AN INTRODUCTION TO MATROID THEORY

KAISER TRAVIS CHAN

Abstract. In this paper, we explore matroids, which generalize properties of linear independence of vectors and cycles in graphs. We investigate properties and operations on matroids, then we look at how we can represent matroids using matrices.

1. Introduction

Matroid theory has been used to find properties that generalize concepts in linear algebra and graph theory. It has first been studied by mathematician Hassler Whitney in 1935, as shown in his paper called *On the abstract properties of linear independence* [Whi35]. Whitney started by giving a definition of a matroid using the concept of a rank function, then derived properties of sets called independent sets, bases, and circuits from the rank function.

Here, we follow the more standard approach of first defining matroids using *independent sets* as defined in 3.1, which from its name, follows from linearly independent sets of vectors in linear algebra. These independent sets are some subsets of what we call the *ground set* of the matroid. We build on this idea in Section 4 where we look at properties of *bases* and *circuits*, which are pulled from linear algebra (basis sets) and graph theory (cycles of graphs), respectively. After that, in Section 5, we use the *rank function* to characterize subsets of the ground set, including conditions for being an independent set, base, or circuit. We derive the properties of the rank function from our definition of matroids that uses independent sets.

We also look at examples of matroids, as shown in Section 3, which showcase how our idea of matroids applies to vectors and graphs, and for vectors we show that it works with both linear and affine independence. In the other sections, we also look into properties of matroids applied into these examples. Additionally, in Section 6, we look at how we can geometrically visualize some matroids defined with affine independence, among other interesting matroids.

We also see what we can do with more than one matroid and what we can do to get another matroid from one or two matroids. We do this in Section 7, and look at *dual* matroids (the term is pulled from duals of graphs), *minors* of matroids, and the *direct sum* of matroids. Lastly, in Section 8, we look at how matroids can be *represented* using matrices.

2. Background

We first start with prerequisites from some fields that motivate our study of matroids.

2.1. **Linear algebra.**

We discuss the idea of linearly independent sets of vectors based on [Axl25].

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Definition 2.1. A set $S = \{v_1, v_2, v_3, \ldots, v_n\}$ of vectors over a field $\mathbb{F}$ is *linearly independent* if, for $a_1, a_2, a_3, \ldots, a_n \in \mathbb{F}$, the equation

$$\sum_{i=1}^{n} a_i v_i = 0$$

only has the trivial solution $a_1 = a_2 = a_3 = \cdots = a_n = 0$. In other words, no vector in S can be expressed as a linear combination of other vectors in S.

Example. Note that by definition, the zero vector is linearly independent with all other vectors.

We also define span and basis sets, both related to linear independence of vectors.

Definition 2.2. The *span* of a set $S = \{v_1, v_2, v_3, \ldots, v_n\}$ of vectors over a field $\mathbb{F}$ is defined as the set of vectors that can be obtained from a linear combination of the vectors in S:

$$\operatorname{span}(v_1, v_2, v_3, \ldots, v_n) = \left\{ \sum_{i=1}^{n} a_i v_i \mid a_1, a_2, a_3, \ldots, a_n \in \mathbb{F} \right\}.$$

Definition 2.3. A *basis* of a vector space V is a set of vectors in V that is linearly independent and has span V.

Note that a basis of a vector space is also a linearly independent set of vectors with maximal cardinality.

We also define affine dependence of vectors [KP09, Chapter 1], which is similar to linear independence:

Definition 2.4. A multiset $S = \{v_1, v_2, v_3, \ldots, v_n\}$ of vectors over a field $\mathbb{F}$ is *affinely dependent* if there exist $a_1, a_2, a_3, \ldots, a_n \in \mathbb{F}$ where

$$\sum_{i=1}^{k} a_i v_i = 0, \qquad \sum_{i=1}^{k} a_i = 0,$$

and all of $a_1, a_2, a_3, \ldots, a_k$ are not equal to zero. The set S is *affinely independent* if it is not affinely independent.

Lastly, we define the rank of a matrix as follows.

Definition 2.5. The *rank* of a matrix A is defined as the dimension of the span of the column vectors of A.

2.2. **Graph theory.** We also look into some definitions from graph theory. [Hil]

Definition 2.6. In a graph G, a *walk* is a finite sequence of edges $v_0v_1, v_1v_2, \ldots, v_{n-1}v_n$, where $v_0, v_1, \ldots, v_n$ are vertices of G. The walk is a *trail* if the edges in the sequence are distinct, and it is a *path* if the vertices $v_0, v_1, \ldots, v_n$ are distinct except the case where $v_0 = v_n$. The path is a *cycle* if $v_0 = v_n$.

Definition 2.7. Let G be a graph with n vertices. Then, a *spanning tree* with n vertices is a connected subgraph of G with $n - 1$ edges and n vertices.

Example. Some spanning trees of the complete graph K_4 is shown in Figure 1.

One can also note that a spanning tree does not contain any cycles.

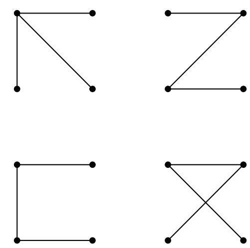

Figure 1. Some spanning trees of K_4.

3. Definition and examples of matroids

There are many ways we can define a matroid, all of which are related to independence, so we start off with the following definition and build the other definitions from this.

Definition 3.1. A *matroid* is an ordered pair $\mathcal{M} = (E, \mathcal{I})$, where E is a finite set called the *ground set* and $\mathcal{I}$ is the set of *independent sets*, which is composed of some subsets of E such that:

- I-1. The empty set is independent.
- I-2. If $A \in \mathcal{I}$, then a set $B \subseteq A$ is also independent.
- I-3. (*independence augmentation property*) If $A, B \in \mathcal{I}$ such that $|B| > |A|$, then there is an element $x \in B \setminus A$ where $A \cup \{x\}$ is also independent.

A subset of E that is not independent is called a *dependent set*.

Notation. We also denote the ground set of a matroid $\mathcal{M}$ with $E(\mathcal{M})$.

We saw earlier that linear algebra and graph theory motivated the study of matroids. Let us look into these two fields and see what can be classified as a matroid in each of these fields.

3.1. Vector and affine matroids.

Firstly, we construct matroids using linear independence of vectors. We see that linear independence also satisfies the properties stated in Definition 3.1.

Theorem 3.2. *Let A be a matrix over a field $\mathbb{F}$, and let E be the set of column vectors of A. Then, let $\mathcal{I}$ be the sets of subsets of E that are linearly independent over $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid and is called a* vector matroid*, denoted by $M[A]$.*

Proof. We show that $(E, \mathcal{I})$ is a matroid by checking if it satisfies the properties in Definition 3.1. Note that $\mathcal{I}$ satisfies I-1 and I-2. So, we verify as follows that I-3 is also satisfied, as shown in [Oxl03].

