SEMANTIC AND SYNTACTIC CONSTRUCTIONS OF THE HYPERREAL NUMBERS

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Abstract. The construction of the hyperreal number system is discussed in this paper. We explore both semantic and syntactic construction which are two independent construction method that creates the same system. We will analyze each construction and compare and contrast both and combine the two.

1. Introduction

The Archimedean Property states that *An ordered field F has the Archimedean Property if, given any positive x and y in F there is an integer $n > 0$ so that $nx > y$.* The set of real numbers follows the Archimedean property due to it's Least Upper Bound property. Infinitesimal is a value that is bigger than zero and smaller than any real number, similarly, infinite number is a value that is bigger than any real number. This doesn't mean that the infinite number is equivalent to the conept of infinity(∞). Since the Archimedean property asserts that repetitive addition of x will be greater than y at some point, regardless of the value of x and y, infinitesimal and ifnite number cannot be defined in the real number system without violating the Archimedean property, since the Archimedean property is strictly against the nature of infinitesimals and infinite values. However, historically, the idea of infinitesimal have been used within the real number system. When calculus was initially constructed by Leibniz and Newton, it was based on informal definition of infinitesimal. The use of infinitesimal was intuitively appealing, however it lacked rigor and proper construction. Later, the use of infinitesimal was replaced with δ-ε method, which formalized the use of infinitesimals with limits.

In 1960, Abraham Robinson created Nonstandard Analysis. The main objective of the construction of hyperreal numbers are to *Preserve the first order logic of $\mathbb{R}$ and include infinitesimal and infinite number into the universe.* There are two construction methods to satisfy this. One is the most well known Robinson's Ultrapower construction, and the other is later found Edward Nelson's Internal Set Theory. Both are fundamentally different from the ground, which each of them is a semantic and syntactic construction, respectively. A syntactic construction is building the language. However, we don't know what the meaning of the language is yet. We build symbols, axioms and rules, but the interpretation depends on the universe that takes the syntax to construct a model. A semantic construction is essentially building a model using a preexisting language to interpret what truth will mean inside the structure. We are normally used to syntactic construction to not be constructing a full system independently. One example of syntactic construction is ZFC set theory. We usually use ZFC as a base of a model construction, not itself. However, syntactic construction is not necessarily always weak. Syntactic construction can provide philosophical clarity and generality better. In this case, we will construct same hyperreal number system using

Date: July 14, 2025.

each syntactic and semantic construction. The initial difference of having different ways of reasoning will lead to bigger difference between the two, such as expressiveness and logical control.

Ultrapower constructions build infinitesimals by constructing a model from sequences and logic — a semantic approach rooted in model theory. Internal Set Theory, by contrast, rewrites the rules of reasoning, treating infinitesimals as legitimate objects governed by new axioms. One constructs, the other permits. One explains what the hyperreals are; the other defines how we may reason about them. Both systems are powerful — and their comparison reveals that mathematical truth can emerge from either objects or inference rules.

2. Ultrapower Universe Construction

The Ultrapower Construction is a semantic construction that lies on top of ZFC. We will prove the existence of infinitesimal and infinite number in the universe, and transfer of the elementary rules of Real number system using model theory.

2.1. Introduction to hyperreal numbers.

The objective of this construction is to build an extension of the real numbers that includes hyperreal numbers, such as infinitesimals and infinite quantities. Cauchy defined infinitesimal as a sequence $\langle r_n \rangle$ such that

$$\lim_{n\to\infty} r_n = 0$$

And an infinite number as one satisfying

$$\lim_{n\to\infty} r_n = \infty.$$

We consider the set of all infinite sequences of $\mathbb{R}$, $\mathbb{R}^{\mathbb{N}} = \{(r_0, r_1, r_2, ...) : r_i \in \mathbb{R}, i \in \mathbb{N}\}$. $\mathbb{R}^{\mathbb{N}}$ is constructed as a direct power of $\mathbb{R}$, that is

$$\mathbb{R}^{\mathbb{N}} = \prod_{n\in\mathbb{N}} \mathbb{R}$$

In the hyperreal number system, elementary objects such as hyperreal numbers and further to sets and more are all represented using sequences. The reason behind this construction is dependent on Cauchy's definition of infinitesimal and infinite number using sequences. We will formalize hyperreal number, and to do so, we will define relations between two sequences.

Definition 2.1. Given two sequences $r = \langle r_1, r_2, r_3, ... \rangle$ and $s = \langle s_1, s_2, s_3, ... \rangle$ that $s, r \in \mathbb{R}^{\mathbb{N}}$, we define their agreement set as

$$E_{rs} = \{n : r_n = s_n\}.$$

The agreement set contains the indices where two sequence have the same term.

Intuitively, if two sequences agree on 'almost everywhere', they should represent the same number of hyperreals. If we look at two sequences, such as $r = \langle 1, 1, 1, 1, 0, 1, 1, \ldots \rangle$ and $s = \langle 1, 1, 1, 1, 1, 1, 1, \ldots \rangle$, the agreement set is $\mathbb{N} \backslash \{5\}$. When a set contains all elements of an infinite set except a finite amount of number(s) we will call it a cofinite set. In this case we contain all, except one element. If a set is cofinite in $\mathbb{N}$, then it is large, because it includes almost all elements, leaving out only finitely many. Whenever an agreement set is large, two sequences are *equivalent*, that they are treated as a same hyperreal number, since they are the same 'almost everywhere'.

The concept of largeness introduced is an informal and for the purpose of intuitive appeal. We will formalize the concept of largeness with filters and further constructions, which a set if large if and only if a set belongs to a filter.

2.2. **Filters.** Filters are essential concepts in building the logic and determining truth value. Filters are sets, that picks out element in a set that is satisfying conditions from the set we are taking the filter of.

Definition 2.2. Let I be a non-empty set. Let the power set of I be $\mathcal{P}(I)$ of all subsets of I. A filter $\mathcal{F}$ on I is a nonempty collection $\mathcal{F} \subseteq \mathcal{P}(I)$ of subsets of I satisfying following axioms:

- Intersections: if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$
- Supersets: if $A \subseteq B \subseteq I$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$

We will use the intersection axiom listed above to build up the finite intersection property.

Definition 2.3. A colleciton $\mathcal{H} \subseteq \mathcal{P}(I)$ has the *finite intersection property* if,

$$B_1 \cap \cdots \cap B_n \neq \emptyset$$

for any n and any $B_1, \ldots, B_n \in \mathcal{H}$.

The filter we will focus on is *Nonprincipal Ultrafilter.*

Definition 2.4. If a filter includes the set itself or the complementary of the set for any subset of I, the filter satisfies maximility since a filter cannot be bigger if one satisfies the property. A filter that satisfies maximility is called an ultrafilter.

The earlier definition of hyperreal numbers is based on the direct power of $\mathbb{R}$, which generates the existence of all infinite sequences without explicitly constructing every individual one of them. To extend the concept of proving existence without construction, we will prove theorems based on Zorn's Lemma, which bases on the idea of axiom of choice.

Lemma 2.5 (Zorn's Lemma)**.** *If $(P, \leq)$ is a partially ordered set in which every linearly ordered subset has an upper bound in P, then P contains a $\leq$-maximal element.*

This asserts that there must exist at least one set with the biggest cardinality in any collection of set. Using the fact that ultra filters is a the biggest filter (maximility), we can use Zorn's lemma to prove the existence of an ultra filter in any collection of set.

Theorem 2.6. *Any collection of subsets of I that has the finite intersection property can be extended to an ultrafilter on I.*

Proof. If $\mathcal{H}$ has the fip then the filter $\mathcal{F}^{\mathcal{H}}$ generated by $\mathcal{F}$ is proper. Let P be the collection of all proper filters on I that includes $\mathcal{F}^{\mathcal{H}}$, partially ordered by set inclusion $\subseteq$. Then every linearly ordered subset of P has an upper bound in P, since the union of this chain is in P. Hence by Zorn's lemma, P has a maximal element, which is thereby a maximal proper filter on I and thus an ultrafilter.

