5}$$

We look at terms constant in y in the right hand side of (9.3) while using (9.4)

$$[y^0]\frac{1}{K(x,y,t)}\left(\frac{1}{t} - 2\bar{x} - 2R(x,t)\right) = \frac{1}{\sqrt{\Delta(x,t)}}\left(\frac{1}{t} - 2\bar{x} - 2R(x,t)\right).$$

Comparing with the right hand side of (9.3), we arrive at

$$[y^0]\left(xyW(x,y,t) - \bar{x}W(\bar{x}\bar{y},y,t) + \bar{y}W(x,\bar{x}\bar{y},t) + \frac{1}{t}\right) = -\bar{x}W_d(\bar{x},t) + \frac{1}{t} \tag{9.6}$$
$$= \frac{1}{\sqrt{\Delta(x,t)}}\left(\frac{1}{t} - 2\bar{x} - 2R(x,t)\right) \tag{9.7}$$

where $W_d(z,t) = \sum_{n\geq 0} z^n[(x^ny^n)]W(x,y,t)$. Now we factorize $\Delta(x,t)$ in terms of its roots X_0, X_1 and X_2

$$\Delta(x,t) = \Delta_0(t)\Delta_+(x,t)\Delta_-(\bar{x},t)$$
$$\Delta_0(t) = 4t^2X_2, \quad \Delta_+(x,t) = 1 - x/X_2, \quad \Delta_-(\bar{x},t) = (1-\bar{x}X_0)(1-\bar{x}X_1).$$

The roots are explicitly given by

$$X_0 = t + 2t^2\sqrt{t} + 6t^4 + \cdots,$$
$$X_1 = t - 2t^2\sqrt{t} + 6t^4 + \cdots,$$

$$X_2 = \frac{1}{4t^2} - 2t - 12t^4 + \cdots .$$

We re-write (9.6) as

$$\sqrt{\Delta_-(\bar{x},t)}\left(\frac{x}{t} - W_d(\bar{x},t)\right) = \frac{\frac{x}{t} - 2 - 2xR(x,t)}{\sqrt{\Delta_0(t)\Delta_+(x,t)}}.$$

Looking only at non-negative powers of x, we arrive at

$$-x = \frac{t}{\sqrt{\Delta_0(t)}}\left(\frac{2xR(x,t) + 2 - x/t}{\sqrt{\Delta_+(x,t)}} - 2\right).$$

Hence, we can derive the generating function for $W(x,0,t)$ as $R(x,t) = xtW(x,0,t)$. This is in turn enables us to find $W(x,y,t)$ from 9.12. (We skip deriving the explicit expressions).

To recapitulate, the idea of the algebraic kernel method is to consider the symmetries of the kernel to derive multiple functional equations involving $W(x,y,t)$. Then, upon carefully analyzing the roots of the kernel (and various other expressions), analyzing the coefficients of the resulting expressions we find the expression for $W(x,y,t)$.

10. Acknowledgments

I would like to thank Dr. Simon Rubinstein Salzedo for providing me this opportunity. I would also like to thank my mentor Rachana Madhukara for her guidance and valuable feedback. I also thank my fellow students at Euler Circle.

References

[ASS25] Jean-Paul Allouche, Jeffrey Shallit, and Manon Stipulanti. Combinatorics on words and generating dirichlet series of automatic sequences. *Discrete Mathematics*, 348(8):114487, 2025.

[BF02] Cyril Banderier and Philippe Flajolet. Basic analytic combinatorics of directed lattice paths. *Theoretical Computer Science*, 281(1-2):37–80, 2002.

[BM05] Mireille Bousquet-Mélou. Walks in the quarter plane: Kreweras' algebraic model. 2005.

[BMPS18] Yuliy Baryshnikov, Stephen Melczer, Robin Pemantle, and Armin Straub. Diagonal asymptotics for symmetric rational functions via acsv. *arXiv preprint arXiv:1804.10929*, 2018.

[BP11] Yuliy Baryshnikov and Robin Pemantle. Asymptotics of multivariate sequences, part iii: quadratic points. *Advances in Mathematics*, 228(6):3127–3206, 2011.

[FFP07] Philippe Flajolet, Éric Fusy, and Carine Pivoteau. Boltzmann sampling of unlabelled structures. In *2007 Proceedings of the Fourth Workshop on Analytic Algorithmics and Combinatorics (ANALCO)*, pages 201–211. SIAM, 2007.

[Fla03] Philippe Flajolet. Singular combinatorics. *arXiv preprint math/0304465*, 2003.

[FO82] Philippe Flajolet and Andrew Odlyzko. The average height of binary trees and other simple trees. *Journal of Computer and System Sciences*, 25(2):171–213, 1982.

[FO90] Philippe Flajolet and Andrew Odlyzko. Singularity analysis of generating functions. *SIAM Journal on discrete mathematics*, 3(2):216–240, 1990.

[FR12] Guy Fayolle and Kilian Raschel. Some exact asymptotics in the counting of walks in the quarter plane. *Discrete Mathematics & Theoretical Computer Science*, (Proceedings), 2012.

[FS09] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. cambridge University press, 2009.

[GS25] Jerzy Grunwald and Grzegorz Serafin. Explicit bounds for bell numbers and their ratios. *Journal of Mathematical Analysis and Applications*, 549(2):129527, 2025.

[Har08] John M Harris. *Combinatorics and graph theory*. Springer, 2008.

[Hay56] Walter K Hayman. A generalisation of stirling's formula. 1956.

[HR18] Godfrey H Hardy and Srinivasa Ramanujan. Asymptotic formulaæ in combinatory analysis. *Proceedings of the London Mathematical Society*, 2(1):75–115, 1918.

[Knu73] Donald E Knuth. *The Art of Computer Programming: Sorting and Searching, Fundamental Al gorithms, volume 1*. Addison-Wesley Professional, 1973.

[Knu98] Donald E Knuth. *The Art of Computer Programming: Sorting and Searching, volume 3*. Addison-Wesley Professional, 1998.

[Mel21] Stephen Melczer. *An Invitation to Analytic Combinatorics*. Springer, 2021.

[New62] DJ Newman. A simplified proof of the partition formula. *Michigan Math. J.*, 9(1):283–287, 1962.

[Pól37] George Pólya. Kombinatorische anzahlbestimmungen für gruppen, graphen und chemische verbindungen. 1937.

[Pro15] Helmut Prodinger. Analytic methods. *Handbook of enumerative combinatorics, Discrete Math. Appl.(Boca Raton)*, pages 173–252, 2015.

[Pro21] Helmut Prodinger. Philippe flajolet's early work in combinatorics. *arXiv preprint arXiv:2103.15791*, 2021.

[PWM24] Robin Pemantle, Mark C Wilson, and Stephen Melczer. *Analytic combinatorics in several variables*, volume 212. Cambridge University Press, 2024.

[Rad38] Hans Rademacher. On the partition function p (n). *Proceedings of the London Mathematical Society*, 2(1):241–254, 1938.

[Red27] J Howard Redfield. The theory of group-reduced distributions. *American Journal of Mathematics*, 49(3):433–455, 1927.

Email address: kanad.bhattacharya.725@gmail.com