Elliptic Curves Over Finite Fields

Jonathan Yu

July 14, 2025

Introduction

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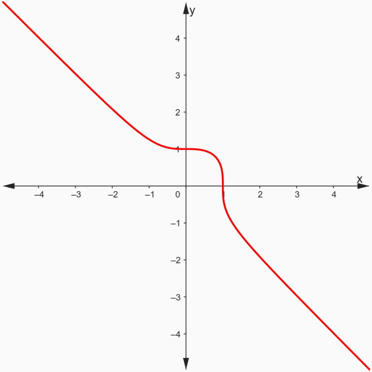

$x^3 + y^3 = 1$

Fermat's Last Theorem

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Theorem (Fermat's last theorem)

For any integer $n > 2$, there are no positive integer solutions to the equation

$$a^n + b^n = c^n.$$

Corollary

For any integer $n > 2$, there are no nonzero integer solutions to the equation

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if both x and y are nonzero.

- Only rational points on $x^3 + y^3 = 1$: $(0, 1), (1, 0)$

Preliminaries

Definition (Characteristic)

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  - Finite number of elements
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- **Frobenius endomorphism**

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- Set of $\mathbb{Q}$-rational points: $L(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2 : y - x = 0\}$

Definition (General Weierstrass Equation)

A **general Weierstrass equation** over a field K is

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where $a_1, a_2, a_3, a_4, a_6 \in K$.

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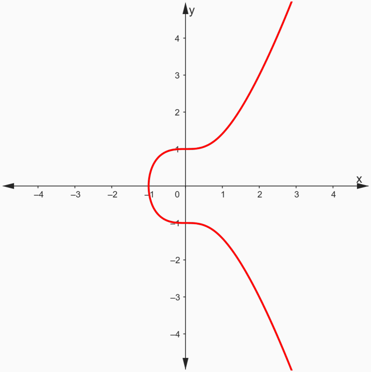

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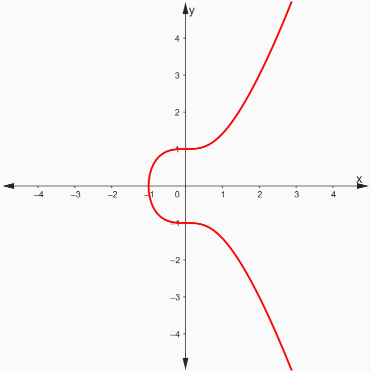

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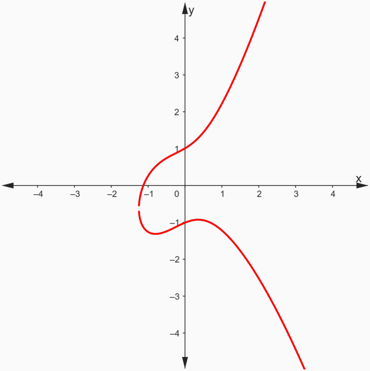

$y^2 - xy = x^3 + \frac{1}{2}x^2 + \frac{3}{16}x + \frac{65}{64}$

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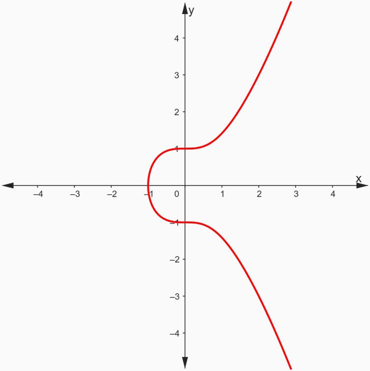

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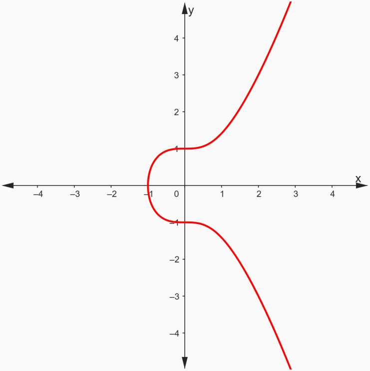

