Topics in Schubert Calculus

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Abstract

In this paper, we will give a brief treatment of the fundamental concepts of Schubert Calculus, which is the study of cells in the Grassmannian, while focusing on its non-combinatorial aspects. We will begin by introducing the required knowledge of projective space. We will then tackle the Grassmannian, Schubert cells, and Schubert varieties, which are the most important constructions in the paper, and how they relate to Young Diagrams as well as a brief aside to introduce flags. Next we will introduce cohomology groups and the cohomology ring of the Grassmannian with a focus towards its relationship with symmetric functions. Finally, in introducing the Littlewood Richardson tableau, we can connect these two concepts in such a way that we can finally solve linear intersection problems, which is how we will conclude the paper.

1 Introduction

First developed in the late 19-th century by the mathematician Herman Schubert, Schubert Calculus provides useful methods for solving linear intersection problems. However, at the time of their creation these methods lacked a solid foundation in mathematics. See [6] to see how Schubert approached the problem initially. Because these methods were still very important, there was a lot of effort put into building a theoretical basis for Schubert Calculus. It was so important that it was even listed as one of the famous 23 problems Hilbert proposed as the turn of the century. A classic example of a problem that can be solved with Schubert Calculus, and what we hope to show in this paper, is the following:

How many lines intersect 4 given lines in 3-dimensional space?

Using something Schubert called the "Principle of Conservation of Number", which essentially states that number of solutions of an intersection problem in any number of parameters (in this case lines) under variation of the parameters is invariant in the case that no solutions become infinite. For this specific problem, that manifested in Schubert finding a specific arrangement of the four lines that did not have infinite solutions, but was easier to work with, and then stated that the solution for that situation was in fact the same in general. This lacks a rigorous foundation, so our goal is to see what rigorous mathematics solves the same problems this idea could. Specifically, our approach to solve this problem is encapsulated by the following idea. Let us say that X_i is the space of all lines L intersecting line l_i for each $i = 1, 2, 3, 4$. Then the intersection $X_1 \cap X_2 \cap X_3 \cap X_4$ is the set of solutions to our problem. In this case, each X_i is and example of a Schubert variety, which is an algebraic and geometric object that is crucial to solving these types of problems and one that we will cover in far more detail later in the paper.

2 Motivation and Prerequisites

Before tackling the Grassmannian, we first briefly introduce some definitions and concepts regarding projective space and projective varieties. It is worth noting that we will only consider projective n-

Figure 1: We see that even though the rails are parallel, they appear to meet at some point on the horizon. This is our point "at infinity".

spaces over $\mathbb{C}$ in this paper, however other texts, such as [1], treat it more generally when discussing Schubert calculus.

Definition 1. We define the projective n-space over $\mathbb{C}$, denoted $\mathbb{P}^n$, as the set of equivalence classes of $n+1$ tuples $(a_0, a_1, \ldots, a_n)$ of elements in $\mathbb{C}$ under the equivalence relation $(a_0, a_1, \ldots, a_n) \sim (\lambda a_0, \lambda a_1, \ldots, \lambda a_n)$ for all $\lambda \in \mathbb{C}\backslash\{0\}$. These equivalence classes are called the points of $\mathbb{P}^n$ and any $(n+1)$-tuple $(a_0, \ldots, a_n)$ in the equivalence class P is called a set of homogeneous coordinates for point P.

It also helps to conceptualize n-dimensional projective space as the set of lines through the origin in $\mathbb{C}^{n+1}$. This characterization becomes more important when dealing with the Grassmannian as it allows us to relate some projective space $\mathbb{P}^n$ to some Grassmannian.

Example. In the projective plane, $\mathbb{P}^2$, the 3-tuples $(1, 0, 2)$ and $(2, 0, 4)$ are both in the same equivalence class, and thus refer to the same point in $\mathbb{P}^2$.

Another helpful way to conceptualize projective space is to have some notion of a point "at infinity". In essence, we can think of $\mathbb{P}^n$ as $\mathbb{C}^n$ completed by points "at infinity" which is a point where parallel lines meet.

Example. In $\mathbb{P}^1$, any point (x, y) where $y \neq 0$ can be rescaled to the form $(t, 1)$, and all such points can be identified with an element of $\mathbb{C}$. The only point remaining in $\mathbb{P}^1$, $(1, 0)$, is the point "at infinity".

Now, recall that affine varieties are the zero set of some collection of polynomials. From this notion, we can also define the projective variety with one important additional requirement about the collection of polynomials.

Definition 2. A polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ is a homogeneous polynomial of degree d if all of its terms have total degree d. i.e A polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ is a homogeneous polynomial of degree d if $f(\lambda a_0, \lambda a_1, \ldots, \lambda a_n) = \lambda^d f(a_0, a_1, \ldots, a_n)$.

While we will not be using these definition directly for much of the paper, it is still useful to keep in mind. Moreover, having strong notions about projective space and projective varieties can help with understanding the Grassmannian, which is in essence a more general version of projective space. Now, we define projective varieties as follows:

Definition 3. Let T be a set of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$. Then we define the projective variety $V(T)$ to be

$$V(T) = \{(a_0, \ldots, a_n) \in \mathbb{P}^n \mid f(a_0, \ldots, a_n) = 0 \text{ for all } f \in T\}.$$

Note that it is important for the polynomials to be homogeneous in order for the variety to remain well defined. To see this let us consider the following example.

Example. Consider $f(x, y) = x^3 - yx$, which is not a homogeneous polynomial. Then we have that $f(2, 4) = 0$ and $f(1, 2) \neq 0$. However, $(2, 4) = (1, 2)$ in projective space $\mathbb{P}^1$. Our main takeaway is that the value of a non-homogeneous polynomial on a point in projective space is not necessarily well defined. Thus we cannot define a projective variety using non-homogenous polynomials.

We have now gone over our preliminary material, and are now ready to begin tackling the Grassmannian and all of its related constructions.

3 The Grassmannian and Young Diagrams

If we think of projective n-space as the set of all lines through the origin, or 1-dimensional subspaces, in some affine space of dimension $n+1$, then it helps to look at the Grassmannian as a generalization of this concept.