Let A and B be linearly independent subsets of E where $|B| = |A| + 1$. Then, let V be the vector space that is the span of $A \cup B$. Then, $\dim V \geq |B|$. If $A \cup \{x\}$ is linearly dependent for all $x \in B \setminus A$, then V is in the span of A, thus $\dim V \leq |A|$. This means that $|B| \leq \dim V \leq |A|$. However, $|B| > |A|$, so we have a contradiction. Therefore, there exists an element $x \in B \setminus A$ where $A \cup \{x\}$ is linearly independent.

We thus conclude that $(E, \mathcal{I})$ is a matroid. ∎

Let us look at an example of an independent set of a vector matroid.

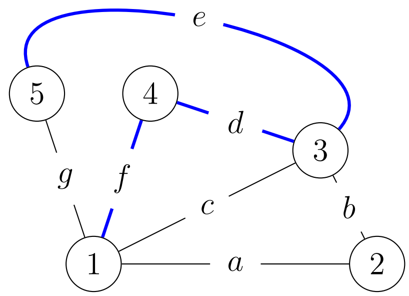

Figure 2. An example graph H, together with one of the independent sets from $M(H)$.

Example. Suppose we have

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then, one independent set of the vector matroid $M[A]$ is $X = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. Notice that all its subsets are linearly independent, satisfying I-2.

Another independent set of the vector matroid is $Y = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. Since $|X| > |Y|$, by I-3, there exists an element $x \in X \setminus Y$ where $Y \cup \{x\}$ is independent. We can see that this element is $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, resulting in $Y \cup \{x\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, which is linearly independent.

We can also do a similar construction of matroids using affine independence:

Theorem 3.3. *Let A be a matrix over a field $\mathbb{F}$, and let E be the set of column vectors of A. Then, let $\mathcal{I}$ be the sets of subsets of E that are affinely independent over $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid and is called an* affine matroid*. [KP09, Chapter 1]*

3.2. **Graphic matroids.** We can also construct matroids from graphs, where the subgraphs without cycles are independent:

Theorem 3.4. *Let $G = (V, E)$ be a graph, and let $\mathcal{I}$ be the set of edges that have no cycles in G. Then, $(E, \mathcal{I})$ is a matroid and is called a* cycle matroid*, denoted by $M(G)$. [Oxl03]*

Example. Consider graph H shown in Figure 2 with edges a, b, c, d, e, f, g. One of the independent sets of $M(H)$ will be the set $\{d, e, f\}$ as the corresponding subgraph does not contain any cycles.

We can also note that some matroids are isomorphic to cycle matroids, and we call these graphic matroids.

Definition 3.5. A *graphic matroid* is a matroid that is isomorphic to a cycle matroid of a graph.

3.3. **Other types of matroids.** We also present other examples of matroids with their own names.

Proposition 3.6. *If E is a set with n elements and $\mathcal{I}$ is the set of all subsets A of E such that $|A| = r$ for some integer r where $0 \leq r \leq n$, then $U_{r,n} = (E, \mathcal{I})$ is a matroid and is called a* uniform matroid.

Proposition 3.7. *Let $A_1, A_2, A_3, \ldots, A_n$ be disjoint sets, and let $E = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$. Then, let $\mathcal{I}$ be the set of all subsets of E that contain zero or one element from each of $A_1, A_2, A_3, \ldots, A_n$. Then, $(E, \mathcal{I})$ is a matroid and is called a* transversal matroid.

4. Bases and circuits

We can also define matroids in terms of bases and circuits, which are both based on independent sets. The proofs of the theorems are also found in [KP09, Chapter 1].

4.1. **Bases.** We first define bases of a matroid to match the definition of a basis of a vector space.

Definition 4.1. A *base* of a matroid $\mathcal{M}$ is an independent set of $\mathcal{M}$ with maximal cardinality.

From this, we can prove the following result, motivated by the fact that basis vectors in a vector space have the same cardinality.

Proposition 4.2. *All bases of a matroid $\mathcal{M}$ have the same cardinality, called the* rank *of $\mathcal{M}$ [Duk04], denoted by $r(\mathcal{M})$.*

Proof. Let B_1 and B_2 be bases where $|B_2| > |B_1|$. Then, by property I-3 in Definition 3.1, there exists an element $x \in B_2 \setminus B_1$ where $B_2 \cup \{x\}$ is independent. However, since B_2 is a base, it is an independent set with maximal cardinality, so adding x makes the resulting set dependent, thus a contradiction. Therefore, it is impossible for one to choose two bases from a matroid that have differing cardinalities. ■

We can also define a matroid in terms of its bases as shown below, and we can prove that this definition is consistent with Definition 3.1.

Lemma 4.3. *The set of bases $\mathcal{B}$ of a matroid with ground set E satisfies the following properties:*

B-1. The set $\mathcal{B}$ is not empty.
*B-2. (*exchange property*) If $B_1, B_2 \in \mathcal{B}$ and $x_1 \in B_1 \setminus B_2$, there exists an element $x_2 \in B_2 \setminus B_1$ such that $B_1 \setminus \{x_1\} \cup \{x_2\}$ is also a base.*

Proof. Since the set of independent sets, $\mathcal{I}$, is not empty due to it containing the empty set (according to property I-1), we know that $\mathcal{B}$ is not empty, satisfying property B-1.

Now we prove property B-2. Suppose we have two bases B_1 and B_2 with an element $x_1 \in B_1 \setminus B_2$. Then, $B_1 \setminus \{x_1\}$ is independent by property I-2. We also know that B_2 is independent since it is a base, and by Proposition 4.2, we have $|B_1| = |B_2|$, so $|B_2| > |B_1 \setminus \{x_1\}|$. So, by property I-3, there exists another element $x_2 \in B_2 \setminus (B_1 \setminus \{x_1\})$ where $B_1 \setminus \{x_1\} \cup \{x_2\}$ is independent. Since $x_2 \in B_2 \setminus B_1$ and $|B_1 \setminus \{x_1\} \cup \{x_2\}| = |B_1|$, we know that $B_1 \setminus \{x_1\} \cup \{x_2\}$ is also a base. ■

We also state the following, which claim that the properties we gave for bases fully characterize a matroid. We leave the proof to [KP09, Chapter 1].

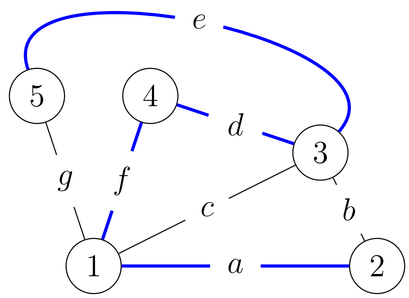

Figure 3. An example graph H, together with one of the bases from $M(H)$.