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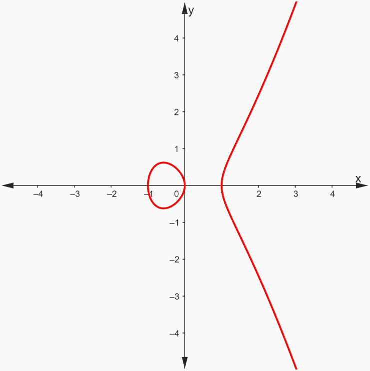

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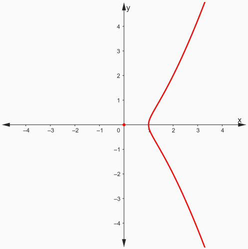

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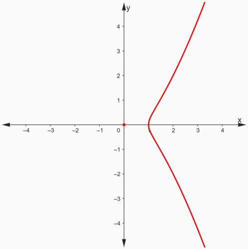

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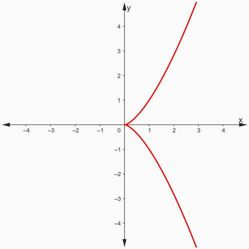

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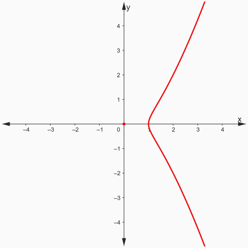

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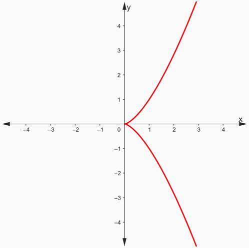

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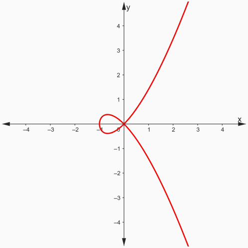

$y^2 = x^3 + x^2$

Short Weierstrass Equation

Proposition

If E is a general Weierstrass equation defined over a field K of characteristic not 2 or 3, then it can be written in the form

$$y^2 = x^3 + ax + b.$$

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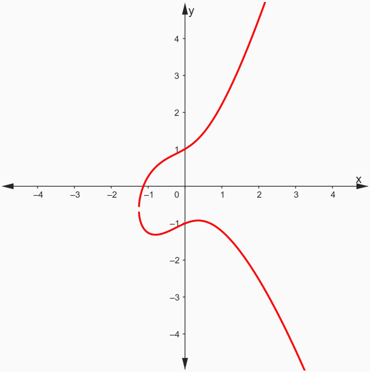

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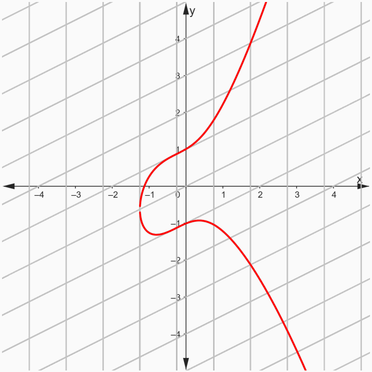

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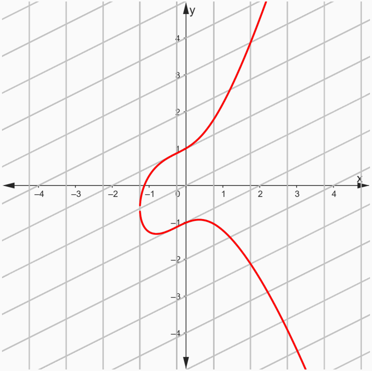

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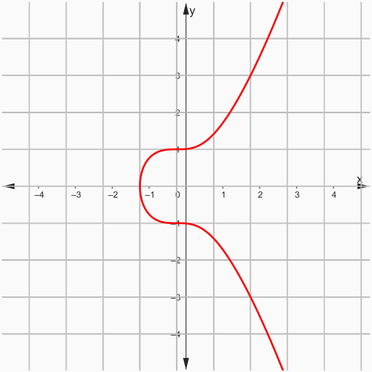

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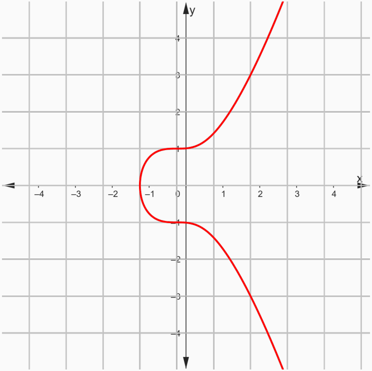