Corollary 2.7. *Any infinite set has a non principal ultrafilter on it.*

proof. If I is infinite, the cofinite filter $\mathcal{F}^{co}$ is proper and has the finite intersecton property, and so is included in an ultrafilter $\mathcal{F}$. But for any $i \in I$ we have $I - \{i\} \in \mathcal{F}^{co} \subseteq \mathcal{F}$, so $\{i\} \in \mathcal{F}^{i}$. Hence $\mathcal{F} \neq \mathcal{F}^{i}$. Thus $\mathcal{F}$ is nonprincipal.

2.3. **The Ultrapower Construction.** Filters will help us to be able to formally classify statements as true or false. As how we introduced the agreement set, two sequences are equal if the agreement set is large. We will take that idea, and the belonging to the filter will be our anchor to measure truth.

Definition 2.8. Let $\mathcal{F}$ be a nonprincipal ultrafilter on the set $\mathbb{N}$. The equivalence relation $\sim_{\mathcal{U}}$ on $\mathbb{R}^{\mathbb{N}}$ is defined by

$$\langle r_n \rangle \sim_{\mathcal{U}} \langle s_n \rangle \quad \text{iff} \quad \{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{F}$$

When this relation holds it may be said that the two sequences agree almost everywhere modulo $\mathcal{F}$, or agree at almost all n.

Remark 2.9. We will denote the agreement set $\{n \in \mathbb{N} : r_n = s_n\}$ as a logical statement $[\![r = s]\!]$. Which,

$$r \sim_{\mathcal{U}} s \quad \text{iff} \quad [\![r = s]\!] \in \mathcal{F}$$

Similarly, we can apply this to other relations, such as inequalities by,

(1) $[\![r < s]\!] = \{n \in \mathbb{N} : r_n < s_n\}$

(2) $[\![r \leq s]\!] = \{n \in \mathbb{N} : r_n < s_n \vee r_n = s_n\}$

Now we will say if a set belongs to the filter, we will say the statement the set contains is true almost everywhere. In standard systems, statement usually have a binary truth values. However, in this case we will measure each statement, that is a subset of $\mathbb{N}$, to be true if it is including 'almost every' elements of $\mathbb{N}$, false if not.

Definition 2.10. The equivalence class of a sequence $r \in \mathbb{R}^{\mathbb{N}}$ under $\sim_{\mathcal{U}}$ will be denoted by $[r]$.

$$[r] = \{s \in \mathbb{R}^{\mathbb{N}} : r \sim_{\mathcal{U}} s\}$$

Each equivalence class represent a hyperreal number. We will represent the number system by combining all equivalence classes in one set. An ultrapower is a quotient of a direct power from a defined equivalence relation by an ultrafilter.

Definition 2.11. The ultrapower construction of $\mathbb{R}$ by previously defined equivalence relation $\sim_{\mathcal{U}}$ is

$$\mathbb{R}^{\mathbb{N}}/\mathcal{U} = {}^*\mathbb{R} = \{[r] : r \in \mathbb{R}^{\mathbb{N}}\}$$

$\mathbb{R}^{\mathbb{N}}/\mathcal{U}$ is a quotient ring of $\mathbb{R}^{\mathbb{N}}$ by the ultrafilter. This circles back to the equivalence class mentioned before, that all sequence $r \in \mathbb{R}^{\mathbb{N}}$ will be classified to one of the equivalence class. Which follows:

(1) $[r] + [s] = [r \oplus s] = [\langle r_n + s_n \rangle]$

(2) $[r] \cdot [s] = [r \odot s] = [\langle r_n \cdot s_n \rangle]$

(3) $[r] < [s] \quad \text{iff} \quad \{n \in \mathbb{N} : r_n < s_n\} \in \mathcal{F}$

With those definition we defined an $\mathcal{L}$-structure, which contains relation $(<)$ and functions $(\cdot, +)$ and the universe ${}^*\mathbb{R}$.

$$\mathcal{L}\text{-structure } = \langle {}^*\mathbb{R}, <, \cdot, + \rangle.$$

Theorem 2.12. *The $\mathcal{L}$-structure ${}^*\mathcal{R}$ is an ordered field with the constant 0 and 1.*

(1) ${}^*\mathbb{R}$ contains a multiplicative and additive inverse.

- *Additive inverse: $-[r] = [-r]$, therefore, $[-r] + [r] = [0]$*

- *Multiplicative inverse: Suppose that $r \not\sim_{\mathcal{U}} 0$, which $\{n \in \mathbb{N} : r_n \neq 0\}$. We define a sequence s_n that:*

$$s_n = \begin{cases} \textit{If } r \sim_{\mathcal{U}} 0, s_n \textit{ does since 0 does not have a multiplicative inverse.} \\ \textit{If } r \not\sim_{\mathcal{U}} 0, [s_n]^{-1} = [s_n^{-1}]. \end{cases}$$

Even if the sequence is not equivalent to 0, there is a possibility that there exists a term $r_n = 0, n \in \mathbb{N}$. When this occur, taking the inverse will lead us to be having an undefined value in our sequences. This might look like an issue, however, the fact that almost every term is a real number resolves this issue. Therefore,

$$[r] \cdot [s] = [r \odot s] = [1]$$

(2) Ordering on ${}^*\mathbb{R}$ is a disjoint union of following sets:

$$[\![r = s]\!], \quad [\![r < s]\!], \quad [\![r > s]\!],$$

Due to finite intersection property.

(3) If $\{[r] : [0] < [r]\}$, that the sequences in the class $[r]$ are positive almost everywhere, ${}^*\mathbb{R}$ is closed under addition and multiplication.

Proof. Let $[r], [s] \in \mathcal{F}$. majority of the terms of each sequences r_n and s_n are r and s respectively. Similarly, $[r] + [s]$ will have terms that are $r + s$ almost everywhere, due to the axiom of intersections 2.2. Thus,

$$\textit{If } [r], [s] \in \mathcal{F}, \textit{ then } [r] + [s] \in \mathcal{F}.$$

Which means $[r] + [s]$ will form its' own equivalence class $\{t \in \mathbb{R}^{\mathbb{N}} : r + s \sim_{\mathcal{U}} t\} = [r \oplus s]$. Therefore,

$$[r \oplus s] \in {}^*\mathbb{R}.$$

We can define similarly with multiplication with the same process which will result $[r] \cdot [s] \in \mathcal{F}$, that is,

$$[r \odot s] \in {}^*\mathbb{R}.$$

Therefore, ${}^*\mathbb{R}$ is closed under addition and multiplication.

2.4. **Construction of hyperreal numbers.** As mentioned earlier, hyperreal numbers are an extension of Real numbers. The previous methods of the use of sequences to build relations up to defining a structure, is to build a system that includes all infinitesimal infinite value and real numbers. We will define real numbers using the a constant sequence.

Definition 2.13. A real number $r \in \mathbb{R}$ is defined as a constant sequence $r = \langle r, r, \ldots \rangle$ in ${}^*\mathbb{R}$. Which,

$$^*r = [r] = [\langle r, r, \ldots \rangle], \ ^*r \in {}^*\mathbb{R}$$

Converting previous statements to this definition, we have,

$$\begin{aligned} ^*(r+s) &= {}^*r + {}^*s, \\ ^*(r \cdot s) &= {}^*r \cdot {}^*s, \\ ^*r < {}^*s &\text{ iff } r < s, \\ ^*r = {}^*s &\text{ iff } r = s. \end{aligned}$$

Definition 2.14. Let ε be an infinitesimal, $\varepsilon = \langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle = \langle \frac{1}{n} : n \in \mathbb{N} \rangle$. Then,

$$[\![0 < \varepsilon]\!] = \left\{ n \in \mathbb{N} : \frac{1}{n} < r \right\}$$

thus $[0] < [\varepsilon] \in {}^*\mathbb{R}$. To satisfy the quality of being smaller than any real number,

$$[\![\varepsilon < r]\!] = \left\{ n \in \mathbb{N} : \frac{1}{n} < r \right\}.$$

There are only finite amount of $n \in \mathbb{N}$ that is excluded in this set. Therefore, this set is cofinite. Since non principal filters are a collection of all cofinite sets, $[\varepsilon] \in {}^*\mathbb{R}$. $[\varepsilon]$ *is a positive infinitesimal.*

Definition 2.15. Let $[\omega] = \langle 1, 2, 3, \ldots \rangle$. Then,

$$[\![r < \omega]\!] = \{ n \in \mathbb{N} : r < n \}$$

This set is cofinite by Eudoxus-Archimedes principal ($\forall r \in \mathbb{R}, \ \exists n$ s.t. $r < n$). There are finite amount of r that $r > n$. Thus, $[\omega] \in {}^*\mathbb{R}, \ [\omega]$ is a positive infinite value.

