Foundations of Schubert Calculus

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Introduction

First developed in the late 19th century by the mathematician Herman Schubert, Schubert Calculus provides methods for solving linear intersection problems. While useful, these methods lacked a solid foundation. As a result, there was a lot of effort to build a theoretical basis for Schubert Calculus, with it even being one of the famous 23 problems Hilbert proposed as the turn of the century.

Projective Space

Definition

We define the projective n-space over $\mathbb{C}$, denoted $\mathbb{P}^n$, as the set of equivalence classes of $n+1$ tuples $(a_0, a_1, \ldots, a_n)$ under the equivalence relation $(a_0, a_1, \ldots, a_n) \sim (\lambda a_0, \lambda a_1, \ldots, \lambda a_n)$ for all $\lambda \in \mathbb{C} \backslash \{0\}$. These equivalence classes are called the points of $\mathbb{P}^n$ and any $(n+1)$-tuple $(a_0, \ldots, a_n)$ in the equivalence class P is called a set of homogeneous coordinates for point P.

Example

In the projective plane, $\mathbb{P}^2$, the 3-tuples $(1, 0, 2)$ and $(2, 0, 4)$ are both in the same equivalence class, and thus refer to the same point in $\mathbb{P}^2$.

Projective Varieties

Definition

Recall that the affine variety is the zero set of some collection of polynomials. Now let T be a set of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$. Then we define the projective variety $V(T)$ to be

$$V(T) = \{(a_0, \ldots, a_n) \in \mathbb{P}^n \mid f(a_0, \ldots, a_n) = 0 \text{ for all } f \in T\}$$

The Grassmannian

Definition

The Grassmannian, $Gr(n,k)$, is the set of all k-dimensional subspaces of $\mathbb{C}^n$. Elements in $Gr(n,k)$ are referred to as the "points" of $Gr(n,k)$.

Example

The matrix:

$$\begin{bmatrix} 0 & 3 & -4 & 8 & -2 & 5 \\ 0 & 1 & 5 & -3 & -4 & 2 \\ 0 & 0 & 0 & 7 & 2 & -1 \end{bmatrix}$$

is a point in $Gr(6,3)$. Its column vectors form a 3-dimensional subspace of $\mathbb{C}^6$. Moreover, from this interpretation we can see that the point remains unchanged under standard row operations. Thus every point in the Grassmannian can be written in reduced row echelon form.

Partitions

Definition

A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a sequence of weakly decreasing non-negative integers. We denote the size of the partition by $|\lambda| = \sum_{i=1}^{k} \lambda_i$.

Example

The partition $\lambda = (4, 3, 1)$ is associated with the following element in $Gr(7, 3)$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 1 & * & * & 0 & * & 0 \end{bmatrix}$$

.

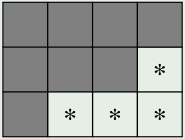
Young Diagrams

Definition

The Young Diagram of partition λ is the left aligned partial grid of boxes contained within an $k \times (n-k)$ rectangle such that the i-th row from the top has λ_i boxes. The larger $k \times (n-k)$ rectangles is called the ambient rectangle, and is denoted as B.

Example

For $\lambda = (4, 3, 1)$ we obtain the following Young diagram:



Schubert Cells

Definition

For a partition λ contained in an ambient rectangle B, the Schubert cell, denoted as Ω_λ°, is the set of points in $Gr(n, k)$ whose row echelon matrix has a corresponding partition of λ. Formally,

$$\Omega_\lambda^\circ = \{V \in Gr(n,k) \mid \dim(V \cap \langle e_1, \ldots, e_r \rangle) = i \text{ for } \\ n - k + i - \lambda_i \leqslant r \leqslant n - k + i - \lambda_{i+1} \text{ for all } i\}$$

Note that $\Omega_\lambda^\circ \cong \mathbb{C}^{k(n-k)-|\lambda|}$.

Schubert Varieties

Example

Consider the matrix in $Gr(4,2)$:

$$\begin{bmatrix} 0 & 1 & 3 & -2 \\ 1 & 0 & -1 & 4 \end{bmatrix}$$

. The Plücker embedding then gives us the points $(-1,-3,2,-1,4,10)$

Definition

The standard Schubert variety corresponding to partition λ, denoted Ω_λ, is the set

$$\Omega_\lambda = \{V \in Gr(n,k) \mid \dim(V \cap \langle e_1, \ldots, e_{n-k+i-\lambda_i}\rangle) \geqslant i \text{ for all } i\}$$

Schubert Varieties

The following example is from Gillespie's paper [1].