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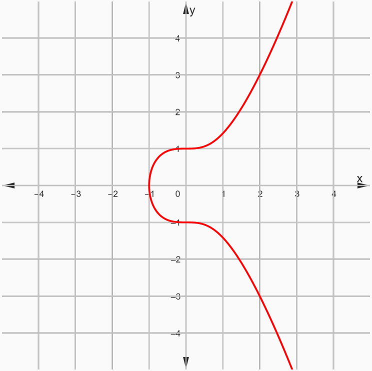

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The rational points on the rational line $ax + by + c = 0$ *are given by* $\left(t, -\frac{a}{b}t - \frac{c}{b}\right)$ *if* $b \neq 0$ *and* $\left(-\frac{c}{a}, t\right)$ *if* $b = 0$.

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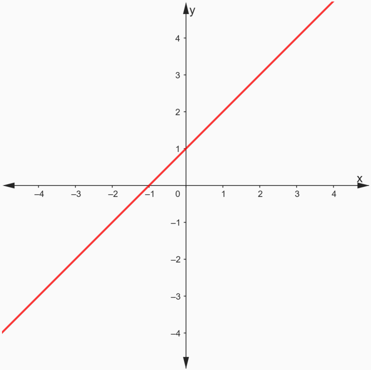

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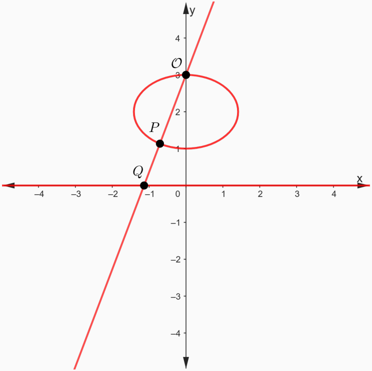

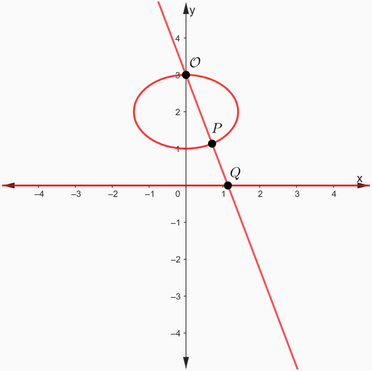

Rational Points on Singular Weierstrass Curves

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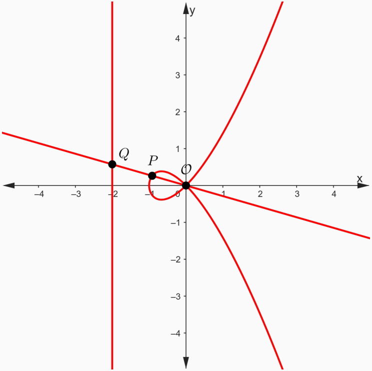

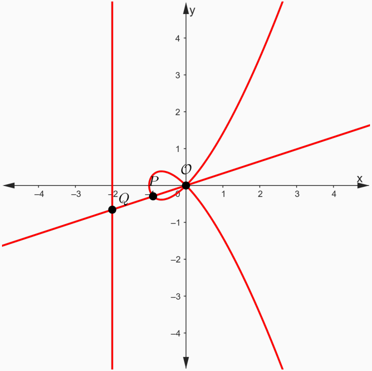

Geometry of Elliptic Curves

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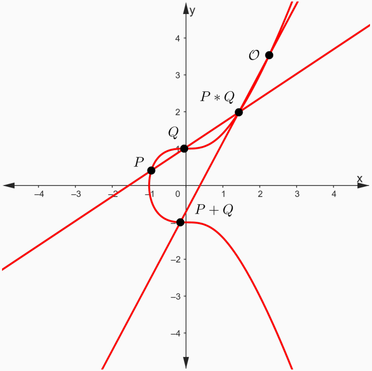