Definition 4. The Grassmannian, $Gr(n, k)$, is the set of all k-dimensional subspaces of $\mathbb{C}^n$. Elements in $Gr(n, k)$ are referred to as the "points" of $Gr(n, k)$.

From this we see that $\mathbb{P}^n = Gr(n + 1, 1)$. Now notice that every point in the Grassmannian can be described as the span of a full rank $k \times n$ matrix.

Example. The matrix:

$$\begin{bmatrix} 0 & 3 & -4 & 8 & -2 & 5 \\ 0 & 1 & 5 & -3 & -4 & 2 \\ 0 & 0 & 0 & 7 & 2 & -1 \end{bmatrix}$$

is a point in $Gr(6, 3)$.

Also notice that we can also apply elementary row operations on the matrix without changing the point of the Grassmannian it corresponds to. Thus every point in the Grassmannian can be reduced to some row echelon form. In this paper, we will use the conventions from [2], which have pivot points be in order from left to right and bottom to top. We illustrate the concept and notation with an example.

Example. Our matrix above has the reduced row echelon form

$$\begin{bmatrix} 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 1 & 0 & * & * \\ 0 & 1 & 0 & 0 & * & * \end{bmatrix}$$

where the $*$ entries are certain complex numbers.

Let us summarize our observations regarding points in the Grassmannian.

Remark 5. It follows that each point of $Gr(n, k)$ can be represented by a unique full rank $k \times n$ matrix in reduced row echelon form.

We will now introduce the Schubert cell, which will allow us to begin to state many of our intersection problems in terms of intersections of varieties. Along side it we will develop the concept of a Young diagram as it connects to elements in the Grassmannian. We begin with introducing some useful constructions for grouping together and identifying elements in the Grassmannian.

Definition 6. A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a sequence of weakly decreasing non-negative integers. We denote the size of the partition by $|\lambda| = \sum_{i=1}^{k} \lambda_i$.

By setting each λ_i to be the distance from the pivot points in row i of the reduced echelon matrix representing a point $V \in Gr(n,k)$ to the edge of a $k \times k$ staircase cut from the upper left corner of the matrix we are able to associate a partition to point V in the Grassmannian.

Example. The partition $\lambda = (4,3,1)$ is associated with the following element in $Gr(7,3)$:

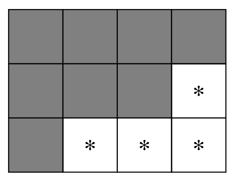
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 1 & * & * & 0 & * & 0 \end{bmatrix}$$

.

Now we introduce Young diagrams, which are important to the development of Schubert cells, as they provide an environment that we can pick partitions from that will be compatible with the constructions we will introduce.

Definition 7. The Young Diagram of partition λ is the left aligned partial grid of boxes contained within an $k \times (n-k)$ rectangle such that the i-th row from the top has λ_i boxes. The larger $k \times (n-k)$ rectangles is called the ambient rectangle, and is denoted as B.

Example. For $\lambda = (4,3,1)$ we obtain the following Young diagram:



We are now ready to introduce the Schubert cell, which provides a lot of insight into the structure of the Grassmannian that we can use to solve intersection problems.

Definition 8. For a partition λ contained in an ambient rectangle B, the Schubert cell, denoted as Ω_λ°, is the set of points in $Gr(n,k)$ whose row echelon matrix has a corresponding partition of λ. Formally,

$$\Omega_\lambda^\circ = \{V \in Gr(n,k) \mid \dim(V \cap \langle e_1, \ldots, e_r \rangle) = i \text{ for } n-k+i-\lambda_i \leqslant r \leqslant n-k+i-\lambda_{i+1} \text{ for all } i\}$$

Note that $\Omega_\lambda^\circ \cong \mathbb{C}^{k(n-k)-|\lambda|}$.

It is important to see that $n-k+i-\lambda_i$ corresponds to the location of the i-th pivot column in row i counted from the right. Similarly $n-k+i-\lambda_{i+1}$ corresponds to the location of the $i+1$-th pivot column in row i.

Before introducing Schubert varieties, we briefly discuss some properties of the Grassmannian. An important property of the Grassmannian is that it is a projective variety, which we can show by use of the Plücker embedding, which allows us to associate points in $\mathbb{P}^{\binom{n}{k}-1}$ to elements of the Grassmannian $Gr(n,k)$. For the following our treatment follows that of [5] with some small changes to notation. To get a point in $\mathbb{P}^{\binom{n}{k}-1}$ from a point in $Gr(n,k)$, consider an ordering on the k-element subsets S of $\{1,2,\ldots,n\}$. We will use this to label the homogeneous coordinates x_S of a point in $\mathbb{P}^{\binom{n}{k}-1}$. Let $V \in Gr(n,k)$ which we recall can be represented as a $k \times n$ matrix. We define x_S to be the determinant of the $k \times k$ submatrix (or minor) whose columns were determined by the elements of S. These determine a valid point in $\mathbb{P}^{\binom{n}{k}-1}$ since row operations only change determinants by a constant factor, and the coordinates cannot all be zero since the matrix V has rank k. To understand this process lets work through an example.

Example. Consider the matrix in $Gr(4,2)$:

$$\begin{bmatrix} 0 & 1 & 3 & -2 \\ 1 & 0 & -1 & 4 \end{bmatrix}$$

. In this case, the determinants of all 2×2 minors in the matrix give a point in $\mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$. Let us say the columns are labeled from left to right as $1, 2, 3$, and 4. We will pick an arbitrary ordering of these 2×2 minors, say $(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34})$. These are the determinants of the minors with columns i and j. The Plücker embedding then gives us the points $(-1, -3, 2, -1, 4, 10)$

The image of $Gr(n, k)$ in $\mathbb{P}^{\binom{n}{k}-1}$ is a projective variety and is cut out by the set of polynomials

$$\sum_{l=1}^{k+1} (-1)^l x_{i_1,\dots,i_{k-1},j_l} x_{j_1,\dots,j_{l-1},j_{l+1},\dots j_{k+1}}$$

We will not provide a rigorous proof, but we will work through an example.