Theorem 4.4. *Let E be a set and $\mathcal{B}$ be the set of subsets of E that satisfy the properties in Lemma 4.3. Then, let $\mathcal{I}$ be the set of subsets of E that are subsets of an element of $\mathcal{B}$. Then, $(E, \mathcal{I})$ is a matroid.*

Corollary 4.5. *A set of subsets of a ground set of a matroid is the set of bases if and only if it satisfies the properties listed in Lemma 4.3.*

Let us look at examples of bases in both vector and graphic matroids.

Example. In a vector matroid on a matrix A, the bases are the maximal linearly independent sets of columns of A, which are the basis sets of A.

Let us once again investigate our matrix from earlier,

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then, one base of the vector matroid $M[A]$, or basis of A, is $X = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, and another is $Y = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. We can exchange the second elements of each of the bases, which results in X becoming Y and Y becoming X.

Example. In a graphic matroid $M(G)$ where G has n vertices, the bases are the spanning trees with n vertices that are subgraphs of G. For instance, in graph H shown in Figure 3 with edges a, b, c, d, e, f, g, one of the bases of the graphic matroid $M(G)$ is $\{a, d, e, f\}$ since it is a spanning tree and a subgraph of H.

4.2. **Circuits.** We also look into circuits, which are defined as follows:

Definition 4.6. A *circuit* is a dependent set of a matroid with minimal cardinality.

Just like bases, circuits also have a list of properties that they satisfy, as shown in the following lemma.

Lemma 4.7. *The set of circuits $\mathcal{C}$ of a matroid with ground set E satisfies the following properties:*

C-1. The empty set is not a circuit.
C-2. If $A \in \mathcal{C}$, then all proper subsets of A are not circuits.
*C-3. (*circuit elimination property*) If $C_1, C_2 \in \mathcal{C}$, where $C_1 \neq C_2$, and $x \in C_1 \cap C_2$, then $(C_1 \cup C_2) \setminus \{x\}$ contains a circuit.*

Proof. Firstly, by property I-1, we know that the empty set is independent, so it is not a circuit, thus property C-1 is true. Also, since circuits have minimal cardinality, no subset of a circuit is a circuit, satisfying property C-2.

Now, we prove C-3 as shown in [KP09, Chapter 1]. Assume for the sake of contradiction that $(C_1 \cup C_2) \setminus \{x\}$ does not contain a circuit. Then, it is independent. By C-2, we know that if an element e belongs to $C_2 \setminus C_1$, then $C_2 \setminus \{e\}$ is independent since C_2 is a circuit.

Let I be an independent subset of $C_1 \cup C_2$ that has maximum cardinality such that $C_2 \setminus \{e\}$. Since C_2 is a circuit, we know that $e \notin I$. Also, since I is not a subset of C_1, there exists an element $f \in C_1 \setminus I$. We stated earlier that $e \in C_2 \setminus C_1$, so e and f are distinct elements. Therefore,

$$|I| \leq |(C_1 \cup C_2) \setminus \{e, f\}| = |C_1 \cup C_2| - 2 < |(C_1 \cup C_2) \setminus \{x\}|.$$

So, $|I| < |(C_1 \cup C_2) \setminus \{x\}|$, which means we can apply property I-3 to get that there exists an element $g \in (C_1 \cup C_2 \setminus \{x\}) \setminus I$ where $I \cup \{g\}$ is independent. However, I has maximal cardinality, giving us a contradiction.

Therefore, $(C_1 \cup C_2) \setminus \{x\}$ contains a circuit. ■

Just like with bases, we also state the following, which claim that the properties we gave for circuits fully characterize a matroid. We once again leave the proof to [KP09, Chapter 1].

Theorem 4.8. *Let E be a set and $\mathcal{C}$ be the set of subsets of E that satisfy the properties in Lemma 4.7. Then, let $\mathcal{I}$ be the set of subsets of E that do not contain an element of $\mathcal{C}$. Then, $(E, \mathcal{I})$ is a matroid.*

Corollary 4.9. *A set of subsets of a ground set of a matroid is the set of circuits if and only if it satisfies the properties listed in Lemma 4.7.*

We also define the following:

Definition 4.10. A *loop* of a matroid with ground set E is an element x of E such that $\{x\}$ is a circuit [KP09, Chapter 1].

We now explore circuits through examples in vector and graphic matroids.

Example. In a vector matroid of a matrix A, the circuits are the minimal linearly dependent sets of column vectors of A. For instance, we can consider our matrix from earlier,

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then, two of the circuits of $M[A]$ are $X = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $Y = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.
By C-3, there exists an element $e \in X \cap Y$ where $(X \cup Y) \setminus \{e\}$ contains a circuit, and in

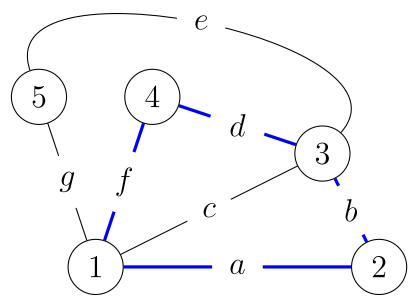

Figure 4. An example graph H, together with one of the circuits from $M(H)$.

this case, this element is $e = \begin{bmatrix}1\\0\\0\end{bmatrix}$. We then have $(X \cup Y) \setminus \{e\} = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$, which is itself a circuit.

Example. In a graphic matroid $M(G)$, the circuits are the cycles in G with at least one edge. We know that these are of minimal cardinality because if we remove an edge from the cycle, then the resulting subgraph has no cycles. For example, in graph H shown in Figure 4 with edges a, b, c, d, e, f, g, one of the circuits of $M(H)$ is $\{a, b, d, f\}$.

5. The rank function

In Whitney's paper, [Whi35], the first definition of matroids that was given used the rank function; however, we use a different definition here that is consistent with the original one. We will also see that this definition of matroids is consistent with Definition 3.1, which uses independent sets.

Definition 5.1. The *rank function* $r_{\mathcal{M}} : 2^E \to \mathbb{N} \cup \{0\}$ of a matroid $\mathcal{M}$ with ground set E is defined such that, if $A \subseteq E$, then $r(A)$ is the cardinality of the largest independent set contained in A. If the matroid being referred to is clear, we usually shorten $r_{\mathcal{M}}$ to r.

