Semantic and Syntactic Constructions of hyperreal numbers

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Goal of Nonstandard Analysis

The objective of Nonstandard Analysis is to extend the first order logic of the real number system, with the inclusion of infinitesimal and infinite number. There are various method to this construction, but first we will define infinitesimal and infinite number.

Infinitesimal and infinite number

Infinitesimal: ‘Number’ that is smaller then each positive real number and is larger than each negative real number. We will only consider nonzero infinitesimal

Infinite Number: ’Number’ that is bigger than any real number. We are not referring to ∞.

Infinitesimal and infinite number cannot be included into the real number system due to the Archimedean property.

Syntactic and Semantic Constructions

Syntactic: Axiomatic system with given lagnuage: symbols, rules, axioms. We don't know what the language means in a universe yet.

Semantic: The meaning or interpretation of the syntax. How symbols and axioms are interpreted in a mathematical structure.

To construct the number system satisfying previously mentioned conditions, we can either use the Ultrapower construction, or Internal Set Theory(IST).

Ultrapower construction is a semantic construction that is built upon the syntax of ZFC set theory.

IST is it's own syntactic construction that is an extension of ZFC. IST is powerful enough to build the hyperrreal numbers with syntax alone.

Ultrapower Construction

The ultrapower construction uses Cauchy's definition of inifnitesimal and infinite number. Which are $\langle r_n \rangle$ such that,

$$\lim_{n\to\infty} r_n = 0.$$

And an infinite number as one satisfying

$$\lim_{n\to\infty} r_n = \infty.$$

Infinitesimal and Infinity is an inifnite sequence that satisfies the limits above.

To align with that perspective we will now define each hyperreal number as a sequence.

We consider the set of all infinite sequences of $\mathbb{R}$, $\mathbb{R}^{\mathbb{N}} = \{(r_0, r_1, r_2, \ldots) : r_i \in \mathbb{R}, i \in \mathbb{N}\}$. $\mathbb{R}^{\mathbb{N}}$ is constructed as a direct power of $\mathbb{R}$, that is

$$\mathbb{R}^{\mathbb{N}} = \prod_{n \in \mathbb{N}} \mathbb{R}.$$

Any infinite sequence of $\mathbb{R}$ belongs to $\mathbb{R}^{\mathbb{N}}$.

Filters and Ultrafilters

A filter $\mathcal{F}$ on a nonempty set I (e.g. $\mathbb{N}$) is a nonempty collection of subsets of I satisfying:

- If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$ (closed under intersection)
- If $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$ (upward closed)
- $\emptyset \notin \mathcal{F}$

An ultrafilter $\mathcal{U}$ is a maximal filter: for every subset $A \subseteq I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, but not both.

- A nonprincipal ultrafilter on $\mathbb{N}$ contains all cofinite sets and no finite sets.

Ultrafilters are crucial for collapsing sequences into equivalence classes.

Equivalence Classes of Sequences

Given a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, define an equivalence relation $\sim$ on $\mathbb{R}^{\mathbb{N}}$ by:

$$(r_n) \sim (s_n) \iff \{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{U}$$

That is, two sequences are equivalent if they agree on a set in $\mathcal{U}$.

Each equivalence class represents a single hyperreal number:

$[(r_n)] =$ the class of all sequences equal to (r_n) almost everywhere (w.r.t. $\mathcal{U}$)

Ultrapower Quotient Ring Construction

The set of equivalence classes under ($\sim$), denoted:

$$^*\mathbb{R} := \mathbb{R}^{\mathbb{N}}/\mathcal{U}$$

forms the hyperreal number system.

Addition, multiplication, and order are defined pointwise:

$$[(r_n)] + [(s_n)] := [(r_n + s_n)], \quad [(r_n)] \cdot [(s_n)] := [(r_n s_n)]$$

$$[(r_n)] < [(s_n)] \iff \{n \in \mathbb{N} : r_n < s_n\} \in \mathcal{U}$$

The field $^*\mathbb{R}$ extends $\mathbb{R}$, contains infinitesimals and infinite numbers, and satisfies the Transfer Principle via Łoś's Theorem.