Example

Let $\lambda = (2)$. Consider the Schubert variety Ω_λ in $\mathbb{P}^5 = Gr(6, 1)$. Its ambient rectangle is a 1×5 row of boxes. Moreover, $V \in \Omega_\lambda$ if $\dim(V \cap \langle e_1, e_2, e_3, e_4 \rangle) \geqslant 1$. Since V is a 1-dimensional subspace of $\mathbb{C}^6$, it is contained in $\langle e_1, e_2, e_3, e_4 \rangle$. Expressed in homogeneous coordinates, the first two entries must be 0. Thus we find that each point of Ω_λ can be written in one of the following ways:

$$\begin{gathered}(0,0,1,*,*,*)\\(0,0,0,1,*,*)\\(0,0,0,0,1,*)\\(0,0,0,0,0,1)\end{gathered}$$

From this we can see that it is in fact possible to rewrite Ω_λ as a disjoint union of Schubert cells.

Introducing Flags

Definition

A complete flag, F, in $\mathbb{C}^n$ is a chain of subspaces

$$\{0\} = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n$$

Where $\dim(F_i) = i$.

Definition

Two subspaces V and W of $\mathbb{C}^n$ are transverse if

$$\dim(V \cap W) = \max(0, \dim(V) + \dim(W) - n)$$

Definition

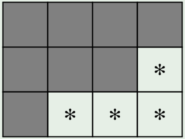
The standard F is defined to be the flag in which $F_i = \langle e_1, \ldots, e_i \rangle$ and the opposite flag E by $E_i = \langle e_n, \ldots, e_{n-i+1} \rangle$.

Definition

Two partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k)$ are said to be complementary in the $k \times (n-k)$ ambient rectangle if and only if $\lambda_i + \mu_{k+1-i} = n - k$ for all i. If λ and μ are complementary, we write $\mu^c = \lambda$.

Example

We see that the complement of $\mu = (4, 3, 1)$ is $\lambda = (3, 1)$



Computing Intersections - The Duality Theorem

Theorem

(Duality Theorem). Let F and E be transverse flags in $\mathbb{C}^n$, and let λ and μ be partitions with $|\lambda| + |\mu| = k(n-k)$. In $Gr(n,k)$, the intersection $\Omega_\lambda(F) \cap \Omega_\mu(E)$ has 1 element if λ and μ are complementary partitions, and is empty otherwise. Moreover, if λ and μ are any partitions with $\mu_{k+1-i} + \lambda_i > n-k$ for some i then the intersection is empty.

First Proof

We will now use our new techniques to prove that there is a unique line passing through two distinct points in $\mathbb{P}^n$. By working in $Gr(n+1,2)$, these two distinct points become distinct 1-dimensional subspaces F_1 and E_1 in $\mathbb{C}^{n+1}$. The Schubert condition tells us that the 2-dimensional subspace that contains them must satisfy

$$\dim(V \cap F_1) \geqslant 1 \text{ and } \dim(V \cap E_1) \geqslant 1.$$

This requires that the partition $\lambda = (\lambda_1)$ where $(n+1) - 2 + 1 - \lambda_1 = 1$ requires that $\lambda_1 = n - 1$. The problem is reduced to computing the intersection of the following:

$$\Omega_{(n-1)}(F) \cap \Omega_{(n-1)}(E).$$

F and E are any two transverse flags extending F_1 and E_1 respectively. Notice that $\lambda = (n-1)$ complements itself in the $2 \times (n-1)$ ambient rectangle. Thus, by the Duality Theorem there exists a unique point in the intersection.

Cell Decomposition of the Grassmannian

Schubert cells give a cell complex structure on the Grassmannian. Define X_0 to be the 0-dimensional Schubert variety Ω_B, where B is the ambient rectangle, and Ω_B is the partition that corresponds to the partition that covers all of B. Because we are working over $\mathbb{C}$ there are no cells of odd dimension. Next we have $X^2 = X^0 \cup \Omega^\circ_{\lambda^1}$ where $\lambda^1 = (n-k, n-k, \ldots, n-k-1)$ is obtained by removing the bottom right corner from the ambient rectangle. We can continue and form X^4 by attaching the two 4-cells given by removing the two outer corner squares in both possible ways from λ^1. By continuing like this, we see that the $2m$-th cell is formed by attaching Schubert cells with partition size $|\lambda| = k(n-k) - m$.