We now define matroids using the rank function as follows:

Lemma 5.2. *The rank function r of a matroid with ground set E satisfies the following properties:*

R-1. For a subset A of E, we have $0 \leq r(A) \leq |A|$.
R-2. If $A \subseteq B \subseteq E$, then $r(A) \leq r(B)$.
*R-3. (*submodularity property*) If $A, B \subseteq E$, we have $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.*

Proof. Note that R-1 is true because the size of the largest independent subset is always nonnegative and cannot be greater than $|A|$. Also, R-2 is true because if $r(A) > r(B)$, then A contains an independent set with cardinality larger than the largest independent subset of B, which cannot be the case since $A \subseteq B$. We now prove property R-3 as shown in [KP09, Chapter 1].

Let $X_\cap$ and $X_\cup$ be inclusion-wise maximal independent subsets of $A \cap B$ and $A \cup B$, respectively, such that $X_\cap \subseteq X_\cup$. Then, by I-2, since $X_\cup \cap A \subseteq X_\cup$, we know that $X_\cup \cap A$ is an independent subset of A. Thus, we have $r(X_\cup \cap A) = |X_\cup \cap A|$. Also, since $X_\cup \cap A \subseteq A$, we have by R-2 that $r(X_\cup \cap A) \leq r(A)$, or $|X_\cup \cap A| \leq r(A)$. Similarly, $|X_\cup \cap B| \leq r(B)$.

Therefore, we have

$$\begin{aligned} r(A) + r(B) &\geq |X_\cup \cap A| + |X_\cup \cap B| \\ &= |(X_\cup \cap A) \cup (X_\cup \cap B)| + |(X_\cup \cap A) \cap (X_\cup \cap B)| \\ &= |X_\cup \cap (A \cup B)| + |X_\cup \cap (A \cap B)| \\ &= |X_\cup| + |X_\cap| \\ &= r(A \cup B) + r(A \cap B), \end{aligned}$$

completing our proof of R-3. ■

5.1. **Independent sets, bases, and circuits in terms of the rank function.** Before we characterize the independent sets of a matroid using the rank function, we first state and prove the following lemma based on [KP09, Chapter 1].

Lemma 5.3. *Let E be a finite set, and let $r : 2^E \to \mathbb{N} \setminus \{0\}$ satisfy the three properties in Lemma 5.2. Then, let $A, B \subseteq E$ where for every element $x \in B \setminus A$, we have $r(A \cup \{x\}) = r(A)$. Then, we have $r(A \cup B) = r(A)$.*

Proof. We prove the lemma by inducting on $k = |B \setminus A|$. Suppose $B \setminus A = \{x_1, x_2, \ldots, x_k\}$. Note that if $k = 1$, then $B \setminus A = \{x_1\}$, and since we know from the condition that $r(A \cup \{x_1\}) = r(A)$, our claim is true and thus the base case is satisfied.

Now, assume the lemma is true for $|B \setminus A| = k - 1$ where $k \geq 2$; that is, $r(A) = r(A \cup \{x_1, x_2, \ldots, x_{k-1}\})$. We will prove that it is also true for $|B \setminus A| = k$. Note that $r(A) = r(A \cup \{x_k\})$ by our condition. This means that

$$r(A) + r(A) = r(A \cup \{x_1, x_2, \ldots, x_{k-1}\}) + r(A \cup \{x_k\}).$$

Additionally, note that $(A \cup \{x_1, x_2, \ldots, x_{k-1}\}) \cup (A \cup \{x_k\}) = A \cup \{x_1, x_2, \ldots, x_k\}$ and $(A \cup \{x_1, x_2, \ldots, x_{k-1}\}) \cap (A \cup \{x_k\}) = A$, so by R-3, we have

$$r(A \cup \{x_1, x_2, \ldots, x_{k-1}\}) + r(A \cup \{x_k\}) \geq r(A \cup \{x_1, x_2, \ldots, x_k\}) + r(A).$$

By R-2, since $A \subseteq A \cup \{x_1, x_2, \ldots, x_k\}$, we have $r(A \cup \{x_1, x_2, \ldots, x_k\}) \geq r(A)$, so

$$r(A \cup \{x_1, x_2, \ldots, x_k\}) + r(A) \geq r(A) + r(A).$$

We would then have:

$$\begin{aligned} r(A) + r(A) &= r(A \cup \{x_1, x_2, \ldots, x_{k-1}\}) + r(A \cup \{x_k\}) \\ &\geq r(A \cup \{x_1, x_2, \ldots, x_k\}) + r(A) \\ &\geq r(A) + r(A). \end{aligned}$$

However, $r(A) + r(A) = r(A) + r(A)$, so the inequalities listed above are actually equalities. Therefore, $r(A) + r(A) = r(A \cup \{x_1, x_2, \ldots, x_k\}) + r(A)$, or $r(A) = r(A \cup \{x_1, x_2, \ldots, x_k\}) = r(A \cup (B \setminus A)) = r(A \cup B)$, thus completing the inductive step.

Therefore, the lemma is true for all such values of $B \setminus A$ by the principle of mathematical induction. ■

Now, we can use Lemma 5.3 to prove the following, which shows the condition for independent sets based on the rank function. We once again base the proof on [KP09, Chapter 1].

Theorem 5.4. *Let E be a finite set, and let $r : 2^E \to \mathbb{N} \setminus \{0\}$ satisfy the three properties in Lemma 5.2. Then, let $\mathcal{I}$ be the set of $A \subseteq E$ where $r(A) = |A|$. Then, $(E, \mathcal{I})$ is a matroid with rank function r.*

Proof. We prove that the given conditions satisfy the properties listed in Definition 3.1. Firstly, $r(\varnothing) = 0 = |\varnothing|$ by R-1, so $\varnothing$ is independent, satisfying property I-1.

We now prove that I-2 holds given our setup. Consider an element A of $\mathcal{I}$, and let A' be a subset of A. Note that applying R-3 on sets A' and $A \setminus A'$ gives

$$r(A) + r(\varnothing) \leq r(A') + r(A \setminus A')$$

since $A' \cup (A \setminus A') = A$ and $A' \cap (A \setminus A') = \varnothing$. Additionally, by R-1 we have $r(A') \leq |A'|$ and $r(A \setminus A') \leq |A \setminus A'|$, so we have

$$\begin{aligned} |A| = r(A) &= r(A) + r(\varnothing) \\ &\leq r(A') + r(A \setminus A') \\ &\leq |A'| + |A \setminus A'| = |A|. \end{aligned}$$

Since $|A| = |A|$, the inequalities listed are equalities, so $r(A') + r(A \setminus A') = |A'| + |A \setminus A'|$. However, $r(A') \leq |A'|$ and $r(A \setminus A') \leq |A \setminus A'|$, so $r(A') = |A'|$, therefore $A' \in \mathcal{I}$. Since $A' \subseteq A$, we have proved property I-2.