As previously mentioned, all non-zero hyper real numbers will have multiplicative inverse. Using infinite value and infinitesimal,

$$\omega \cdot \varepsilon = 1, \quad [\omega] = [\varepsilon]^{-1}, \quad [\varepsilon] = [\omega]^{-1}.$$

By 2.13, $[\![\omega = r]\!] \notin \mathcal{F}$, since the set $\{n \in \mathbb{N} : \omega_n = r_n\}$ can only be either $\emptyset$ or $\{r\}$ if $r \in \mathbb{N}$. Which $^*r \neq [\omega], \ [\omega] \in {}^*\mathbb{R} - \mathbb{R}$. Same as $[\varepsilon], \{n \in \mathbb{N} : \varepsilon_n = r_n\}$ can only either be $\emptyset$ or $\{r\}, \ r = \frac{1}{n}, \ n \in \mathbb{N}$.
Our construction of infinitesimal and infinite number satisfies Cauchy's definition of infinite number and infinitesimal(2.1),

$$\lim_{n \to \infty} [\omega] = \infty, \quad \lim_{n \to \infty} [\varepsilon] = 0.$$

2.5. **Enlargements of Sets.** Set $A \subseteq \mathbb{R}$ can be enlarged to a set $^*A \subseteq {}^*\mathbb{R}$:
For each sequence $r \in \mathbb{R}^{\mathbb{N}}$, which

$$[r] \in {}^*A \text{ iff } \{n \in \mathbb{N} : r_n \in A\} \in \mathcal{F}.$$

Which means, the enlarged set $^*A$ includes all sequences that are same almost everywhere from the constant sequence of r, for all $r \in A$. This directly implies that it will include all the equivalence class of $\langle r \rangle$.

Remark 2.16. $[r_n] = [r]$, it is used in the purpose to show the indices of the sequence. For example, when we want to show $r \sim_{\mathcal{U}} s$, then if we use $[\![r = s]\!] \in \mathcal{F}$, would be $[r]$, and when we use $\{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{F}$, would be $[r_n]$.

2.5.1. *Nonstandard Elements.* When the set A is enlarged to ${}^*A$, then, $A \subseteq {}^*A$. There exists nonstandard element ${}^*A - A$. For example, $\mathbb{Q} \in \mathbb{R}$ and $\{n \in \mathbb{N} : [\varepsilon] \in \mathbb{Q}\} = \mathbb{N} \in \mathcal{F}$, since the terms of $\varepsilon = \langle \frac{1}{n} : n \in \mathbb{N} \rangle$ are all rational number. Therefore we can denote $[\varepsilon] \in {}^*\mathbb{Q}$. $[\varepsilon]$ is a nonstandard element of rational numbers.

The enlargements of sets are further expanded in section 4.

2.6. **Extension of Functions.** A function $f : \mathbb{R} \to \mathbb{R}$ extends to ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$. We will define the function ${}^*f$, the extension of f.

Definition 2.17. For each sequence $r \in \mathbb{R}^{\mathbb{N}}$, let $f \circ r$ be $\langle f(r_1), f(r_2), \ldots \rangle$. Which we can denote,

$${}^*f([r]) = [f \circ r].$$

Which expands to

$${}^*f([\langle r_1, r_2, \ldots \rangle]) = [\langle f(r_1), f(r_2), \ldots \rangle]$$

The super set axiom(2.2) asserts that the function of a hyperreal number must return a hyperreal number too, since,

$$[\![r = r']\!] \subseteq [\![f \circ r = f \circ r']\!].$$

Therefore,

$${}^*f({}^*r) = {}^*s \text{ iff } [\![f \circ r = s]\!] \in \mathcal{F}.$$

2.6.1. *Partial Functions.* If a function $f : A \to \mathbb{R}$ has a domain restriction which lead to the function's domain be $A \subset \mathbb{R}$. Then f can be extended to a function ${}^*f : {}^*A \to {}^*\mathbb{R}$, which the domain is the enlargement of A(2.5).

Definition 2.18. The extension of a partial function f to ${}^*f$, that the seqnence $r \in \mathbb{R}^{\mathbb{N}}$ and $[r] \in {}^*A$,

$$[\![r \in A]\!] = \{n \in \mathbb{N} : r_n \in A\} \in \mathcal{F}.$$

Let

$$s_n = \begin{cases} f(r_n) & \text{if } n \in [\![r \in A]\!], \\ 0 & \text{if } n \notin [\![r \in A]\!] \end{cases}$$

If $[\![s_n = f(r_n)]\!] \in \mathcal{F}$, then,

$${}^*f([r_n]) = [f(r_n)].$$

2.7. **Enlargement of Relations.**

Definition 2.19. Let P be a k-ary relation on $\mathbb{R}$. Thus, $P \subseteq \mathbb{R}^k$. For given sequences $r^1, \ldots, r^k \in \mathbb{R}^{\mathbb{N}}$,

$$[\![P(r^1, \ldots, r^k)]\!] = \{n \in \mathbb{N} : P(r^1_n, \ldots, r^k_n)\}.$$

Which is a set that includes the indices of terms that is true under the relation.

To expand the relation P to ${}^*P \in {}^*\mathbb{R}$ with $[r^1], \ldots, [r^k] \in ({}^*\mathbb{R})^k$,

$$ {}^*P([r_1], \ldots, [r^k]) \quad \text{iff} \quad [\![P(r^1, \ldots, r^k)]\!] \in \mathcal{F}. $$

This definition is previously informally used, such as,

$$ [r] = [s] \quad \text{iff} \quad [\![r = s]\!] \in \mathcal{F}, $$

$$ [r] < [s] \quad \text{iff} \quad [\![r < s]\!] \in \mathcal{F}, $$

Which essentially have the same construction. We can see ${}^*P$ as a generalization expanding to all relations defined in $\mathbb{R}$.

We have defined the extended version of functions and relation that belongs to ${}^*\mathbb{R}$, which allows us to define the structure and further analyze the logic of ${}^*\mathbb{R}$.

3. Transfer principle and logic of model

The previous section, we have constructed the universe of the hyperreal number, ${}^*\mathbb{R}$. We have formalized the inclusion infinitesimal and infinity into the universe. We will now build the logic of the system using model theory, and show that the hyperreal will extend the first order logic, the elementary rules of the real number system.

3.1. **Model Theoretical Definition of Hyperreals.** We will now rigorously define the constructions from eariler using model theory.

Let $\mathcal{L} = \{+, \cdot, 0, 1, <\}$ be defined as the language of ordered fields. The $\mathcal{L}$ structure $\mathfrak{R} = \{\mathbb{R}; +, \cdot, 0, 1, <\}$.

Definition 3.1. A language $\mathcal{L}$ is given by specifying the following data:

(1) A set of function symbols $\mathscr{F}$ and positive integers m_f for each k_f-ary function $f \in \mathscr{F}$;
(2) A set of relation symbols $\mathscr{P}$ and postivie integers k_R for each k_P-ary relation $P \in \mathscr{P}$;
(3) A set of constant symbols $\mathscr{C}$.

The previous definition is always given by the syntactic construction we are using, which is ZFC at this case. Therefore, the langauge in our new model ${}^*\mathfrak{R}$ maintains the same language as $\mathfrak{R}$. Which $\mathcal{L} = \{+, \cdot, 0, 1, <\}$, and $k_+ = 2, k_\cdot = 2,$ and $k_< = 2$.

Now we can interpret the language into each models. Before interpretation, we have the language without meanings, those are syntaxes. Interpreted symbol now include truth values, precise definition of each functions, relation, and constants. We will be giving definition to each language as an interpretation process, which will be called semantics. The concept of syntax and semantic will later be expanded into comparison of structures.

Definition 3.2. An $\mathcal{L}$-structure $\mathfrak{M}$ is given by the following:

(1) A nonempty set M is called the universe or domain of $\mathfrak{M}$;
In ${}^*\mathfrak{R}$, the universe is ${}^*\mathbb{R}$.