Example. Revisiting our example from above, we see that it is in fact that case that

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = -10 - (-12) + (-2) = 0$$

as desired.

In other texts, the Plücker embedding is often described using more advanced concepts. See [1] or [4] for a more complete treatment of the Plücker embedding. With these details we can now define Schubert varieties as closed subvarieties of the Grassmannian.

Definition 9. The standard Schubert variety corresponding to partition λ, denoted Ω_λ, is the set

$$\Omega_\lambda = \{V \in Gr(n, k) \mid \dim(V \cap \langle e_1, \dots, e_{n-k+i-\lambda_i}\rangle) \geqslant i \text{ for all } i\}$$

Again, $n - k + i - \lambda_i$ is the location of the pivot column in the i-th row.

To gain a better understanding of what this definition means, let us work through an example.

Example. Let $\lambda = (2)$. Consider the Schubert variety Ω_λ in $\mathbb{P}^5 = Gr(6, 1)$. The ambient rectangle is in this case a 1×5 row of boxes and our condition for and element $V \in \Omega_\lambda$ is that $\dim(V \cap \langle e_1, e_2, e_3, e_4\rangle) \geqslant 1$. Since V is a 1-dimensional subspace of $\mathbb{C}^6$, this means that V must be contained in $\langle e_1, e_2, e_3, e_4\rangle$. Expressed in homogeneous coordinates, the first two entries must be 0. Thus we find that each point of Ω_λ can be written in one of the following ways:

$$\begin{gathered}(0, 0, 1, *, *, *)\\(0, 0, 0, 1, *, *)\\(0, 0, 0, 0, 1, *)\\(0, 0, 0, 0, 0, 1)\end{gathered}$$

From this we can see that it is in fact possible to rewrite Ω_λ as a disjoint union of Schubert cells.

In the previous two concepts we have introduced, we limit ourself to intersection with the span of the standard basis. To work with Schubert cells and varieties in more general settings, it is important to introduce the notion of a flag, which will be the last topic we will discuss in this section.

Definition 10. A complete flag, F, in $\mathbb{C}^n$ is a chain of subspaces

$$\{0\} = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n$$

Where $\dim(F_i) = i$.

We now define some special properties that a pair of flags can possess, and show that some nice results follow in the case where that property holds.

Definition 11. Two subspaces V and W of $\mathbb{C}^n$ are transverse if

$$\dim(V \cap W) = \max(0, \dim(V) + \dim(W) - n)$$

Moreover, two flags F^1 and F^2 are transverse if every pair of subspaces F_i^1 and F_j^2 are transverse.

It is in fact possible to show that two flags are transverse using a weaker condition.

Lemma 12. *Two complete flags $F, E \subset \mathbb{C}^n$ are transverse if and only if $F_{n-i} \cap E_i = \{0\}$.*

Proof. If two complete flags $F, E \subset \mathbb{C}^n$ are transverse, then $\dim(F_{n-i}) = n - i$ and $\dim(E_i) = i$ by definition of complete flags. Thus we have that $\dim(F_{n-i}) + \dim(E_i) - n = n - i + i - n = 0$ and since $\dim(F_{n-i} \cap E_i) = \dim(\{0\}) = 0$ we find that it is the case that $\dim(F_{n-i} \cap E_i) = \max(0, \dim(F_{n-i}) + \dim(E_i) - n) = 0$. Now suppose that for two complete flags $F, E \subset \mathbb{C}^n$ we have that $F_{n-i} \cap E_i = \{0\}$ for all i. We proceed by induction on n. When $n = 1$, first note that then each flag is a chain of two subspaces, those being $\{0\}$ and $\mathbb{C}$. Thus $F_i \cap E_j = \{0\}$ for i and j not both 1. In this case, the value $i + j - n$ is less than or equal to 0. Thus the dimension condition in Definition 11 is satisfied. In the case $i = j = 1$, we have $F_1 \cap E_1 = \mathbb{C} \cap \mathbb{C} = \mathbb{C}$ and $i + j - n = 1$. Thus the dimension condition is satisfied for all cases. Now, for induction we assume the following:

1. If $F_{n-i-1} \cap E_i = \{0\}$ for all i, then F and E are transverse flags.
2. $F_{n-i} \cap E_i = \{0\}$ for all i.

In particular, we notice that $F_{n-1} \cap E_1 = \{0\}$ and $\mathbb{C}^n = F_n$, so it follows that $F_n = F_{n-1} \oplus E_1$. Now we will quotient both flags by E_1, which will reduce F_n to F_{n-1} and reduce the dimension of each E_i by 1. This will get us a new pair of flags

$$E' : \{0\} = E_1/E_1 \subset E_2/E_! \subset \cdots \subset E_n/E_1$$
$$F' : \{0\} = F_0 \subset F_1 \subset \cdots \subset F_{n-1}$$

Since $F'_{n-i-1} \cap E'_i = F_{n-(i+1)} \cap E_{i+1}/E_1$. Since $F_{n-i-1} \cap E_{i+1} = \{0\}$ by our second part of our inductive hypothesis, it follows that $F'_{n-i-1} \cap E'_i = \{0\}$. By the first part of our inductive hypothesis, it follows that F' and E' are transverse flags. More precisely,

$$\dim(F_i \cap E_j/E_1) = \max(0, i + j - 1 - (n - 1)) = \max(0, i + j - n)$$

Now, for any $i \neq n$ we know that $E_1 \not\subset F_i$, so

$$\dim(F_i \cap E_j) = \dim(F_i \cap E_j/E_1) = \max(0, i + j - n).$$

If $i = n$ and $j \neq 0$ then $E_1 \subset F_n$ so we have

$$\dim(F_n \cap E_j) = 1 + \dim(F_{n-1} \cap E_j/E_1) = 1 + \max(0, n - 1 + j - n) = 1 + j - 1 = j.$$

Finally we have the case where $i = n$ and $j = 0$. This is trivial since $F_n \cap E_0 = \{0\}$. Therefore, for all i and j it is the case that

$$\dim(F_i \cap E_j) = \max(0, i + j - n).$$

Thus we conclude that F and E are transverse. □

Let us explore flags a little farther. While flags do allow us to generalize a lot of notions, there are still specific flags that we will focus on for many of the results in this paper.