We now prove that I-3 also holds. Consider two elements A and B of $\mathcal{I}$ where $|B| > |A|$. Also, suppose for the sake of contradiction that for all $x \in B \setminus A$, we have $A \cup \{x\} \notin \mathcal{I}$. Then, since $r(A \cup \{e\}) \neq |A \cup \{e\}| = |A| + 1$, we know by R-1 that $|A| + 1 > r(A \cup \{e\})$. Also, by R-2, we have $r(A \cup \{e\}) \geq r(A) = |A|$. Putting these together, we have

$$|A| + 1 > r(A \cup \{e\}) \geq r(A) = |A|$$

, so $r(A \cup \{e\}) = |A|$. Thus, by Lemma 5.3, we have $r(A) = r(A \cup B)$. We also have by R-2 that $r(A \cup B) \geq r(A)$, so

$$r(B) \leq r(A \cup B) = r(A) = |A| < |B|,$$

. Therefore, $r(B) < |B|$, which means $I_2 \notin \mathcal{I}$, thus giving us a contradiction. Therefore, $I_2 \in \mathcal{I}$, satisfying I-3.

We also need to prove that r is the rank function of $\mathcal{M} = (E, \mathcal{I})$. To do this, we consider a subset A of E. Note that if A is independent in $\mathcal{M}$, then $r(A) = |A| = r_{\mathcal{M}}(A)$. If A is not independent in $\mathcal{M}$, then we let I be an independent subset of X with maximal cardinality. Then, for every $x \in X \setminus I$ we have $I \cup \{x\} \neq I$, so $r(I \cup \{x\}) = r(I)$. Thus, by Lemma 5.3, we have that $r(X) = r(I) = r_{\mathcal{M}}(I)$. Therefore, for all subsets A of E, we have $r(A) = r_{\mathcal{M}}(A)$, so r is the rank function of the matroid $\mathcal{M}$. ■

Corollary 5.5. *Let E be a set. A function r with domain 2^E is the rank function of a matroid with ground set E if an only if r satisfies the properties listed in Lemma 5.2.*

Given that we characterized independent sets in terms of the rank function in 5.4, we also give conditions for the bases and circuits as follows [KP09, Chapter 2]:

Proposition 5.6. *Let $\mathcal{M}$ be a matroid with r as its rank function. Then, for any subset A of $E(\mathcal{M})$, we have:*

1. *A is a base if and only if $|A| = r(A) = r(\mathcal{M})$.*
2. *A is a circuit if and only if $A \neq \varnothing$ and for all $x \in A$ we have $r(A \setminus \{x\}) = |A| - 1 = r(A)$.*

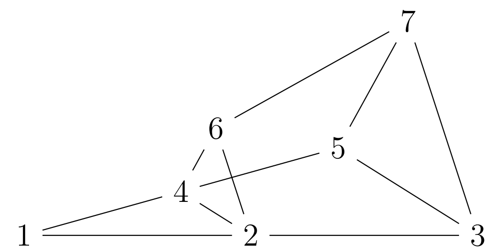

Figure 5. A visualization that does not give a matroid, as described in [KP09, Chapter 1]

5.2. **Examples of rank functions.** Let us now look at how the rank function applies in vector and graphic matroids.

Proposition 5.7. *Let A be a matrix over a field $\mathbb{F}$, and let E be the set of column vectors of A. Then, the rank function of the vector matroid of A is given by the rank of the matrix formed by each subset of E.*

Proposition 5.8. *Let $G = (V, E)$ be a graph. Then, the rank function of $M(G)$ is given by the largest number of edges in each subgraph of G that has no cycles.*

6. Geometric visualizations of matroids

We first explore a property of affine matroids by rank, as described in [KP09, Chapter 1], starting with those with rank three.

Let E be a multiset of vectors in $\mathbb{R}^2$, interpreted as points on the Cartesian plane. Then, a subset $A \subseteq E$ is affinely dependent if it contains two points at the same position, three collinear points, or four or more coplanar points.

Similarly, in an affine matroid of rank four, the vectors, which we again interpret as points, are in $\mathbb{R}^3$. A subset in this case would be affinely dependent if it contains two points at the same position, three collinear points, four coplanar points, or five or more points in space.

Using the idea from affine matroids, we can draw diagrams of matroids with rank at most 4 by assigning each element of the ground set to a point, and these points are positioned so that circuits with two elements correspond to identical points, those with three elements correspond to collinear points, and those with four elements correspond to coplanar points.

Not all visualizations can give matroids, as shown in the following example given in [KP09, Chapter 1].

Example. The diagram shown in Figure 5 does not represent a matroid. To see this, consider subsets $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 5, 6, 7\}$ of the ground set $E = \{1, 2, 3, 4, 5, 6, 7\}$. Then, $r(A) = 3$, $r(B) = 3$, $r(A \cup B) = 4$, and $r(A \cap B) = 4$, thus violating R-3 since $r(A \cup B) + r(A \cap B) > r(A) + r(B)$.

These constructions do not need to be in $\mathbb{R}^n$. For instance, the Fano matroid has its visualization on the projective plane over $GF(2)$ instead of the Cartesian plane $\mathbb{R}^2$:

Proposition 6.1. *Let $E = \{1, 2, 3, 4, 5, 6, 7\}$ be the set of points in the Fano plane, which is the projective plane over $GF(2) = \mathbb{Z}/2\mathbb{Z}$. Then, let $\mathcal{I}$ be the set of sets of points that are*

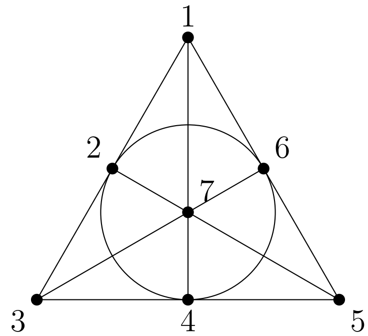

Figure 6. The Fano matroid F_7.

not collinear in the Fano plane. Then, $(E, \mathcal{I})$ *is a matroid and is called the* Fano matroid F_7*, depicted in Figure 6.* [1] *[KP09, Chapter 1]*

7. Duals, minors, and the direct sum

Now that we investigated properties of matroids, we look at what we can do with matroids to get other matroids. Specifically, we will explore dual matroids, minors of matroids, and the direct sum or union of matroids.

7.1. Duals of graphs and matroids.

To get some background, we define duals of graphs as follows.

Definition 7.1. Let G be a graph. The *dual* of G, denoted by G^*, is constructed by placing a vertex representing every face or region of G, then drawing an edge between pairs of vertices that represent adjacent faces.

Example. An example graph H together with its dual graph is shown in Figure 7. Note that since an edge also connects two faces, each edge of H^* corresponds to an edge in H, and we label them like so (e.g. a' corresponds to a).