(2) A function interpretation of $\mathfrak{M}$ is $f^{\mathfrak{M}} : M^{k_f} \to M$ for each $f \in \mathscr{F}$;
In $^*\mathfrak{R}$*, functions act componentwise on sequences.*
(i.e. for any hyperreal number, $[\langle r_1, r_2, r_3, \ldots\rangle]$*, all function on the hyperreal number is applied as,* $[\langle f(r_1), f(r_2), f(r_3), \ldots\rangle]$*.)*

(3) A relation interpretation of $\mathfrak{M}$ is set $P^{\mathfrak{M}} \subseteq M^{k_P}$ for each $P \in \mathscr{P}$;
In $^*\mathfrak{R}$*, relation is defined as,* $\{n \in \mathbb{N} : r_n < s_n\} \in \mathcal{F}$.

(4) A constant interpretation of $\mathfrak{M}$ is an element $c^{\mathfrak{M}} \in M$ for each $c \in \mathscr{C}$.
Constants 0 *and* 1 *in* $^*\mathfrak{R}$ *interpreted as,* $[0] = [\langle 0, 0, 0, \ldots\rangle]$ *and* $[1] = [\langle 1, 1, 1, \ldots\rangle]$.

Now we can declare that,

$$\mathcal{L}\text{-structure } {}^*\mathfrak{R} = \{{}^*\mathbb{R}; \mathscr{F}^{*\mathfrak{R}}, \mathscr{P}^{*\mathfrak{R}}, \mathscr{C}^{*\mathfrak{R}}\} = \{{}^*\mathbb{R}; +, \cdot, <, [0], [1]\}.$$

Notice the similarity between the language of $\mathfrak{R}$ and $^*\mathfrak{R}$, this is because they are based on the same syntax. We will further build the logic of this structure, building formulae with the language $\mathcal{L}$.

3.2. **Terms, Formulae, and Sentences.** As previously mentioned, a language contains a set of function, relation and constant symbols. Using those symbols we generate a formula with logic symbols and variables. Logic symbols contain the followings:

(1) Equality symbol: $=$
(2) Boolean connectives: $\wedge, \vee, \neg$
(3) quantifies: $\forall, \exists$
(4) parenthesis: (,)

Logic symbols are available for any structures. Qunaitifers only range over the universe, and we call variables without being defined with a quantifier, a free variable. We can only quantify over elements, which is the reason why we only aim to transfer first order logic.

Definition 3.3. $\mathcal{L}$*-terms* are a string of symbols of variables and language $\mathcal{L}$ which,

- Each variable symbol v_i such that $i \in \mathbb{N}$ are $\mathcal{L}$-terms;
- Each constant symbol $c \in \mathscr{C}$ are $\mathcal{L}$-terms;
- If $t_1, \ldots, t_{k_f}$ are $\mathcal{L}$-terms, and $f \in \mathscr{F}$, then $f(t_1, \ldots, t_{k_f})$ is a $\mathcal{L}$-term.

One example of a term from the language $\mathcal{L}$ of $\mathcal{L}$-structure $^*\mathfrak{R}$ is $+(v_1, v_2)$, which can be simplified to $v_1 + v_2$. We used variables and function of the language.

3.2.1. *Interpretation of Terms.* A term can be divided into subterms. Using that we will interpret each subterm and combine them to make a whole interpreted term. Terms are an expression, that will potentially return a value. Depending on the model of interpretation, a term could be undefined, or return a value.

Definition 3.4. Let $\mathfrak{M}$ be an $\mathcal{L}$-structure and let t be a term with variables from the tuple $\bar{v} = (v_{i_1}, \ldots, v_{i_m})$. We define the interpretation of t in $\mathfrak{M}$, written $t^{\mathfrak{M}} : M^m \to M$, as a function from m-tuples of elements in M to elements of M, via the following inductive rules:
Let $\bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M^m$. Then:

(1) If s is a constant symbol c, then:

$$s^{\mathfrak{M}}(\bar{a}) = c^{\mathfrak{M}}$$

(2) If s is a variable v_{i_j}, then:

$$s^{\mathfrak{M}}(\bar{a}) = a_{i_j}$$

(3) If s is a compound term $f(t_1, \ldots, t_{n_f})$, where f is an n_f-ary function symbol and $t_1, \ldots, t_{n_f}$ are terms, then:

$$s^{\mathfrak{M}}(\bar{a}) = f^{\mathfrak{M}}(t_1^{\mathfrak{M}}(\bar{a}), \ldots, t_{n_f}^{\mathfrak{M}}(\bar{a}))$$

Thus, $t^{\mathfrak{M}}$ is defined for every term t by recursively interpreting each of its components in the structure $\mathfrak{M}$ with it's universe.

We will now introduce formulae. The key distinction between term and formulae is that term will interpret, and return value and formulae will interpret and return truth value. Formulae take in the first order logic of a model that expresses what is true inside the model.

Definition 3.5. We say that φ is an atomic $\mathcal{L}$ -formula if φ is either,
i) $t_1 = t_2$, where t_1 and t_2 are terms, or
ii) $P(t_1, \ldots, t_{n_P})$, where $P \in \mathscr{P}$ and $t_1, ..., t_{n_P}$ are terms.

The set of $\mathcal{L}$-formulae is the smallest set $\mathcal{W}$ containing the atomic formulae such that
i) if φ is in $\mathcal{W}$, then $\neg\varphi$ is in $\mathcal{W}$,
ii) if φ and ψ are in $\mathcal{W}$, then $(\varphi \wedge \psi)$ and $(\varphi \vee \psi)$ are in $\mathcal{W}$, and
iii)if φ is in $\mathcal{W}$,then $\exists v_i \varphi$ and $\forall v_i \psi$ are in $\mathcal{W}$.

From that definition, we can generate formulae. However, we do not know if they are true or not, because they are still a syntax, we haven't interpreted them yet. Formula without free variables are sentences.

3.2.2. *Interpretation of Formulae.*

Definition 3.6. Let φ be a formula with free variables from $\bar{v} = (v_{i_1}, \ldots, v_{i_m})$, and let $\bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M^m$. We define when a structure $\mathfrak{M}$ satisfies φ at $\bar{a}$ (written $\mathfrak{M} \models \varphi(\bar{a})$) by induction:

(1) If φ is $t_1 = t_2$, then $\mathfrak{M} \models \varphi(\bar{a})$ iff $t_1^{\mathfrak{M}}(\bar{a}) = t_2^{\mathfrak{M}}(\bar{a})$.
(2) If φ is $R(t_1, \ldots, t_n)$, then $\mathfrak{M} \models \varphi(\bar{a})$ iff $(t_1^{\mathfrak{M}}(\bar{a}), \ldots, t_n^{\mathfrak{M}}(\bar{a})) \in R^{\mathfrak{M}}$.
(3) If φ is $\neg\psi$, then $\mathfrak{M} \models \varphi(\bar{a})$ iff $\mathfrak{M} \not\models \psi(\bar{a})$.
(4) If φ is $\psi \wedge \theta$, then $\mathfrak{M} \models \varphi(\bar{a})$ iff $\mathfrak{M} \models \psi(\bar{a})$ and $\mathfrak{M} \models \theta(\bar{a})$.
(5) If φ is $\psi \vee \theta$, then $\mathfrak{M} \models \varphi(\bar{a})$ iff $\mathfrak{M} \models \psi(\bar{a})$ or $\mathfrak{M} \models \theta(\bar{a})$.
(6) If φ is $\exists v_j \ \psi(\bar{v}, v_j)$, then $\mathfrak{M} \models \varphi(\bar{a})$ iff there exists $b \in M$ such that $\mathfrak{M} \models \psi(\bar{a}, b)$.
(7) If φ is $\forall v_j \ \psi(\bar{v}, v_j)$, then $\mathfrak{M} \models \varphi(\bar{a})$ iff for all $b \in M$, $\mathfrak{M} \models \psi(\bar{a}, b)$.

If $\mathfrak{M} \models \varphi(\bar{a})$, we say that $\varphi(\bar{a})$ is *true in* $\mathfrak{M}$.