Definition 13. The standard F is defined to be the flag in which $F_i = \langle e_1, \ldots, e_i \rangle$ and the opposite flag E by $E_i = \langle e_n, \ldots, e_{n-i+1} \rangle$.

Let us briefly show that these two flags are transverse. $F_{n-i} = \langle e_1, \ldots, e_{n-i} \rangle$ and $E_i = \langle e_n, \ldots, e_{n-i+1} \rangle$. We see that $F_{n-i} \cap E_i = \{0\}$, thus the standard and opposite flags are transverse.

Lemma 14. *For any pair of transverse flags F' and E', there is an element in the general linear group $g \in GL_n$ such that $gF' = F$ and $gE' = E$ where F and E are the standard and opposite flags respectively.*

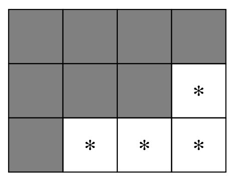
We won't go into a rigorous proof, but will provide some intuition. Let F denote the standard basis and V denote some other complete flag. Pick a basis for V so that $V_k = \langle v_1, \ldots, v_k \rangle$. Consider the matrix $g \in GL_k$ whose columns are the vectors v_i. This matrix will take F_k to V_k. Therefore, when considering intersections of Schubert varieties it suffices to do all of our computations using the standard and opposite flags, and then multiple the results by the appropriate $g \in GL_n$.

For future sections, the definition of standard and opposite flags will be useful to have available, since much of the logic hinges on noticing and manipulating the indices in the basis elements that span these flags.

With all of that, we are now ready to begin introducing our first major tool for computing intersections of Schubert varieties and use it to solve a simple problem. First we start by introducing some relationships between partitions.

Definition 15. Two partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k$ are said to be complementary in the $k \times (n-k)$ ambient rectangle if and only if $\lambda_i + \mu_{k+1-i} = n-k$ for all i. If λ and μ are complementary, we write $\mu^c = \lambda$.

Example. We see that the complement of $\mu = (4,3,1)$ is $\lambda = (3,1)$



We can develop some geometric intuition for this concept by noticing that if we rotate the Young diagram of λ and place it in the lower right corner of the ambient rectangle, its complement is the partition μ formed by the space remaining in the ambient rectangle.

Theorem 16. *(Duality Theorem). Let F and E be transverse flags in $\mathbb{C}^n$, and let λ and μ be partitions with $|\lambda| + |\mu| = k(n-k)$. In $Gr(n,k)$, the intersection $\Omega_\lambda(F) \cap \Omega_\mu(E)$ has 1 element if λ and μ are complementary partitions, and is empty otherwise. Moreover, if λ and μ are any partitions with $\mu_{k+1-i} + \lambda_i > n-k$ for some i then the intersection is empty.*

Proof. Let us start by proving the second statement. Suppose that for some i it is the case that $\mu_{k+1-i} + \lambda_i > n-k$. Assume, for sake of contradiction, that there is a subspace $V \in \Omega_\lambda(F) \cap \Omega_\mu(E)$. We know that $\dim(V) = k$ and that

$$\dim(V \cap \langle e_1, e_2, \ldots, e_{n-k+i-\lambda_i} \rangle) \geqslant i \tag{1}$$

$$\dim(V \cap \langle e_n, e_{n-1}, \ldots, e_{n+1-(n-k+(k+1+i)-\mu_{k+1-i}} \rangle) \geqslant k+1-i \tag{2}$$

By simplifying 2 and reversing the order of the generators we get

$$\dim(V \cap \langle e_{i+\mu_{k+1-i}}, \ldots, e_{n-1}, e_n \rangle) \geqslant k+1-i \tag{3}$$

Because $\mu_{k+1-i}+\lambda_i > n-k$ it follows that $i+\mu_{k+1-i}+\lambda_i > i+n-k$ and that $i+\mu_{k+1-i} > i+n-k-\lambda_i$. Therefore, the two subspaces we are intersecting with V in (1) and (3) are disjoint. Thus it follows that the dimension of V is at least $k+1-i+i = k+1$ which is a contradiction. Now we can continue to the first statement. If $|\lambda| + |\mu| = k(n-k)$ but λ and μ are not complementary, then $\mu_{k+1-i} + \lambda_i > n-k$ for some i. Finally, suppose that λ and μ are complementary. Then (1) and (2) still hold, but now we have the added constraint that $n-k+i-\lambda_i = i+\mu_{n+1-i}$ for all i. Therefore $\dim(V \cap \langle e_{e_i+\mu_{n+1-i}} \rangle) = 1$ for all $i = 1, \ldots, k$. Since $\dim(V) = k$ it then must equal the span of these basis elements

$$V = \langle e_{1+\mu_n}, e_{2+\mu_{n-1}}, \ldots, e_{k+\mu_{n+1-k}} \rangle$$

which is our unique solution. □

We end this section by applying our new techniques to prove that there is a unique line passing through two distinct points in $\mathbb{P}^n$. Working in $Gr(n+1,2)$, these two distinct points become two distinct 1-dimensional subspaces F_1 and E_1 in $\mathbb{C}^{n+1}$. The Schubert conditions (the properties of the Schubert variety) tells us that the 2-dimensional subspace that contains them must satisfy

$$\dim(V \cap F_1) \geqslant 1 \text{ and } \dim(V \cap E_1) \geqslant 1.$$

These are the conditions for a partition $\lambda = (\lambda_1)$ where $(n+1) - 2 + 1 - \lambda_1 = 1$. We see this means that $\lambda_1 = n-1$. Thus, we see that we are intersecting the following Schubert varieties

$$\Omega_{(n-1)}(F) \cap \Omega_{(n-1)}(E)$$

where F and E are any two transverse flags extending F_1 and E_1 respectively. Notice that $\lambda = (n-1)$ complements itself in the $2 \times (n-1)$ ambient rectangle. Thus by the Duality Theorem there exists a unique point in the intersection. i.e There is a unique line connecting two distinct points in $\mathbb{P}^n$. The next step in developing this problem solving framework is to revisit the Schubert cell and the structure it provides to the Grassmannian.