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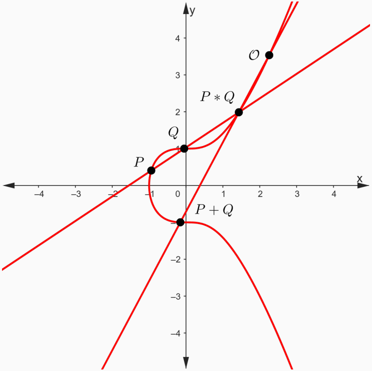

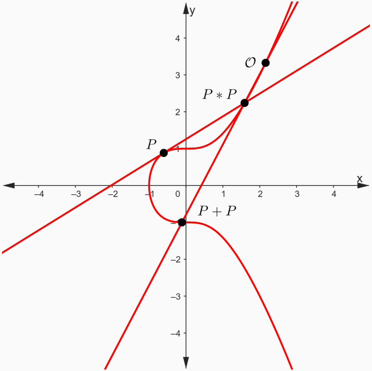

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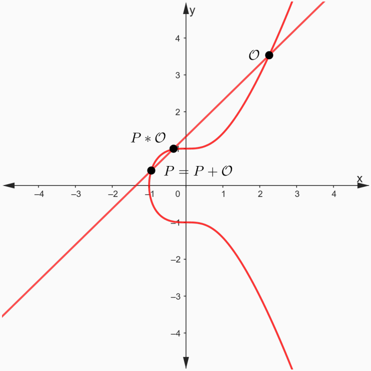

Identity Element

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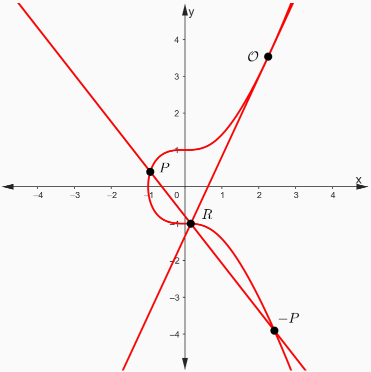

Inverse Element

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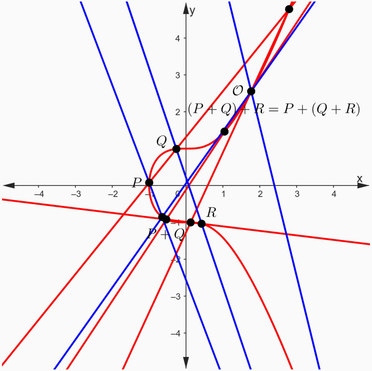

Associative Property

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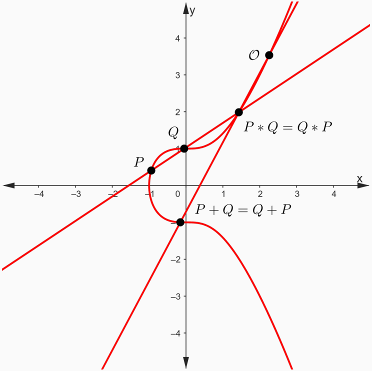

Commutative Property

Hasse’s Theorem

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Let E be an elliptic curve and $\mathbb{F}_q$ a finite field of order $q = p^n$ for some prime p and positive integer n. Then

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- The $+1$ is because of point at infinity

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x	$x^3 + x + 1 \pmod 5$	y	$\#$ of y
0	$0^3 + 0 + 1 = 1$	1, 4	2
1	$1^3 + 1 + 1 = 3$	none	0
2	$2^3 + 2 + 1 = 11 \equiv 1$	1, 4	2
3	$3^3 + 3 + 1 = 31 \equiv 1$	1, 4	2
4	$4^3 + 4 + 1 = 69 \equiv 4$	2, 3	2

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- $|9 - (5+1)| = 3 \leq 2\sqrt{5}$

Proof.

- Consider Frobenius endomorphism: $\varphi_q : E \to E$, where $\varphi_q(x, y) = (x^q, y^q)$ and $\varphi(\mathcal{O}) = \mathcal{O}$, where $\mathcal{O}$ is point at infinity

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- $|a| \leq 2\sqrt{q}$ $\square$

Riemann Hypothesis for Elliptic Curves

Theorem

Let $a = \#E(\mathbb{F}_q) - (q+1)$. *Let* α *and* β *be the roots of the characteristic polynomial* $T^2 - aT + q = 0$. *Then* $|\alpha| = |\beta| = \sqrt{q}$.

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