Internal Set Theory

Internal Set Theory

Internal Set Theory(IST) is an extension of ZFC, which conserves the axioms and rules of ZFC with some addition.

We introduce a new predicate, *Standard*. Since it is a syntactic symbol we do not have any meaning yet. Therefore, we will define it through axioms later.

Standard is not defined in ZFC. Therefore, we should obey a few rules to avoid illegal sets.

- The predicate *Standard* cannot be used inside a set, specifically cannot be used to define a set.
- *Standard* is permitted to use anywhere else than that, as a formula, or outside a set, etc

Internal and External Formulas

An internal formula only includes the language of ZFC, specifically do not use or mention the predicate *standard.*

$$\{x|x < 83\}$$

An external formula that uses a language including *standard*.

$$\exists x(standard(x) \wedge x > 57)$$

Abbr.

$$\forall^{\text{st}} := \forall x(standard(x))$$

$$\exists^{\text{st}} := \exists x(standard(x))$$

Axioms

The Axioms of Internal Set Theory adds 3 additional axioms to the traditional ZFC Axioms to maintain conservatism of ZFC.

1. Axiom of Transfer Principle:

$$\forall^{\mathrm{st}} x\, \varphi(x) \implies \forall x\, \varphi(x),$$

   where φ is an internal formula.

2. Axiom of Idealization

$$\forall^{\mathrm{st}} A \subseteq_{\mathrm{fin}} X\, \exists x\, \forall a \in A\, \varphi(x,a) \iff \exists x\, \forall^{\mathrm{st}} a \in X\, \varphi(x,a),$$

   where φ is an internal formula, X is a standard set, and the quantification $\forall^{\mathrm{st}} A \subseteq_{\mathrm{fin}} X$ ranges over all standard finite subsets A of X.

3. Axiom of Standardization

$$\forall X\, \exists^{\mathrm{st}} Y\, \forall^{\mathrm{st}} x\, \big(x \in Y \iff (x \in X \wedge \varphi(x))\big),$$

   where φ is an internal formula.

Generating Infinitesimal

We will use the Axiom of Idealization. An internal formula φ is defined as,

$$\varphi(x, r) := 0 < x < r,\ \ r \in \mathbb{R}^+.$$

Then applying that to the axiom,

$$\forall^{\text{st}} A \subseteq_{fin} \mathbb{R}^+, \exists x \forall r \in A(0 < x < r) \iff \exists x \forall^{\text{st}} r > 0(0 < x < r),$$

which results,

$$x := \varepsilon.$$

Generating Infinite Number

If we define $H := \frac{1}{\varepsilon}$, then, for every $n \in \mathbb{N}$, since $0 < \varepsilon < \frac{1}{n}$,

$$n < \frac{1}{\varepsilon} = H$$

Therefore,

$$\forall^{\text{st}} n \in \mathbb{N}\ \exists H, n < H.$$

Both shows that we can build infinitesimal and infinite number, also extend the first order logic of $\mathbb{R}$ using the Axiom of Transfer Principle, from the syntactic construction, IST.

Connecting Semantics and Syntax

Semantic Ultrapower Models Interpret IST

Although Internal Set Theory (IST) is syntactic, it can be modeled semantically using an ultrapower construction.

In such a model:

- ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathcal{U}$
- The set of standard reals corresponds to constant sequences $[(r, r, r, \dots)]$
- The predicate `standard(x)` is interpreted as:

$$\text{“}\exists r \in \mathbb{R} \text{ such that } x = [(r, r, r, \dots)]\text{”}$$

This shows that the axioms of IST are sound in the ultrapower model.

From Semantic Construction to Syntactic Theory

- The ultrapower ${}^*\mathbb{R}$ gives a model where infinitesimals exist and the Transfer Principle holds.
- IST provides an axiomatic, syntactic framework that can replicate the same reasoning within ZFC + a new predicate.
- IST does not require ultrafilters or models — but its consistency is proven using them.
- IST is a syntactic shadow of what semantic ultrapowers construct.

Thus, both approaches ultimately describe the same hyperreal world, however, different construction.