3.3. **Embedding.** The goal of building the model of $^*\mathfrak{R}$ is to show the first order logical continuity from $\mathfrak{R}$ to $^*\mathfrak{R}$. To directly show this, we can use the $\mathcal{L}$-embedding function, that will later show the logical connection between the two models.

Definition 3.7. $\mathfrak{R}$ and $^*\mathfrak{R}$ are $\mathcal{L}$-structures with universes $\mathbb{R}$ and $^*\mathbb{R}$ each, which an $\mathcal{L}$-embedding $\eta : \mathfrak{R} \to {}^*\mathfrak{R}$ is an injective map $\eta : \mathbb{R} \to {}^*\mathbb{R}$. Let $\eta : \mathbb{R} \hookrightarrow {}^*\mathbb{R}$ be $\eta(r) = [(r, r, r, r, \ldots)]$ which $r \in \mathbb{R}$. η is a proper embedding function since,

(1) Injectivity: each r represents an unique constant sequence. If $r \neq s$ then, $\eta(r) \neq \eta(s)$.
(2) Function preservation: $\eta(f^{\mathfrak{R}}(a_1, \ldots, a_{k_f})) = f^{^*\mathfrak{R}}((\eta(a_1), \ldots, \eta(a_{k_f}))$ for all $f \in \mathscr{F}$ and $a_1, \ldots, a_n \in M$ (2.6);
(3) Relation Preservation: $(a_1, \ldots, a_{k_R}) \in P^{\mathfrak{R}}$ iff $(\eta(a_1), \ldots \eta(a_{k_P})) \in R^{^*\mathfrak{R}}$ for all $P \in \mathscr{P}$ and $a_1, \ldots, a_{m_j} \in M(2.7)$;
(4) Constant Preservation: $\eta(c^{\mathfrak{R}}) = c^{^*\mathfrak{R}}$ for $c \in \mathscr{C}$.

As we defined earlier, clearly $\mathbb{R} \subseteq {}^*\mathbb{R}$, and the inclusion map is the $\mathcal{L}$ embedding η, therefore $^*\mathfrak{R}$ is an *extension* of $\mathfrak{R}$.

We will look at interpreted formulae being transferred between two models, one and extension of the one. We will start off with non quantifier formulae, which represent the rules or condition elements of the set can satisfy.

Proposition 3.8. *Let $\mathfrak{M} \subseteq \mathfrak{N}$ be structures for the same language, let $a \in M$, and let $\varphi(v)$ be a quantifier-free formula. Then:*

$$\mathfrak{M} \models \varphi(a) \quad \textit{if and only if} \quad \mathfrak{N} \models \varphi(a).$$

Proof. We proceed by induction on the structure of terms and formulae.
Claim. *For any term $t(v)$ and any $b \in M$, we have $t^{\mathfrak{M}}(b) = t^{\mathfrak{N}}(b)$. Proof of claim:*

- *If t is a constant symbol c, then $t^{\mathfrak{M}}(b) = c^{\mathfrak{M}} = c^{\mathfrak{N}} = t^{\mathfrak{N}}(b)$.*
- *If t is a variable v_i, then $t^{\mathfrak{M}}(b) = b_i = t^{\mathfrak{N}}(b)$.*
- *If $t = f(t_1, \ldots, t_n)$ and by the inductive hypothesis $t_i^{\mathfrak{M}}(b) = t_i^{\mathfrak{N}}(b)$ for all i, then since $\mathfrak{M} \subseteq \mathfrak{N}$ and $f^{\mathfrak{M}} = f^{\mathfrak{N}}|_{M^n}$, it follows that*

$$t^{\mathfrak{M}}(b) = f^{\mathfrak{M}}(t_1^{\mathfrak{M}}(b), \ldots, t_n^{\mathfrak{M}}(b)) = f^{\mathfrak{N}}(t_1^{\mathfrak{N}}(b), \ldots, t_n^{\mathfrak{N}}(b)) = t^{\mathfrak{N}}(b).$$

Main proof: *We now prove the proposition by induction on formulae.*

- *If φ is $t_1 = t_2$, then*

$$\mathfrak{M} \models \varphi(a) \iff t_1^{\mathfrak{M}}(a) = t_2^{\mathfrak{M}}(a) \iff t_1^{\mathfrak{N}}(a) = t_2^{\mathfrak{N}}(a) \iff \mathfrak{N} \models \varphi(a).$$

- *If φ is $R(t_1, \ldots, t_n)$, then since $\mathfrak{M} \subseteq \mathfrak{N}$ and $R^{\mathfrak{M}} = R^{\mathfrak{N}} \cap M^n$:*

$$\mathfrak{M} \models \varphi(a) \iff (t_1^{\mathfrak{M}}(a), \ldots, t_n^{\mathfrak{M}}(a)) \in R^{\mathfrak{M}} \iff (t_1^{\mathfrak{N}}(a), \ldots, t_n^{\mathfrak{N}}(a)) \in R^{\mathfrak{N}} \iff \mathfrak{N} \models \varphi(a).$$

- *If $\varphi = \neg\psi$, assume by induction that the proposition holds for ψ. Then:*

$$\mathfrak{M} \models \neg\psi(a) \iff \mathfrak{M} \not\models \psi(a) \iff \mathfrak{N} \not\models \psi(a) \iff \mathfrak{N} \models \neg\psi(a).$$

- *If $\varphi = \psi_1 \wedge \psi_2$, assume it holds for ψ_1 and ψ_2. Then:*

$$\mathfrak{M} \models \varphi(a) \iff \mathfrak{M} \models \psi_1(a) \textit{ and } \mathfrak{M} \models \psi_2(a) \iff \mathfrak{N} \models \psi_1(a) \textit{ and } \mathfrak{N} \models \psi_2(a) \iff \mathfrak{N} \models \varphi(a).$$

Since quantifier-free formulae are built from atomic formulae using negation and conjunction, the result follows by structural induction.

Since $\mathfrak{R} \subseteq {}^*\mathfrak{R}$, The previous statement works directly as $\mathfrak{M}$ as $\mathfrak{R}$ and $\mathfrak{N}$ as ${}^*\mathfrak{R}$. Up to here, we built the logic using only model theoretic properties. Later on, we will imply the construction of ultrapower and use of ultrafilter into building more specific and transfer focused logic.

3.4. **Łoś's Theorem and Transfer Principle.** In the previous section we introduced filters, and emphasized how truth are defined using ultrafilter. We will bring that idea back to further expand how we define truthfulness with formulae. This will lead to the proof of the transfer principle, that the first order logic of $\mathbb{R}$ is extended to ${}^*\mathbb{R}$.

Theorem 3.9 (Łoś's Theorem)**.** *Let $\mathcal{M}$ be an L-structure, and let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Let $\prod_{\mathcal{U}} \mathcal{M}$ be the ultrapower of $\mathcal{M}$ with respect to $\mathcal{U}$. For any first-order L-formula $\varphi(x_1, \ldots, x_n)$ and any sequences $a_1, \ldots, a_n \in M^{\mathbb{N}}$, we have:*

$$\prod_{\mathcal{U}} \mathcal{M} \models \varphi([a_1], \ldots, [a_n]) \iff \{i \in \mathbb{N} \mid \mathcal{M} \models \varphi(a_1(i), \ldots, a_n(i))\} \in \mathcal{U}.$$

Proof. We proceed by structural induction on the formula φ.

Base Case: Atomic formulae

Suppose $\varphi(x_1, \ldots, x_n) = R(t_1, \ldots, t_k)$ where R is a k-ary relation symbol and each t_j is a term.

Then:

$$\prod_{\mathcal{U}} \mathcal{M} \models R([t_1], \ldots, [t_k]) \iff \{i \in \mathbb{N} \mid \mathcal{M} \models R(t_1(i), \ldots, t_k(i))\} \in \mathcal{U}.$$

This follows directly from the definition of the relation R in the ultrapower structure.

Inductive Steps

Assume the theorem holds for formulae φ and ψ. We verify it for logical connectives and quantifiers.