4 Cohomology and Symmetric Functions

Let us first review some preliminary definitions regarding homology and cohomology, mostly revolving around CW-complexes and their relation to the Grassmannian. Our treatment will mostly draw from [3] with additional remarks regarding projective space and the Grassmannian. Computations with the Grassmannian involving its cohomology groups are best done using its cellular homology and cohomology. Therefore we will begin by developing a CW-complex structure on the Grassmannian.

Definition 17. An n-cell is a topological space homeomorphic to the unit open ball in $\mathbb{R}^n$. Sometimes called an open n-disk. An n-disk is the closure of the unit open ball in $\mathbb{R}^n$

Definition 18. We construct a CW-complex using the following procedure:

1. Begin with a discrete set X^0, called the 0-skeleton, whose points are regarded as 0-cells.
2. Inductively form the n-skeleton, X^n, from X^{n-1} by attaching the n-cells e^n_α through the maps $\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$ where D^n_α is a collection of n-disks. We will do this by setting X^n as the quotient space of the disjoint union $X^{n-1} \bigsqcup_\alpha D^n_\alpha$ under the relation $x \sim \varphi_\alpha(x)$ for all $x \in \partial D^n_\alpha$. Thus the n-skeleton $X^n = X^{n-1} \bigsqcup_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.
3. The CW-complex is $X = \bigcup_n X_n$, which could just be $X = X^n$ if the inductive process stops at some $n < \infty$.

For this paper, we are not too concerned with the specific definitions of the boundary maps. It is more important to us that we know they exist and what some of their properties are.

Example. Let us consider the cell structure of the projective plane $\mathbb{P}^2$. Let $X^0 = \{(0,0,1)\}$. Then we attach a 2-cell, a copy of $\mathbb{C}$, similar to a balloon in order to get X^2. Finally, we attach a 4-cell, a copy of $\mathbb{C}^2$, to get X^4. Imagine higher dimensional balloons.

It is important to realize that Schubert cells give a cell complex structure on the Grassmannian. We will give a brief sketch of the proof. Define X_0 to be the 0-dimensional Schubert variety Ω_B, where B is the ambient rectangle, and Ω_B is the partition that corresponds to the partition that covers all of B. Because we are working over $\mathbb{C}$ there are no cells of odd dimension. Next we have $X^2 = X^0 \cup \Omega^\circ_{\lambda^1}$ where $\lambda^1 = (n-k, n-k, \ldots, n-k-1)$ is obtained by removing the bottom right corner from the ambient rectangle. The closure of $\Omega^\circ_{\lambda^1}$ is $\Omega^\circ_B = X^0$, so the closure maps the boundary Ω°_B of X^2 to X^0 see [8] for a full construction. We can continue and form X^4 by attaching the two 4-cells given by removing the two outer corner squares in both possible ways from λ^1. By continuing like this, we see that the $2m$-th cell is formed by attaching Schubert cells with partition size $|\lambda| = k(n-k) - m$. From this, we obtain the CW decomposition of the Grassmannian into cells

$$X^0 \subset X^2 \subset \cdots \subset X^{2k(n-k)}.$$

Example. The Grassmannian $Gr(4,2)$ has the cellular decomposition

$$Gr(4,2) = \Omega^\circ_{(2,2)} \cup \Omega^\circ_{(2,1)} \cup \Omega^\circ_{(2,0)} \cup \Omega^\circ_{(1,1)} \cup \Omega^\circ_{(1,0)} \cup \Omega^\circ_{(0,0)}$$

With $X^0 = \Omega^\circ_{(2,2)}$, X^2 is formed by attaching $\Omega^\circ_{(2,1)}$ to X^0, X^4 by attaching $\Omega^\circ_{(2,0)} \cup \Omega^\circ_{(1,1)}$ to X^2, X^6 by attaching $\Omega^\circ_{(1,0)}$ to X^6, and finally X^8 by attaching $\Omega^\circ_{(0,0)}$ to X^6.

Definition 19. For a CW complex $X = X^0 \subset \cdots \subset X^n$ let

$$C_k = \mathbb{Z}^{\# k-\text{cells}}$$

be the free abelian group generated by the k-cells $B^{(k)}_\alpha = (D^{(k)}_\alpha)^\circ$. The cellular boundary map $d_{k+1} : C_{k+1} \to C_k$ as

$$d_{k+1}(B^{(k+1)}_\alpha) = \textstyle\sum_\beta \deg_{\alpha\beta} \cdot B^{(k)}_\alpha,$$

where $\deg_{\alpha\beta}$ is the degree of the composite map

$$\partial \overline{B^{(k+1)}_\alpha} \to X^k \to \overline{B^{(k)}_\beta}.$$

The first map above is the cellular attaching map from the boundary of the closure of the ball $\overline{B^{(k+1)}_\alpha}$ to the k-skeleton, and the second map is the quotient map formed by collapsing $X^k \backslash B^{(k)}_\beta$ to a point. This composite map from a k-sphere to another k-sphere has a degree. See [3], section 2.2, p. 134.

The cellular boundary maps make the groups C_k into a chain complex, a sequence of maps

$$0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$$

for which $d_i \circ d_{i+1} = 0$ for all i. Because of this property we can consider the following quotient groups

$$H_i(X) = \ker(d_i)/\operatorname{Im}(d_{i+1})$$

for all i. These are abelian groups and are called the cellular homology groups of the space X.

Example. Let us revisit $\mathbb{P}^2$. Its decomposition consisted of a point, a 2-cell, and a 4-cell. Thus its cellular chain complex can be written as

$$\cdots \to 0 \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0$$

and the homology groups are $H_0 = H_2 = H_4 = \mathbb{Z}$ and $H_1 = H_3 = 0$

From this, we can swiftly define cellular cohomology by dualizing the chain complex above.