Take a spanning tree of H, for example $\{a, c, f, g\}$. Its complement is then $\{b, d, e\}$, which corresponds to $\{b', d', e'\}$. Note that $\{b', d', e'\}$ is also a spanning tree of H^*. We can observe with other spanning trees that the spanning trees of H^* are complements of those of H.

Our observation that the spanning trees of G^* are complements of the spanning trees of G is true for every graph [Oxl03]. So, we generalize this definition to matroids.

Theorem 7.2. *Let* $\mathcal{M} = (E, \mathcal{I})$ *be a matroid with* $\mathcal{B}$ *as the set of its bases. Then, let* $\mathcal{B}^* = \{E \setminus x \mid x \in \mathcal{B}\}$ *be the set of complements of the elements of* $\mathcal{B}$*. Then,* $\mathcal{B}^*$ *is a set of bases of another matroid with ground set* E*.*

We thus define dual matroids based on Theorem 7.2:

Definition 7.3. The *dual* of a matroid $\mathcal{M}$ with $\mathcal{B}$ as the set of its bases, denoted as $\mathcal{M}^*$, is defined so that its set of bases, $\mathcal{B}^* = \{E \setminus \{x\} \mid x \in \mathcal{B}\}$, is the set of complements of the elements of $\mathcal{B}$. The bases of $\mathcal{M}^*$ are called *cobases* and its circuits are called *cocircuits.* The rank function of $\mathcal{M}^*$, denoted by r^*, is called the *corank function.*

[1] Note that despite the fact that $\{2, 4, 6\}$ are connected by a circle in our diagram, they are still treated as collinear.

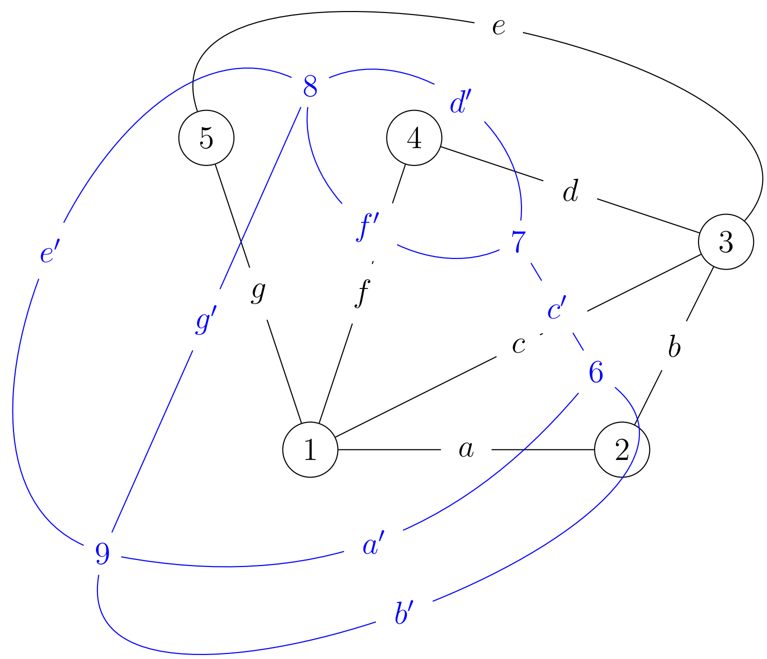

Figure 7. An example graph H with its dual H^* in blue.

Let us look at duals of uniform matroids as an example of this concept.

Example. In the uniform matroid $U_{r,n}$, the bases are the subsets of the ground set with r elements, so those of $U^*_{r,n}$ are the subsets of the ground set with $n - r$ elements, therefore $U^*_{r,n} = U_{n-r,n}$. Thus, the dual of a uniform matroid is also a uniform matroid.

Note that we call $\mathcal{M}^*$ the dual of $\mathcal{M}$ due to the following property.

Proposition 7.4. *Let $\mathcal{M}$ be a matroid. Then, $(\mathcal{M}^*)^* = \mathcal{M}$.*

We also discuss properties of the corank function of a matroid [KP09, Chapter 2].

Proposition 7.5. *Let $\mathcal{M}$ be a matroid. Then, $r(\mathcal{M}) + r^*(\mathcal{M}^*) = |E(\mathcal{M})| = |E(\mathcal{M}^*)|$.*

In fact, the above property is a special case of the following proposition, which generalizes this to any subset of the ground set of the matroid. This also gives us a formula for the corank of a subset of the ground set.

Proposition 7.6. *Let $\mathcal{M}$ be a matroid with ground set E, and A be a subset of E. Then,*

$$r^*(A) = |A| - r(\mathcal{M}) + r(E \setminus A).$$

We also state the following, which gives some information about cocircuits of a matroid according to [Oxl03].

Theorem 7.7. *Let $\mathcal{M}$ be a matroid. We then have the folowing:*

(1) *A set C^* is a cocircuit of $\mathcal{M}$ if and only if C^* has a non-empty intersection with every base of M while having minimal cardinality.*
(2) *A set B is a base of $\mathcal{M}$ if and only if B has a non-empty intersection with every cocircuit of M while having minimal cardinality.*

We will also soon see that the duals of matroids when other operations are applied give special properties.

7.2. **Minors of matroids.** We first define two operations on matroids, namely deletion and contraction.

Definition 7.8. Let $\mathcal{M}$ be a matroid with ground set E. When a subset A is *deleted* from E, the resulting matroid $\mathcal{M} \setminus A$ (or $\mathcal{M}|(E \setminus A)$) has the ground set $E \setminus A$ and the independent sets are the subsets of $E \setminus A$ that are also independent in $\mathcal{M}$.

Definition 7.9. Let $\mathcal{M}$ be a matroid with ground set E. When a subset A is *contracted* from E, the resulting matroid $\mathcal{M}/A$ is the same as the matroid obtained from deleting A in the dual matroid $\mathcal{M}^*$.

We are now ready to define minors of matroids in terms of these two operations.

Definition 7.10. A *minor* of a matroid $\mathcal{M}$ is a matroid that can be obtained from $\mathcal{M}$ by a sequence of deletions and contractions.

We will see that any minor can be expressed as the original matroid with one deletion and one contraction applied to it. We first prove the following:

Proposition 7.11. *Let $\mathcal{M}$ be a matroid with ground set E. Let $A \subseteq E$ and $X \in E \setminus A$. Then, we have*

$$r_{\mathcal{M} \setminus A}(X) = r_{\mathcal{M}}(X), \tag{7.1}$$

$$r_{\mathcal{M}/A}(X) = r_{\mathcal{M}}(X \cup A) - r_{\mathcal{M}}(A). \tag{7.2}$$

Proof. Since X does not contain any elements that are in A, the largest independent sets contained in X in the matroids $\mathcal{M}$ and $\mathcal{M} \setminus A$ are the same, so their cardinalities are equal. Therefore, $r_{\mathcal{M} \setminus A}(X) = r_{\mathcal{M}}(X)$, satisfying Equation 7.1.