- Negation: For $\varphi = \neg\psi$, we have:

$$\prod_{\mathcal{U}} \mathcal{M} \models \neg\psi([a]) \iff \prod_{\mathcal{U}} \mathcal{M} \not\models \psi([a]).$$

  By the inductive hypothesis:

$$\iff \{i \in \mathbb{N} \mid \mathcal{M} \models \psi(a(i))\} \notin \mathcal{U} \iff \{i \in \mathbb{N} \mid \mathcal{M} \models \neg\psi(a(i))\} \in \mathcal{U}.$$

- Conjunction: For $\varphi = \psi \wedge \theta$, we have:

$$\prod_{\mathcal{U}} \mathcal{M} \models \psi \wedge \theta([a]) \iff \prod_{\mathcal{U}} \mathcal{M} \models \psi([a]) \text{ and } \prod_{\mathcal{U}} \mathcal{M} \models \theta([a]).$$

  Apply the inductive hypothesis to each part and use closure of $\mathcal{U}$ under intersection.

- Universal Quantifier: Let $\varphi = \forall x\, \psi(x, a_1, \ldots, a_n)$. Then:

$$\prod_{\mathcal{U}} \mathcal{M} \models \forall x\, \psi(x, [a_1], \ldots, [a_n])$$

  holds if and only if for every $[b] \in \prod_{\mathcal{U}} M$, we have:

$$\prod_{\mathcal{U}} \mathcal{M} \models \psi([b], [a_1], \ldots, [a_n]).$$

  By the inductive hypothesis, this is equivalent to:

$$\{i \in \mathbb{N} \mid \mathcal{M} \models \psi(b(i), a_1(i), \ldots, a_n(i))\} \in \mathcal{U} \text{ for all } b \in M^{\mathbb{N}}.$$

Hence:

$$\{i \in \mathbb{N} \mid \mathcal{M} \models \forall x\, \psi(x, a_1(i), \ldots, a_n(i))\} \in \mathcal{U}.$$

- Existential Quantifier: Follows similarly by noting that

$$\prod_{\mathcal{U}} \mathcal{M} \models \exists x\, \psi(x, [a]) \iff \text{there exists } [b] \text{ such that } \prod_{\mathcal{U}} \mathcal{M} \models \psi([b], [a]).$$

Corollary 3.10 (Transfer Principle)**.** *By induction, Łoś's Theorem holds for all first-order formulae.*

Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$, and $[a_1], \ldots, [a_n] \in {}^*\mathbb{R}$. Then for any first order formula $\varphi(x_1, \ldots, x_n)$,

$${}^*\mathbb{R} \models \varphi([a_1], \ldots, [a_n]) \iff \{i \in \mathbb{N} : \mathbb{R} \models \varphi(a_1(i), \ldots, a_n(i))\} \in \mathcal{U}$$

where i is the index of each term from the sequence. This states that for n-ary formula φ in $\mathbb{R}$, when the sequence is evaluated componentwise, if the formula is satisfied by almost all terms, then the formula is true in hyperreal numbers too. This states that any first order logic is true in either real or hyperreal numbers, if and only if it is true in both.

We have officially stated that all first order logic, elementary thoerems in real number system should be true and provable in the hyperreal number system too. We have satisfied the condition of extension of first order logic from $\mathbb{R}$ to ${}^*\mathbb{R}$. Now we will introduce how we utilize this in the hyperreal numbers.

3.5. **$*$-Transform.** Using the transfer principle, we can move the first order statements from $\mathbb{R}$ to ${}^*\mathbb{R}$. We will discuss the notation of transfered formulae in ${}^*\mathbb{R}$.

The goal is to translate the formula of $\mathbb{R}$ into ${}^*\mathbb{R}$. We will break down formula into terms. If a term τ is in a formula φ, then:

(1) If $\tau = f(\tau_1, \ldots, \tau_m)$ then ${}^*\tau = {}^*f({}^*\tau_1, \ldots, {}^*\tau_m)$.
(2) If τ is a variable or a constant, then ${}^*\tau = \tau$.

Using the terms $*$-transform, we will define the $*$-transform of relation too. For relation $P \in \mathscr{P}$, the $*$-transform of it is ${}^*P$. To summarize, to $*$-transform a formula is:

(1) replace all term τ by ${}^*\tau$
(2) replace P by ${}^*P$
(3) replace the bound of quantifiers of the formula φ (e.g. $(\forall x \in X)\varphi$) of the form, to ${}^*X$.
(4) Logical symbols will stay the same since logical symbol is an universal language to all structures

Example $*$-transform of the *Eudoxus-Archimedes Principle*:

$$\forall x \exists m (x < m \text{ and } m \in \mathbb{N})$$

To

$$\forall x \exists m (x < m \text{ and } m \in {}^*\mathbb{N}).$$

Remark 3.11. For generally well known functions and relations we conventionally do not attach $*$ to them, such as $<, \leq, =, \neq$ etc and $\sin(x), e^x, +, \cdot$ etc.

Now we have built a full model of hyperreal numbers and satisfied all the conditions. We first built the universe that includes infinitesimal and infinite number using ultrapower construction, then we applied ultrapower into model theoretical properties that proves the extension and transform of real first order logic into the hyperreals. Up to here, this is Robinson's Nonstandard Analysis. We will look into another method of construction that results the same system.

4. Internal Set Theory

After Robinson created Nonstandard Analysis, Edward Nelson contributed to the development of the hyperreal numbers system by creating Internal Set Theory, which is built directly in the purpose to strengthen Non Standard Analysis. Internal Set Theory is an Extension of ZFC Set Theory, likewise how we extended $\mathbb{R}$ to $^*\mathbb{R}$. This is an independent construction of the hyperreal numbers from ultrapower and model theoretic construction. We use fundamentally different approach, as we extend the language of the system, the use of internal set theory will be a syntactic construction. The construction of IST will be very brief since it doesn't require much construction from the scratch. Internal set theory is an addition to ZFC, not a change. Therefore, true statements in ZFC must remain true in IST, and the properties transfer too.

We are going to split the language into two categories. Because IST is an extension of ZFC, we are allowed to be talking inside the given universe that obeys ZFC, and outside of that universe. When we talk inside the universe we will use *Internal Language.* We must obey ZFC rule strictly when using Internal Language. When we are talking about the universe from outside, we will use *External Language.* We do not have to speak the language of ZFC in External, however we cannot contradict rules and theorems from internal language. Both language inherently have different purpose. External language is used to describe the universe, and Internal is for building and using the universe. Internal Language is not aware of any existence of external language, however, external language is aware of internal language since they must not contradict it. Intuitively, we do not have to know all the neural pathway for our brain to function, describing and functioning is separate for models too.

Any formulae, properties, theorems from ZFC are transferred to IST. Mainly, we know the existence of $^*\mathbb{R}, \mathbb{R}, \mathbb{N}, \mathbb{Q}, \mathbb{Z}$, since they are all constructed upon ZFC. Another, the Real Number system is fundamentally a syntactic construction, although philosophical debate still exists regarding the real number system as a mathematical object. Which means the whole real number system, theorems, functions, relations are accessible in IST.

4.1. **Standard.** In IST, standardness is an important concept that distinguishes internal and external language. As we extend ZFC, we will add another predicate to the language, *standard(x).* The predicate standard belongs to IST but does not belong to ZFC, since it is an undefined predicate under ZFC. Standard is included in the external language, that we will use to describe the model, specifically ZFC. From this, we can naturally come up with definition of internal sets and external sets.

Definition 4.1. An internal formula only includes the language of ZFC, specifically do not use or mention the predicate *standard.* An internal formula should still remain defineness in ZFC alone.

Definition 4.2. An external formula that uses a language including *standard.*

4.1.1. *Sets of IST. An external formula can never define a set.* There are two reasons for this.
First, ZFC has a set theoretic ontology, which the entire mathematical universe is made of sets. The universe is only made out of objects, that we can only generate each elements of the universe using sets. Which means in ZFC, further out to IST, every construction is made out of sets. As mentioned, external language only have the purpose of describing the ZFC model itself, and cannot manipulate the model. Generating sets will intervene with ZFC, which is outside of the role of external language.
Second, which is a more technical reason, that creating a set with external language will violate the rules of ZFC. Only internal comprehension is permitted inside a set. Therefore, external formula cannot form a set.

Which means,

$$\{x|standard(x)\}.$$

Is illegal. However, external formulae outside of that can include $standard(x)$. Such as,

$$\exists x(standard(x) \wedge x > 57)$$

Which is a statement, not a set. We can also define,

$$\forall^{\mathrm{st}} x := \forall x(standard(x))$$

$$\exists^{\mathrm{st}} x := \exists x(standard(x))$$

as a quantifier, because that does not violate the comprehension rules of ZFC, and simply extends the language with statements.