Definition 20. Let $C^k = \hom(C_k, \mathbb{Z})$ for each k. The boundary maps $d^*_k : C^{k-1} \to C^k$ are defined as

$$d^*_k f(c) = f(d_k(c))$$

for any $f \in C^k$ and $c \in C_k$. These coboundary maps form a cochain complex, from which we define the cohomology groups as

$$H^i(X) = \ker(d^*_{i+1})/\operatorname{Im}(d^*_i)$$

for all i. Finally, we have the direct sum of the cohomology groups

$$H^*(X) = \bigoplus_i H^i(X)$$

which has a ring structure when equipped with the cup product, the dual of intersections of cycles in homology. In this setting it roughly corresponds to intersections of cohomology classes.

Definition 21. Recall that a graded ring is a ring S together with a set of subgroups S_d, $d \geqslant 0$ such that $S = \bigoplus_{d \geqslant 0} S_d$ as an abelian group, and $st \in S_{d+e}$ for all $s \in S_d, t \in S_e$.

The is an equivalent definition of cohomology in the Grassmannian known as the Chow ring. While we will not go into it in this paper, the following theorem makes use of some properties made apparent by this interpretation.

Theorem 22. *The cohomology ring $H^*(Gr(n,k))$ has a $\mathbb{Z}$-basis given by the classes*

$$\sigma_\lambda := [\Omega_\lambda(F)] \in H^{2|\lambda|}(Gr(n,k))$$

for λ a partition fitting inside the ambient rectangle. The cohomology $H^*(Gr(n,k))$ is a graded ring. i.e $\sigma_\lambda \cdot \sigma_\mu \in H^{2|\lambda|+2|\mu|}(Gr(n,k))$ and we have

$$\sigma_\lambda \cdot \sigma_\mu = [\Omega_\lambda(F) \cap \Omega_\mu(E)]$$

where F and E are the standard and opposite flags. We remark that σ_λ is independent of the choice of flag F since any two Schubert varieties of the same partition shape are equivalent under change of basis.

Now consider the following. Let $\lambda^1, \ldots, \lambda^m$ be partitions such that $\sum_{i=1}^m |\lambda_i| = k(n-k)$. Because the cohomology ring is graded, the product $\sigma_{\lambda^1} \cdot \sigma_{\lambda^2} \cdots \sigma_{\lambda^m} \in H^{2k(n-k)}(Gr(n,k))$. However, the only generator of this cohomology group is σ_B, which is the class of a single point $\Omega_B(F)$, where B is the ambient rectangle. Therefore the intersection of the Schubert varieties corresponding to the λ^i for m generic flags is a finite union of points. In particular, the number of points is the coefficient c in the product

$$\sigma_{\lambda^1} \cdot \sigma_{\lambda^2} \cdots \sigma_{\lambda^m} = c^B_{\lambda^1,\ldots,\lambda^m} \sigma_B$$

In order to learn how to calculate these coefficients, we need to introduce an important type of symmetric function, the Schur functions, and develop a connection between them, the Young tableau, and elements in the cohomology ring.

Definition 23. The ring of symmetric functions $\Lambda_{\mathbb{C}}(x_1, x_2, \ldots)$ is the ring of bounded degree formal power series $f \in \mathbb{C}[[x_1, x_2, \ldots]]$ which are symmetric under permuting variables. More formally,

$$f(x_1, x_2, \ldots) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots)$$

for any permutation $\pi : \mathbb{Z}^+ \to \mathbb{Z}^+$ and $\deg(f) < \infty$.

We will be interested in a specific class of symmetric functions, called Schur functions, which we will define using the Young tableau. First we introduce some ways to modify partitions.

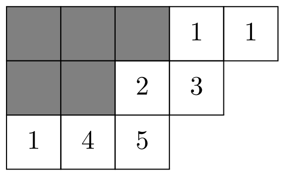
Definition 24. A skew shape is the difference, υ/λ, formed by removing the Young diagram of a partition λ from a strictly larger partition υ. A skew shape is called a horizontal strip if no column contains more than one box.

Now we introduce the semistandard Young tableau, which is essentially a type of skew shape combined with a sequence of integers called the contents that have certain conditions imposed upon them.

Definition 25. A semistandard Young tableau of a skew shape υ/λ is a filling of the boxes of the Young diagram of shape υ/λ with positive integers such that within each row the integers weakly increase from left to right and within each column the integers strictly increase from top to bottom. The content of a semistandard Young tableau is denoted $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ where μ_i denoted the number of boxes labeled with integer i. The reading word of the tableau is the word formed by concatenating the rows from bottom to top.

Remark 26. We can also consider the semistandard Young Tableau of a single partition υ. Just consider the definition using the skew shape, but set the second partition λ as a partition with entries all 0.

Example. The following is a semistandard Young Tableau of shape υ/λ and content μ where $\upsilon = (5, 4, 2)$, $\lambda = (2, 2, 0)$ and $\mu = (3, 1, 1, 1, 1)$.



In this case, the reading word of the semistandard Young Tableau is 1452311.

Now that we have defined skew shapes and semistandard Young Tableaux, we have enough background knowledge to define the Schur functions and begin discussing their relationship to cohomology classes.

Definition 27. Given a semistandard Young tableau T of skew shape υ/λ. The Schur function for skew shape υ/λ is given by

$$s_{\upsilon/\lambda} = \textstyle\sum_T x^T$$

where the sum ranges over all possible semistandard Young Tableau of skew shape υ/λ T and $x^T = x_1^{m_1} x_2^{m_2} \ldots$ where m_i is the number of occurrences of the integer i in T. In the case that λ is empty, we say $s_{\upsilon/\lambda} = s_\upsilon$ is the Schur function of shape υ.

An important aspect of Schur functions is that they are symmetric functions as well. For completeness and to better understand Schur functions let us prove this result.