Now, we prove Equation 7.2 as follows, based on the proof in [KP09, Chapter 2], using the property of the corank function described in Proposition 7.6. Firstly, we have the following, with the second equality coming from Equation 7.1:

$$\begin{aligned} r_{\mathcal{M}/A}(X) &= |X| + r_{\mathcal{M}^* \setminus A}(E \setminus A \setminus X) - r_{\mathcal{M}^* \setminus A}(E \setminus A) \\ &= |X| + r^*_{\mathcal{M}}(E \setminus (A \cup X)) - r^*_{\mathcal{M}}(E \setminus A). \end{aligned}$$

Now, by Proposition 7.6, we have

$$\begin{aligned} r^*_{\mathcal{M}}(E \setminus (A \cup X)) &= |E \setminus (A \cup X)| + r_{\mathcal{M}}(A \cup X) - r_{\mathcal{M}}(E), \\ r^*_{\mathcal{M}}(E \setminus A) &= |E \setminus A| + r_{\mathcal{M}}(A) - r_{\mathcal{M}}(E). \end{aligned}$$

This means that our expression becomes

$$\begin{aligned} r_{\mathcal{M}/A}(X) &= |X| + r^*_{\mathcal{M}}(E \setminus (A \cup X)) - r^*_{\mathcal{M}}(E \setminus A) \\ &= |X| + (|E \setminus (A \cup X)| + r_{\mathcal{M}}(A \cup X) - r_{\mathcal{M}}(E)) - (|E \setminus A| + r_{\mathcal{M}}(A) - r_{\mathcal{M}}(E)). \end{aligned}$$

■

Now we prove the following lemma, based on [KP09, Chapter 2], which indicates that deletions and contractions are actually commutative, both with themselves and with each other, and that we can combine deletions and contractions into their unions.

Lemma 7.12. *Let $\mathcal{M}$ be a matroid with ground set E, and let A and B be disjoint subsets of E. We then have*

$$(\mathcal{M} \setminus A) \setminus B = \mathcal{M} \setminus (A \cup B) = (\mathcal{M} \setminus B) \setminus A, \tag{7.3}$$

$$(\mathcal{M}/A)/B = \mathcal{M}/(A \cup B) = (\mathcal{M}/B)/A, \tag{7.4}$$

$$(\mathcal{M}/A) \setminus B = (\mathcal{M} \setminus B)/A. \tag{7.5}$$

Proof. Note that Equation 7.3 follows from the definition of deletion, because each sequence of operations removes the same elements and the same independent sets. Equation 7.4 follows from Equation 7.3 but instead considering the corresponding dual matroid.

We now prove Equation 7.5 by showing that $(\mathcal{M}/A) \setminus B$ and $(\mathcal{M} \setminus B)/A$ have the same rank function. Consider a subset X of $E \setminus (A \cup B)$. Then, we have by Equation 7.1 that $r_{(\mathcal{M}/A)\setminus B}(X) = r_{\mathcal{M}/A}(X)$, which by Equation 7.2 is equal to $r_{\mathcal{M}}(X \cup A) - r_{\mathcal{M}}(A)$. We then apply Equation 7.1 to each term to get

$$r_{\mathcal{M}}(X \cup A) - r_{\mathcal{M}}(A) = r_{\mathcal{M}\setminus B}(X \cup A) - r_{\mathcal{M}\setminus B}(A),$$

which by Equation 7.2 is equal to $r_{(\mathcal{M}\setminus B)/A}(X)$. Putting this together, we have

$$\begin{aligned} r_{(\mathcal{M}/A)\setminus B}(X) &= r_{\mathcal{M}/A}(X) \\ &= r_{\mathcal{M}}(X \cup A) - r_{\mathcal{M}}(A) \\ &= r_{\mathcal{M}\setminus B}(X \cup A) - r_{\mathcal{M}\setminus B}(A) \\ &= r_{(\mathcal{M}\setminus B)/A}(X), \end{aligned}$$

therefore $(\mathcal{M}/A) \setminus B$ and $(\mathcal{M} \setminus B)/A$ have the same rank function, thus completing our proof for Equation 7.5. ■

Note that Lemma 7.12 shows that any sequence of deletions and contractions can be written as just one deletion and one contraction by combining all deletions into one deletion involving the union of the sets being deleted, and doing the same for the contractions.

We also state a property, according to [KP09, Chapter 2], of minors of matroids when it comes to their duals. Note that this follows from the fact that for disjoint subsets A, B of $E(\mathcal{M})$, we have $\mathcal{N} = \mathcal{M} \setminus A/B$ if and only if $\mathcal{N}^* = \mathcal{M}^*/X \setminus Y$.

Proposition 7.13. *Let $\mathcal{M}$ and $\mathcal{N}$ be two matroids. Then, $\mathcal{N}$ is a minor of $\mathcal{M}$ if and only if $\mathcal{N}^*$ is a minor of $\mathcal{M}^*$.*

7.3. **Direct sum of matroids.** We now define the union or direct sum of two matroids using the union of disjoint ground sets:

Proposition 7.14. *Let $\mathcal{M}_1 = (E_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{I}_2)$ be matroids, where E_1 and E_2 are disjoint. Then, define $\mathcal{I}$ to be the set of subsets A of $E_1 \cup E_2$ where $A \cap E_1$ and $A \cap E_2$ are independent in $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Then, $(E_1 \cup E_2, \mathcal{I})$ is a matroid and is called the* union *or* direct sum *of $\mathcal{M}_1$ and $\mathcal{M}_2$, denoted by $\mathcal{M}_1 \oplus \mathcal{M}_2$.*

We also present some properties of direct sums, characterizing their circuits, bases, and rank function.

Proposition 7.15. *Let $\mathcal{M}_1 = (E_1, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{I}_2)$ be matroids with disjoint ground sets, and let $\mathcal{B}(\mathcal{M})$ and $\mathcal{C}(\mathcal{M})$ denote the sets of bases and circuits of a matroid $\mathcal{M}$. Then, we have:*

1. *The set of bases of $\mathcal{M}_1 \oplus \mathcal{M}_2$ is the set of $B_1 \cup B_2$ for all possible $B_1 \in \mathcal{B}(\mathcal{M}_1)$ and $B_2 \in \mathcal{B}(\mathcal{M}_2)$.*
2. *The set of circuits of $\mathcal{M}_1 \oplus \mathcal{M}_2$ is the same as $\mathcal{C}(\mathcal{M}_1) \cup \mathcal{C}(\mathcal{M}_2)$.*
3. *For all subsets A of the ground set of $\mathcal{M}_1 \oplus \mathcal{M}_2$, we have*

$$r_{\mathcal{M}_1 \oplus \mathcal{M}_2} = r_{\mathcal{M}_1}(A \cup E_1) + r_{\mathcal{M}_2}(A \cup E_2).$$

Additionally, we state the following proposition which specifies how we can get the dual of the direct sum of two matroids. The proof is found in [KP09, Chapter 2].