Formally, before we interpret the syntax, we don't know what it means, as mentioned in section 3 (3.1). Which means we do not know what it means to be *standard* yet, and what determines one to be true to *standard.* However, we can construct the rules and behavior of standardness solely using syntax, which are axioms. Nelson's intention to this construction is that standard intuitively means that it is a classical mathematical object that already exists, such as real numbers.

4.2. **Axioms of Internal Set Theory.** The Axioms of Internal Set Theory adds 3 additional axioms to the traditional ZFC Axioms to maintain conservatism of ZFC. These axioms are external statements, meaning that they are describing ZFC, not impacting the construction at all. These axioms control how we can understand ZFC. The followings are axiom schema, which is a framework for infinite amount of axiom are generatable from it.

(1) Axiom of Transfer Principle:

$$\forall^{\mathrm{st}} x\, \varphi(x) \quad \Longrightarrow \quad \forall x\, \varphi(x),$$

where φ is an internal formula. Axiom of transfer principle allows internal truths that hold for all standard objects to be extended to all objects.

(2) Axiom of Idealization

$$\forall^{\text{st}} A \subseteq_{\text{fin}} X \, \exists x \, \forall a \in A \, \varphi(x,a) \iff \exists x \, \forall^{\text{st}} a \in X \, \varphi(x,a),$$

where φ is an internal formula, X is a standard set, and the quantification $\forall^{\text{st}} A \subseteq_{\text{fin}} X$ ranges over all standard finite subsets A of X. Axiom of idealization allows internal truths that hold for all standard objects to be extended to all objects.

(3) Axiom of Standardization

$$\forall^{\text{st}} X \, \exists^{\text{st}} Y \, \forall^{\text{st}} x \left(x \in Y \iff (x \in X \wedge \varphi(x)) \right),$$

where φ is an internal formula. This allows us extract internal sets matching external predicates over standard elements.

Remark 4.3. In ZFC, everything is made of sets. Elements are sets, therefore, sets of the universe are first order objects. Therefore, we are allowed to quantify over sets. Second order logic goes further to properties of sets, which will not occur in ZFC nor IST.

4.3. **Internal and External Sets.** As we mentioned before, the use of predicate *standard* determines internal/external formulae. And while internal formula can be used in any form of formulae, external formulae cannot be inside a set. This sets the rule in the construction and the system.

Definition 4.4. An internal set is a legal set under ZFC. It is one category of an internal formula.

Definition 4.5. An external set cannot be defined using internal formula, the definition involves the predicate standard.

To obey the rules of ZFC, we cannot express external set explicitly, however, we can use the axiom of standardization to define it. Technically, external sets are not sets, they are often classes or collection that we can talk about, however does not exists as an object in the system.

We will list some external sets that are illegal to write out in the internal language:

- $\forall^{\text{st}} x, x \in \{x \in \mathbb{N} : x \text{ is even}\}$
- $\mathbb{N}_{\text{st}} := \{x \in \mathbb{N} : \text{standard}(x)\}$

The predicate standard can never be used in the internal language. Therefore, defining external sets in the internal language ZFC is illegal. However, there exists the use of external sets. It is often used in proofs or description that we will describe, not construct. Since external sets are not allowed for the concern of contradiction of ZFC and out of purpose, use of proof is completely valid. When we describe, we will often prove the existence without construct it, which the axiom of standardization is based of the purpose of this. We use internal sets to construct, we use external sets to classify.

4.4. **Standard and Nonstandard Elements.** Now we will prove the inclusion of infinitesimal and infinite number using this construction. This entirely relies on external language which means we will be describing the universe from the outside. We are not directly constructing them, rather describing the existence of them by classifying elements of the universe by standard and nonstandard elements.

We mentioned that all the properties of real number system transfers to the internal language. Naturally, we might recall that the Archimedean property is restricting the real number system to include infinitesimal and infinite number into their universe, which should apply to the internal language too. While IST externally proves the existence of infinitesimal and infinite number elements of $\mathbb{R}$, this does not contradict the internal Archimedean property. The internal version holds universally for all real numbers definable within ZFC's logic, but external language permits classification of objects that lie outside the scope of internal quantifiers. This way, IST extends expressivity without violating consistency.

4.4.1. *Infinitesimal.* Let $x \in \mathbb{R}$,

$$x \text{ is infinitesimal } \iff \forall^{\text{st}} y > 0, |x| < y$$

However, that is a statement, and we don't know yet that such element exists. To prove the existence we will use the axiom of idealization.

$$\forall^{\text{st}} A \subseteq_{\text{fin}} \mathbb{R}^+ \, \exists x \in \mathbb{R} \, \forall y \in A \, |x| < y \quad \iff \quad \exists x \in \mathbb{R} \, \forall^{\text{st}} y > 0, |x| < y,$$

Let $A = \{y_1, \ldots, y_n\} \subseteq \mathbb{R}^+$, then,

$$x = \frac{1}{\max(y_i) + 1}$$

Then we know that x is a multiplicative inverse of 1 added to the biggest element of $\mathbb{R}^+$. Therefore, there must not exist any real number that is smaller than x. Then, the right hand side is true, infinitesimal exists in IST. Such x is defined as infinitesimal, ε.

4.4.2. *Infinite Number.* We will use the existence of infinitesimal to construct infinite number. Let $H := \frac{1}{\varepsilon}$, since $0 < \varepsilon < \frac{1}{n}$, for every $n \in \mathbb{N}$, then,

$$n < \frac{1}{\varepsilon} = H$$

Therefore,

$$\forall^{\text{st}} n \in \mathbb{N} \; \exists H, n < H.$$

We already know the existence of infinitesimal, therefore, we do not have to prove the existence of infinite number. Thus, such H is an infinite number. Infinitesimal and Infinities are nonstandard elements, since they do not exist in classical mathematics. Formally I will show why:
To be a nonstandard element, the element should satisfy the predicate $\neg$standard(x). If we assume that infinitesimal is a standard element, then the internal property

$$\forall y > 0, |x| < y$$

should hold. However, this is impossible since the internal language must obey the rules of ZFC, including Archimedean property. Therefore, infinitesimal must not be standard. Same reason holds for infinite number as well.

5. Connection of Semantic and Syntactic Constructions

We have completed the basic construction of ultrapower and IST. From IST, as we have seen, syntactic model is capable of constructing its own model alone. We have proven the existence of infinitesimal and infinite number externally, and proven the transfer of first order logic from internal to external using the Axiom schema of transfer principle. We have two separate construction to the hyperreal number system, each syntactically and semantically. Two systems are fundamentally the same, are able to prove the same theorems and hold same properties. What we see in both construction is that they use an extension to include infinitesimal and infinite number. Ultrapower construction defined number as infinite sequences, which allows them to follow a new rule, and ignore Archimedean property. IST did not construct infinitesimal and infinite number, we proved the existence of it. Because we are describing the universe from outside point of view, we do not have to speak the language of ZFC. Using the predicate standard, we classified standard and nonstandard elements. The second part of the condition of hyperreal number system, which we must preserve the first order logic of real number system is satisfied in both construction as well. In ultrapower construction we proved the transfer principle using model theoretical properties and filter dependent truth values. In IST, we start with the langauge of real number system since the internal language is ZFC, which includes it. Using the Axiom schema of transfer, which asserts that for all internal formula, if it is true for standard elements it must be true for nonstandard elements too. Since all first order logic of real numbers are standard, it must be universally true, for nonstandard elements too.

We can deduce that the base of both constructions were extension and how we extend it depends on each construction. Ultrapower construction seems more natural since it takes the steps of semantic construction, which we are all used to. One advantage of the Ultrapower construction is that it is more natural and easier to understand and accept, which will greatly benefit for pedagogical purpose. On the other hand, IST construction allows us for deeper view of the logical construction, however it is unintuitive and inefficient for understanding and acceptance. Despite the difficulty, IST provides a more general view of the construction to the reader, by not constructing as a *special case* as the ultrapower construction, but defining a language that can produce such system. IST fundamentally did not add anything, however we developed a way to reason to the existence of infinitesimal and infinite number. IST is better for analysis and logical construction.