Proposition 28. *For any skew shape υ/λ the Schur function $s_{\upsilon/\lambda}$ is symmetric.*

Proof. Recall that any element of the symmetric group S_n can be rewritten as a product of transpositions. Thus it suffices to show that $s_{\upsilon/\lambda}$ is invariant under the transposition $(i, i+1) \in S_n$. Suppose that υ/λ has size n. This means that $\sum_i \upsilon_i - \lambda_i = n$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a sequence of non-negative integers such that there exists a semistandard Young tableau of shape υ/λ and content α. Define $\alpha' = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_k)$ by permuting α_i and α_{i+1} in α. By constructing a bijection $\varphi : T^{\alpha}_{\upsilon/\lambda} \to T^{\alpha'}_{\upsilon/\lambda}$ from the set of semistandard Young tableaux of shape υ/λ and content α to the semistandard Young tableaux of shape υ/λ and content α'. Consider $T \in T^{\alpha}_{\upsilon/\lambda}$. Since the numbering of the cells of T must strictly increase down columns, we will disregard any column in T that contain both i and $i+1$, as well as columns that contain neither. The remaining columns we consider will have exactly one of i or $i+1$. These columns form a diagram whose rows contain some number of cells labeled i and some number of cells labeled $i+1$. In each row swap the i's for the $i+1$'s and vice versa. Then reorder the row to be weakly increasing. Repeat this process for each row. No cells are being introduced or removed, so this procedure doesn't change the overall shape of T. This process results in the tableau $\varphi(T) \in T^{\alpha'}_{\upsilon/\lambda}$. φ is injective since if $\varphi(T) = \varphi(S)$, then we simply reverse the process we applied to show that $T = S$. Similarly, we show that φ is surjective because for any tableau T of content α' and skew shape υ/λ applying φ twice gives T again, so $\varphi(T) \in T^{\alpha}_{\upsilon/\lambda}$ maps by φ to T. Thus φ is a bijection. $\square$

Example. For $\lambda = (2, 1)$, the tableaux

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \cdots$$

are some of the infinitely many semistandard Young tableaux of shape λ. Using this, we can write a couple of the terms of $s_\lambda = x_1^2x_2 + x_1x_2^2 + x_1^2x_2 + x_1x_3x_3 + \cdots$.

One last fact that we will use about Schur functions s_λ is that they form a basis for the vectors space $\Lambda(x_1, x_2, \ldots)$ as λ ranges over all partitions. We will not go into this proof since it requires some additional developments not useful for this paper. We direct the reader to [5] for a complete treatment. Accepting this fact, we are now ready to describe a very important relationship between the cohomology ring of the Grassmannian and a quotient of the ring of symmetric functions.

Theorem 29. *There is a ring isomorphism*

$$H^*(Gr(n,k)) \cong \Lambda(x_1, x_2, \ldots)/(s_\lambda \mid \lambda \not\subset B)$$

where B is the ambient rectangle and $(s_\lambda \mid \lambda \not\subset B)$ is the ideal generated by the Schur functions. The isomorphism sends the Schubert class σ_λ to the Schur function s_λ.

Now, the map that carries $\sigma_\lambda \to s_\lambda$ forms an isomorphism of the underlying vector spaces. Thus, in order to show that there is a ring isomorphism, it remains to show that such an isomorphism preserves the operations in each space. It should carry the cup product of cohomology classes, or the intersection of Schubert varieties, to the product of Schur functions. To do this, it suffices to show that both spaces satisfy the Pieri rule, which for Schur functions tells us how to multiply a one-row shape by any other partition and tells us about the intersection of Schubert varieties with certain partitions for cohomology. We will introduce both rules here.

Theorem 30. *Given a partition λ and a one-row shape (r) the product of the associated Schur functions is*

$$s_{(r)} \cdot s_\lambda = \textstyle\sum_{\upsilon} s_\upsilon$$

where the sum ranges over all partitions υ such that υ/λ is a horizontal strip of size r.

The proof of this theorem is quite technical and involved, so we will not discuss it in this paper. See [7] for a complete proof. We are, however, able to discuss the proof of the equivalent rule for the cohomology ring.

Theorem 31. *(Pieri Rule for Schubert Classes) Let λ and μ be partitions with size $|\lambda| + |\mu| = k(n-k) - r$. Let F and E be the standard and opposite flags, and H be a generic complete flag. Then the intersection*

$$\Omega_\lambda(F) \cap \Omega_\mu(E) \cap \Omega_{(r)}(H)$$

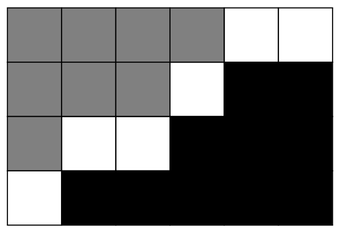
has one element if μ^c/λ has length r and no two boxes in the same column i.e is a horizontal strip, and is empty otherwise.

In order to prove this, we have some preliminary observations we need to cover first. Importantly, note that the matrices that correspond to λ and μ do not have pivots in the same position as was the case in the Duality Theorem. As a result there may be $*$ entries that occur in the matrix representations of Ω_λ and Ω_μ. The subspaces which occur in the intersection $\Omega_\lambda \cap \Omega_\mu$ are spanned by the rows of a matrix that has non-zero elements only in the intermediary sites between pivots. To see this, recall the initial definition of the Schubert variety and the condition required for elements to be included in it. Now notice that the condition that the diagrams for λ and μ do not intersect and that no two of the r boxes between them are contained in the same column is equivalent to the following condition:

$$n-k-\lambda_k \geqslant \mu_1 \geqslant n-k-\lambda_{k-1} \geqslant \cdots \geqslant n-k-\lambda_1 \geqslant \mu_k \geqslant 0 \tag{4}$$

To see this, just think about how this looks on a Young diagram. You will notice that this condition does indeed prevent two r boxes between λ and μ from being in the same column. Now, we set $A_i = F_{n-k+i-\lambda_i}$, $B_i = E_{n-k+i-\mu_i}$ and $C_i = A_i \cap B_{k+1-i}$. Let C be the span of all the C_i's. These are important as they will be used in solving some essential lemmas. To make sure we understand all this, let us go through an example.

Example. Consider the case in $Gr(10,4)$ with $\lambda = (4,3,1)$ and $\mu = (5,3,2)$. In the young diagram, we place λ in the top left and μ in the bottom right.