Cellular Cohomology

The following slides on cohomology are based off of Allen Hatcher's Algebraic Topology textbook [2], but have been limited to pertain to the discussions in this presentation.

Definition

For a CW complex $X = X^0 \subset \cdots \subset X^n$ let

$$C_k = \mathbb{Z}^{\# k\text{-cells}}$$

be the free abelian group generated by the k-cells $B_\alpha^{(k)} = (D_\alpha^{(k)})^\circ$. The cellular boundary map $d_{k+1} : C_{k+1} \to C_k$ as

$$d_{k+1}(B_\alpha^{(k+1)}) = \sum_\beta \deg_{\alpha\beta} \cdot B_\alpha^{(k)},$$

where $\deg_{\alpha\beta}$ is the degree of the composite map

$$\partial \overline{B_\alpha^{(k+1)}} \to X^k \to \overline{B_\beta^{(k)}}.$$

Cellular Cohomology

The cellular boundary maps make the groups C_k into a chain complex, a sequence of maps

$$0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$$

for which $d_i \circ d_{i+1} = 0$ for all i. For more information on these maps, see Tajakka's paper on the subject [4]. Because of this property we can consider the following quotient groups

$$H_i(X) = \ker(d_i)/\operatorname{Im}(d_{i+1})$$

for all i. These are abelian groups and are called the cellular homology groups of the space X.

Example

Let us revisit $\mathbb{P}^2$. Its decomposition consisted of a point, a 2-cell, and a 4-cell. Thus its cellular chain complex can be written as

$$\cdots \to 0 \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0$$

and the homology groups are $H_0 = H_2 = H_4 = \mathbb{Z}$ and $H_1 = H_3 = 0$

Cellular Cohomology

Definition

Let $C^k = \hom(C_k, \mathbb{Z})$ for each k. The boundary maps $d_k^* : C^{k-1} \to C^k$ are defined as

$$d_k^* f(c) = f(d_k(c))$$

for any $f \in C^k$ and $c \in C_k$. These coboundary maps form a cochain complex, from which we define the cohomology groups as

$$H^i(X) = \ker(d_{i+1}^*)/\operatorname{Im}(d_i^*)$$

for all i. Finally, we have the direct sum of the cohomology groups

$$H^*(X) = \bigoplus_i H^i(X)$$

which has a ring structure when equipped with the cup product, the dual of the cap product on homology. In this setting it corresponds to intersections of cohomology classes.

Cellular Cohomology

Theorem

The cohomology ring $H^*(Gr(n,k))$ has a $\mathbb{Z}$-basis given by the classes

$$\sigma_\lambda := [\Omega_\lambda(F)] \in H^{2|\lambda|}(Gr(n,k))$$

for λ a partition fitting inside the ambient rectangle. The cohomology $H^*(Gr(n,k))$ is a graded ring. i.e $\sigma_\lambda \cdot \sigma_\mu \in H^{2|\lambda|+2|\mu|}(Gr(n,k))$ and we have

$$\sigma_\lambda \cdot \sigma_\mu = [\Omega_\lambda(F) \cap \Omega_\mu(E)]$$

where F and E are the standard and opposite flags. We remark that σ_λ is independent of the choice of flag F since any two Schubert varieties of the same partition shape are equivalent under change of basis.

Consider the following. Let $\lambda^1, \ldots, \lambda^m$ be partitions such that $\sum_{i=1}^m |\lambda_i| = k(n-k)$. Because the cohomology ring is graded, the product $\sigma_{\lambda^1} \cdot \sigma_{\lambda^2} \cdots \sigma_{\lambda^m} \in H^{2k(n-k)}(Gr(n,k))$. However, the only generator of the cohomology group is σ_B, which is the class of a single point $\Omega_B(F)$, where B is the ambient rectangle. Therefore the intersection of the Schubert varieties corresponding to the λ^i for m generic flags is a finite union of points. In particular, the number of points is the coefficient c in the product

$$\sigma_{\lambda^1} \cdot \sigma_{\lambda^2} \cdots \sigma_{\lambda^m} = c^B_{\lambda^1,\ldots,\lambda^m} \sigma_B$$

Symmetric Functions

Definition

The ring of symmetric functions $\Lambda_{\mathbb{C}}(x_1, x_2, \dots)$ is the ring of bounded degree formal power series $f \in \mathbb{C}[[x_1, x_2, \dots]]$ which are symmetric under permuting variables. More formally,

$$f(x_1, x_2, \dots) = f(x_{\pi(1)}, x_{\pi(2)}, \dots)$$

for any permutation $\pi : \mathbb{Z}^+ \to \mathbb{Z}^+$ and $\deg(f) < \infty$.