The transfer principle asserts that a first order theorem in $\mathbb{R}$ is true and provable if and only if it is true and provable in ${}^*\mathbb{R}$. Now because this is true for both constructions, any statement that is provable is provable on the other one too.

6. Construction of semantic model of IST

Now we can consider, how can we semantically interpret the syntax of IST, and what are the conditions to be a model to that? Which can lead to how we can construct the ultrapower universe ${}^*\mathbb{R}$ with the syntax of IST to show the connection between Robinson and Nelson's construction which can function well independently from each other. These idea is explored by mathematicians Vladimir Kanovei and Michael Reeken, with using ZFGC (ZF + Global Choice) to construct the ZFC expansion, IST. Global Choice is a stronger version of axiom of choice that allows choice for proper classes, this simplifies the process of forming ultrapower because we can easily form equivalence classes, without violating IST axioms. This is allowed because ZFGC is stronger than ZFC without contradicting it, it is a similar construction as building a model of Peano Arithmetics with ZFC, which we can build a weaker model using a stronger one.

6.1. **Model Construction.** Let $\mathbb{S}$ be a transitive model of ZFC, the standard part of $\mathbb{I}$ that models IST. $\mathbb{S}$ and $\mathbb{I}$ are sets which $\mathbb{S} \subseteq \mathbb{I}$, internally are the universe and meta-theoretically a set-sized model. We define the language of those models to be, $\{\mathbb{S}; \in, <\}$ and $\{\mathbb{I}; \in, <\}$ for each SS and $\mathbb{I}$. Which, $\{\mathbb{S}; \in, <\} \models$ ZFGC and $\{\mathbb{I}; \in, <\} \models$ ZFGC.

One language we will add is a transitive set T which $T \subseteq M$, for the transitive model M. Such set as T is called *innocuous*, which is does not interfere into the construction or create sets outside the universe. T can be seen as a new truth predicate, that we can describe an element about the membership to T. Additionally, let $\texttt{Truth}^M_{\in,<}$ is a set of all closed $\mathcal{L}$-formulae true in $\{M; \in, <\}$, that is a set of syntactic objects. We can define it as,

$$\texttt{Truth}^M_{\in,<} = \{\varphi \in \mathcal{L}_{\in,<}\text{-sentences} : \{M; \in, <\} \models \varphi\}$$

Although $\texttt{Truth}^M_{\in,<}$ is not transitive, it is still innocuous because it does not construct sets outside of the universe. Now from $\mathbb{S}$ we will create the ultrafilter and further form an equivalence class using the ultrafilter. The process of construction is slightly modified from the ultrafilter from Ultrapower construction. $\texttt{Truth}^M_{\in,<}$ is used to define an important set for the construction of ultrafilter, the set $\texttt{Def}_{\in,<}(\mathbb{S})$.

6.2. **Ultrafilter Construction.** Let $\texttt{Def}_{\in,<}(\mathbb{S})$ denote the collection of all subsets of $\mathbb{S}$ that are definable in the structure $\{\mathbb{S}; \in, <\}$ using formulas in the language $\mathcal{L} = \{\in, <\}$ with parameters from $\mathbb{S}$. Define the index set $I = \mathrm{P}_{\text{fin}}(\mathbb{S})$ to be the collection of all finite subsets of $\mathbb{S}$. This forms a proper class within $\mathbb{S}$.

We now define an algebra $\mathcal{A}$ of subsets of I, where each $A \in \mathcal{A}$ is a definable subset of I in the structure $\{\mathbb{S}; \in, <\}$. That is, $A \in \texttt{Def}_{\in,<}(\mathbb{S})$ and $A \subseteq I$.

We construct an ultrafilter $\mathcal{U} \subseteq \mathcal{A}$ satisfying the following three properties:

(A) For every $a \in \mathbb{S}$, the set $\{i \in I : a \in i\}$ belongs to $\mathcal{U}$.
(B) If $P \subseteq \mathbb{S} \times I$ is definable in $\{\mathbb{S}; \in, <\}$, then the set $\{x \in \mathbb{S} : \{i \in I : (x, i) \in P\} \in \mathcal{U}\}$ is also definable in $\{\mathbb{S}; \in, <\}$.
(C) There exists a definable set $U \subseteq \mathbb{S}$ in $\{\mathbb{S}; \in, <, T\}$ such that $U = \{U_x : x \in \mathbb{S}\}$, where $U_x = \{i \in I : (x, i) \in U\}$.

Similarly to the use of ultrafilter in ultrapower construction, we will define the truth value of each formulae from the membership to the ultrafilter. This might seem circular, which if something is defined as true using ultrafilter which is constructed using the set $\texttt{Truth}^M_{\in,<}$. However, because both the construction of the set $\texttt{Truth}^M_{\in,<}$ and ultrafilter is external, we already know what statement is true in $\mathbb{S}$, and what is not. This allows us to produce $\texttt{Truth}^M_{\in,<}$ without ultrafilter. Which the ultrapower is essentially a formal way to show if one statement is true or not, because membership to ultrafilter means it belongs to $\texttt{Truth}^M_{\in,<}$ too.

6.3. **Ultrapower Construction of IST.** To construct a semantic model of IST, we begin with a transitive model $\mathbb{S}$ of ZFC. Our goal is to build a larger universe $\mathbb{I}$ containing $\mathbb{S}$ as the class of standard sets, and to define an IST structure $\{\mathbb{I}; \in, \text{standard}\}$ where all axioms of IST hold. This process uses a modified version of the ultrapower method. The construction is done externally, within ZFGC, which allows us to define the necessary sets and truth predicates.

Consider the set F of all definable functions $f : I^r \to \mathbb{S}$ for $r \in \mathbb{N}$. These are the building blocks of the ultrapower. We can see that in ultrapower construction previously, this function is defined as an infinite sequence using direct power.

Define the equivalence relation:

$$f \sim g \iff \{i \in I : f(i) = g(i)\} \in \mathcal{U}$$

That is, two functions are identified if they agree on a large (membership to ultrafilter) set of indices. The ultrapower universe is the set of equivalence classes $[f]$ under this relation:

$$\mathbb{S} = \{[f] : f \in F\}$$

We interpret $[f] \in^* [g]$ as $\{i \in I : f(i) \in g(i)\} \in \mathcal{U}$. Other relations are defined similarly as well.

In the previous section where IST was purely syntactic, the predicate standard had no meanings since it was not interpreted. We will not interpret standard. Let,

$$\text{standard}([f]) \iff [f] = [\hat{a}] \text{ for some constant function } \hat{a}(i) = a.$$

We can see this definition of standard resembles the ultrapower construction that the constant sequence that represents real numbers. To satisfy the transfer of first order principle from $\mathbb{R}$, we will interpret the universe $\mathbb{S}$ as $\mathbb{R}$ later.

With this setup, the structure $\{^*\mathbb{S}; \in^*, \text{standard}\}$ satisfies all three axioms of IST:

Transfer: Truth in $\mathbb{S}$ extends to truth in $*\mathbb{S}$.
Since $\mathbb{S}$ represent $\mathbb{R}$ in this case, first order logic is transferred.

Idealization: Universal statements over standard finite sets can be replaced by existential statements over all sets.

Standardization: For any definable property, there exists a standard set collecting exactly the standard elements that satisfy it.

$\mathbb{S}$ is identified with the standard part of $^*\mathbb{S}$, and $^*\mathbb{S}$ forms the full IST universe $\mathbb{I}$.

6.4. **Nonstandard Elements.** Now we have fully constructed semantic model of IST, which that we brought the full $^*\mathbb{R}$ into IST. The construction of infinitesimal and Infinite number follows the same as ultrapower construction.(2.14). Standard elements are $x \in \mathbb{R}$, nonstandard elements are $x \in {}^*\mathbb{R} - \mathbb{R}$.

We have fully constructed the system again as well, with satisfying transfer of first order logic of $\mathbb{R}$, and inclusion of infinitesimal and infinite number to the universe. This shows that the semantic model of IST indeed satisfies to be a hyperreal number system as well. We created the same system 3 different ways, and they will all result in the same theorems that are provable. All 3 have unique way of reasoning and expression, and different purpose of use.

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