Proposition 7.16. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be matroids with disjoint ground sets. Then,*

$$(\mathcal{M}_1 \oplus \mathcal{M}_2)^* = \mathcal{M}_1^* \oplus \mathcal{M}_2^*.$$

8. Representability of matroids

Definition 8.1. A matroid with n elements is $\mathbb{F}$-*representable* if each element of the matroid can be mapped to a column vector in a matrix A with n columns over the field $\mathbb{F}$ so that the column vectors corresponding to the elements in each independent set are linearly independent. The matrix A is called the $\mathbb{F}$-*representation* of the matroid. Additionally, a matroid is *representable* if there exists a field $\mathbb{F}$ such that the matroid is $\mathbb{F}$-representable.

Definition 8.2. A matroid is *binary* if it can be represented over the field $GF(2) = \mathbb{Z}/2\mathbb{Z}$ and *ternary* if it can be represented over the field $GF(3) = \mathbb{Z}/3\mathbb{Z}$.

Given this definition, consider the following example of the representability of a uniform matroid. The proof is also in [Oxl03].

Proposition 8.3. *The uniform matroid $U_{2,4}$ is binary but not ternary.*

Proof. We first prove that $U_{2,4}$ is not binary. Suppose for the sake of contradiction that it is representable over $GF(2)$. Then, the representation A has four columns and would become

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Notice that A does not have four nonzero columns, so there is a set of two vectors that is linearly dependent (specifically, the zero vector and any of the other columns are linearly dependent), giving us a contradiction since each column of A must be distinct. Thus, $U_{2,4}$ cannot be represented over $GF(2)$ and is thus not binary.

Additionally, $U_{2,4}$ is ternary because it can be represented by the matrix $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$. ∎

8.1. Representability of graphic matroids.

As usual, we also want to look into graphic matroids. First, let us define the vertex-edge incidence matrix of a graph, which allows us to connect graph theory and linear algebra.

Definition 8.4. Let G be a graph. The *vertex-edge incidence matrix* of G is the matrix that has rows labeled with the vertices of G and the columns labeled with the edges of G. If an edge e in G is a loop (that is, it connects a vertex to itself), then the column corresponding to e is the zero vector. Otherwise, the entry corresponding to vertex v and edge e of G is 1 if v is an end-vertex of e and 0 if it is not [Oxl03].

For us to get an understanding of how this is constructed, let us once again look at our graph H shown in Figure 2.

Example. Consider graph H shown in Figure 2. The vertex-edge incidence matrix would then be as follows:

$$A_H = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

The surprising fact about graphic matroids is that all of them are binary, and we can use the vertex-edge incidence matrix as described in Definition 8.4 to represent such graphic matroids. We show this is true based on [Oxl03].

Theorem 8.5. *Let G be a graph with vertex-edge incidence matrix A_G. Then, the vector matroid $M[A_G]$ viewed over $GF(2)$ has all subsets of E that do not contain the edges of a cycle in G as its independent sets. Then, $M[A_G] = M(G)$, and $M(G)$ is binary.*

Proof. Note that by the definition of representability as described in Definition 8.1, we only need to prove that a subset S of columns of A_G is linearly dependent if and only if S contains a set of edges of graph G that form a cycle.

Suppose that S contains the set of edges of a cycle C in graph G. If C is a loop, then the column corresponding to the sole edge of C is the zero vector, thus S is linearly dependent. Otherwise, each vertex in C is met by exactly two edges of C. This means that the sum of the columns of C is the zero vector when taken modulo 2, leading to S also being linearly dependent in this case. Therefore, if S contains the set of edges of a cycle C in G, then S is linearly dependent.

Now we prove the converse; that is, if S is a linearly dependent set of columns of A_G, then it also contains the set of edges of a cycle in G. Suppose that S is linearly dependent. Then, let $D \subseteq S$ be a circuit of $M[A_G]$ that does not contain the zero vector as a column. We then have that the sum of the columns of D taken modulo 2 is the zero vector, so every vertex that is an end-vertex of an edge of D is an end-vertex of at least two edges in D.

Let d_1 be an edge of D with end-vertices v_0 and v_1. Then, v_1 is also an end-vertex of another edge $d_2 \in D$, which has v_2 as is other end-vertex. We can use this idea to make a sequence $d_1, d_2, d_3, \dots$ of edges of D and another sequence $v_0, v_1, v_2, \dots$ of vertices that are met by these edges. Since we know that G is finite, we will eventually get a vertex v in the sequence that will repeat, and once this happens we get a cycle in D starting at v. This means that D contains the edges of a cycle in G, completing our proof. ■

8.2. **The Vámos matroid.** However, not all matroids are representable by a field. Consider the following example, as described in [FHJK23, KP09].

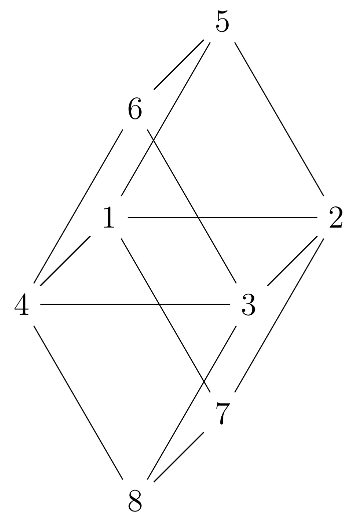

Figure 8. The Vámos matroid V_8.

Definition 8.6. Let $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and

$$A = \{\{1, 2, 3, 4\}, \{1, 4, 5, 6\}, \{1, 4, 7, 8\}, \{2, 3, 5, 6\}, \{2, 3, 7, 8\}\}.$$

Then, there exists a matroid $\mathcal{M}$ where all subsets of E with at most three elements are independent, and the five elements of A are the only circuits. This is called the *Vámos matroid*, depicted by Figure 8, and is denoted by V_8.

A surprising fact about this matroid is it is not representable over any field; we leave the proof to [KP09, Chapter 6].

Proposition 8.7. *The Vámos matroid V_8 is not representable over any field.*

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Philippine Science High School — Main Campus, Senator Miriam P. Defensor-Santiago Avenue, Diliman, Quezon City, Philippines

Email address: ktachan314@gmail.com