Semistandard Young Tableau

Definition

A skew shape is the difference, υ/λ, formed by removing the Young diagram of a partition λ from a strictly larger partition υ. A skew shape is called a horizontal strip if no column contains more than one box.

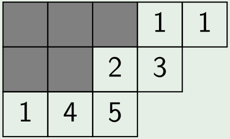
Definition

A semistandard Young tableau of a skew shape υ/λ is a filling of the boxes of the Young diagram of shape υ/λ with positive integers such that within each row the integers weakly increase from left to right and within each column the integers strictly increase from top to bottom. The content of a semistandard Young tableau is denoted $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ where μ_i denoted the number of boxes labeled with integer i. The reading word of the tableau is the word formed by concatenating the rows from bottom to top.

Semistandard Young Tableau

Example

The following is a semistandard Young Tableau of shape υ/λ and content μ where $\upsilon = (5, 4, 2)$, $\lambda = (2, 2, 0)$ and $\mu = (3, 1, 1, 1, 1)$.



In this case, the reading word of the semistandard Young Tableau is 1452311.

Schur Functions

Definition

Given a semistandard Young tableau T of skew shape υ/λ. The Schur function for skew shape υ/λ is given by

$$s_{\upsilon/\lambda} = \sum_T x^T$$

where the sum ranges over all possible semistandard Young Tableau of skew shape υ/λ T and $x^T = x_1^{m_1} x_2^{m_2} \dots$ where m_i is the number of occurrences of the integer i in T. In the case that λ is empty, we say $s_{\upsilon/\lambda} = s_\upsilon$ is the Schur function of shape υ.

Proposition

For any skew shape υ/λ the Schur function $s_{\upsilon/\lambda}$ is symmetric.

For a full proof, see John Naughton's paper [3].

The Isomorphism

Theorem

There is a ring isomorphism

$$H^*(Gr(n,k)) \cong \Lambda(x_1, x_2, \dots)/(s_\lambda \mid \lambda \not\subset B)$$

where B is the ambient rectangle and $(s_\lambda \mid \lambda \not\subset B)$ is the ideal generated by the Schur functions whose partitions do not fit inside of B. The isomorphism sends the Schubert class σ_λ to the Schur function s_λ.

Pieri Rules

Theorem

Given a partition λ and one-row shape (r) the product of the associated Schur function is

$$s_{(r)} \cdot s_\lambda = \sum_\upsilon s_\upsilon$$

where υ is a partition such that υ/λ is a horizontal strip of size r

Theorem

Let λ and μ be partitions such that $|\lambda| + |\mu| = k(n-k) - r$. Let F and E be standard and opposite flags, and H be some generic complete flag. Then the intersection

$$\Omega_\lambda(F) \cap \Omega_\mu(E) \cap \Omega_{(r)}(H)$$

has one element if μ^c/λ has length r and no two boxes in the same column, and is empty otherwise.

The Littlewood Richardson Tableau

Definition

A word $w_1 w_2 w_3 \cdots w_n$ where each $w_i \in \{1, 2, \dots\}$ is Yamanouchi, also lattice or ballot, if every suffix $w_k w_{k+1} \cdots w_n$ contains at least as many letters equal to i as $i + 1$ for all i.

Definition

A Littlewood Richardson Tableau is a semistandard Young tableau whose reading word is Yamanouchi

Definition

A sequence of skew tableaux $T_1, T_2, \dots$ form a chain if their shapes do not overlap and

$$T_1 \cup T_2 \cup \cdots \cup T_i$$

is a partition shape for all i.

The Littlewood Richardson Rule

Theorem

Let $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}$, the product of their corresponding Schur functions can be written in the basis of Schur functions via the formula

$$s_{\lambda^{(1)}} s_{\lambda^{(2)}} \cdots s_{\lambda^{(m)}} = \sum_{\upsilon} c^{\upsilon}_{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(m)}} s_{\upsilon}$$

where $c_{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(m)}}$ is the number of chains of Littlewood Richardson tableaux of contents $\lambda^{(i)}$ with total shape υ.

Corollary

Given partitions λ and μ, the product of the corresponding Schur functions can be written in the basis of Schur functions via the formula

$$s_{\lambda} s_{\mu} = \sum_{\upsilon} c^{\upsilon}_{\lambda,\mu} s_{\upsilon}$$

where $c^{\upsilon}_{\lambda,\mu}$ is the number of Littlewood-Richardson tableaux of skew shape υ/λ and content μ

Zero-dimensional Littlewood Richardson Rule

Theorem

In $H^*(Gr(n,k))$ we have

$$\sigma_{\lambda^{(1)}} \cdots \sigma_{\lambda^{(m)}} = \sum_{\upsilon} c^{\upsilon}_{\lambda^{(1)},\dots,\lambda^{(m)}} \sigma_{\upsilon}$$

where the sum is restricted to partitions υ fitting in the ambient rectangle.

Proof.

By the general Pieri formula for Schur functions,

$$s_{\lambda^{(1)}} s_{\lambda^{(2)}} \cdots s_{\lambda^{(m)}} = \sum_{\upsilon} c^{\upsilon}_{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(m)}} s_{\upsilon}$$

where $c^{\upsilon}_{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(m)}}$ is the number of chains of Littlewood Richardson tableaux of contents $\lambda^{(i)}$ with total shape υ. Because of the isomorphism between Schur functions and the cohomology ring, we get our result. □

Zero-dimensional Littlewood Richardson Rule

Theorem

Let B be the $k \times (n-k)$ ambient rectangle and let $\lambda^{(1)}, \ldots, \lambda^{(m)}$ be partitions fitting inside B such that $|B| = k(n-k) = \sum_i |\lambda_i|$. Also let $F^{(1)}, \ldots, F^{(m)}$ be any m generic flags. Then

$$c^B_{\lambda^{(1)}, \ldots, \lambda^{(m)}} := |\Omega_{\lambda^{(1)}}(F^{(1)}), \ldots, \Omega_{\lambda^{(m)}}(F^{(m)})|$$

is equal to the number of chains of Littlewood Richardson tableaux of contents $\lambda^{(1)}, \ldots, \lambda^{(m)}$ with total shape equal to B.

An Intersection Problem

In this last section, we will bring together all that we have covered so far to approach this problem. We want to know that how many lines intersect 4 given lines in 3-dimensional space? The Schubert variety $\Omega_{(1,0)}(F) \subset Gr(4,2)$ consists of the 2-dimensional subspaces V of $\mathbb{C}^4$ for which $\dim(V \cap F_2) \geqslant 1$. Under the quotient map from $\mathbb{C}^4 \to \mathbb{P}^3$, we can see that this is equivalent to the space of all lines that intersect a given line in at least a point, which is what we want. Using this, we can reduce the problem to finding out properties of the intersection

$$\Omega_{(1)}(F^{(1)}) \cap \Omega_{(1)}(F^{(2)}) \cap \Omega_{(1)}(F^{(3)}) \cap \Omega_{(1)}(F^{(4)})$$

where $F^{(1)}$, $F^{(2)}$, $F^{(3)}$, and $F^{(4)}$ are distinct flags.

An Intersection Problem

Using our results on the cohomology ring of the Grassmannian, we can restate this problem again as

$$\sigma_{(1)} \cdot \sigma_{(1)} \cdot \sigma_{(1)} \cdot \sigma_{(1)} = c \cdot \sigma_{(2,2)}$$

where the value of c is our answer. Finally we use our results about Schur functions and the Littlewood Richardson rule. We find that $c = c^{(2,2)}_{(1),(1),(1),(1)}$ which is the number of ways to fill a 2×2 rectangle using a chain of Littlewood Richardson tableaux each consisting of a single box. Since each tableau of the chain must contain a single 1 as its entry, we label them with subscripts to indicate their step in the chain. From this, we find that there are exactly two chains satisfying these conditions

$$\begin{array}{|c|c|}\hline 1_1 & 1_2 \\ \hline 1_3 & 1_4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1_1 & 1_3 \\ \hline 1_2 & 1_4 \\ \hline \end{array}.$$

Therefore $c^{(2,2)}_{(1),(1),(1),(1)} = 2$, thus there are 2 lines intersecting 4 generic lines in $\mathbb{P}^3$.

Bibliography

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