The white squares are precisely the skew shape μ^c/λ. Using this, and by recalling the definitions for standard and opposite flags, we get the following:

$$\begin{array}{ccc} A_1 = F_3 & B_4 = E_{10} & C_1 = \langle e_1, e_2, e_3\rangle \\ A_2 = F_5 & B_3 = E_7 & C_1 = \langle e_4, e_5\rangle \\ A_3 = F_8 & B_2 = E_5 & C_1 = \langle e_6, e_7, e_8\rangle \\ A_4 = F_{10} & B_1 = E_2 & C_1 = \langle e_{10}, e_9\rangle \end{array}$$

From this we can write the subspaces V in the intersection $\Omega_\lambda \cap \Omega_\mu$ as a span of the rows of a matrix of the form

$$\begin{bmatrix} * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

The following lemma will help us generalize the notions we have developed in the example and introduce some key properties of our constructions.

Lemma 32. *The following are true:*

a. $C = \bigcap_{i=1}^{k}(A_i + B_{k-i})$.

b. $\sum_{i=1}^{k} \dim(C_i) = k + r$.

c. The sum $C = C_1 + \cdots + C_k$ *is a direst sum of nonempty subspaces if and only if the inequalities in (4) hold.*

Proof. For the first part, it is important to notice that the basis vector e_m is in C exactly when it is in some C_j. Moreover, this occurs precisely when $j+\mu_{k+1-j} \leqslant m \leqslant n-k+j-\lambda_i$ for some j. This follows from the definitions for standard and opposite flags. Now we focus on $\bigcap_{i=1}^{k}(A_i + B_{k-i})$. For this, we notice that e_m is in $\bigcap_{i=1}^{k}(A_i+B_{k-i})$ when either $m \leqslant n-k+j-\lambda_j$ or $m > i+\mu_{k-i}$. For completeness, we will operate under the convention that $\lambda_0 = \mu_0 = n-k$ for the next section. We claim that these two conditions for e_m are in fact equivalent. To see this, suppose $j + \mu_{k+1-j} \leqslant m \leqslant n-k+j-\lambda_j$

for some j. Then for all $i < j$ we have $i + \mu_{k+1-j}$ and for all $i > j$ we have that $m \leqslant n - k + j - \lambda_j \leqslant n - k + i - \lambda_i$. Now, consider the smallest j such that $m \leqslant n + j - \lambda_j$. Since this is the smallest such j, it also follows that $m > (j-1) + \mu_{k-(j-1)}$, which is the condition we wanted. Since these conditions are equivalent to each other, it follows that $C = \bigcap_{i=1}^{k}(A_i + B_{k-i})$. For the second part, notice that $C_i = F_{n-k+i-\lambda_i} \cap E_{n-k+i-\mu_i}$ and is spanned by the vectors $e_{k-i+\mu_i+1}, \ldots, e_{n-k+i-\lambda_i}$. Therefore $\dim(C_i) = n - k + i - \lambda_i - (k - i + \mu_i + 1) + 1 = n - 2k + 2i - \lambda_i - \mu_i$. Taking the sum from $i = 1, \ldots, k$ we get

$$
\begin{aligned}
\sum_{i=1}^{k} \dim(C_i) &= \sum_{i=1}^{k} n - 2k + 2i - \lambda_i - \mu_i \\
&= kn - 2k^2 + 2\frac{k(k+1)}{2} - (|\lambda| + |\mu|) \\
&= kn - 2k^2 + k(k+1) - (k(n-k) - r) \\
&= k - r
\end{aligned}
$$

.

Finally, we move on to the last part of the lemma. First notice that if the inequalities in (4) fail then there is some column containing at least two stars, and the corresponding C_i's will intersect in a line, thus the sum will not be direct. If it is the case that the inequalities hold then the columns each contain at most one start, and so the intersection $C_i \cap C_j$ is trivial for all i and j. ☐

Using this lemma, we can then prove yet another lemma which will finally allow us to tackle the original theorem.

Lemma 33. *If $V \in Gr(n,k)$ is in the intersection $\Omega_\lambda \cap \Omega_\mu$, then $V \subset C$. If it is the case that $C_1, \ldots, C_k$ are linearly independent, then $\dim(V \cap C_i) = 1$ for all i and $V = \bigoplus_{i=1}^{k}(V \cap C_i)$*

Proof. By the previous lemma, it suffices to show that $V \subset A_i + B_{k-i}$. If $A_i \cap B_{k-i} \neq \{0\}$ then $A_i + B_{k-i} = \mathbb{C}^n$ so $V \in A_i + B_{k-i}$ trivially. Then suppose that $A_i + B_{k-i} = \{0\}$. Since $V \subset \Omega_\lambda \cap \Omega_\mu$, we know that $\dim(V \cap A_i) \geqslant i$ and $\dim(V \cap B_{k-i})$. Since V is k-dimensional, $V = (V \cap A_i) \oplus (V \cap B_{k-i})$. Now we further assume that all of the C_i's are linearly independent. We know that $\dim(V \cap C_i) \geqslant 1$, since $V \cap A_i$ and $V \cap B_{k+1-i}$ have dimensions of at least i and $k + 1 - i$ respectively, so that A_i and B_{k+1-i} intersect non trivially in V. Now because the C_i's are independent, V contains the direct sum $\bigoplus_{i=1}^{k}(V \cap C_i)$ which has at least dimension k. Since $\dim(V) = k$, we have that $V = \bigoplus_{i=1}^{k}(V \cap C_i)$ and each summand must be of dimension 1. ☐

We will now prove the Pieri rule for Schubert classes by making use of the two previous results.

Proof. (Pieri Rule for Schubert Classes) If the inequalities in (4) fail, then by part (c) of Lemma 32 the space C is not a direct sum of the C_i's, and by part (b) of the same lemma, its dimension is at most $r_k - 1$. In this case, a general space H of dimension $n - k + 1 - r$ will intersect C trivially, as $(r + k - 1) + (n - k + 1 - r) = n$ and we are in $\mathbb{C}^n$. Therefore no $V \subset \Omega_\lambda \cap \Omega_\mu$ is in $\Omega_{(r)}$. Thus the intersection of all three varieties is empty. Now, if the inequalities do hold, then $C = \bigoplus_{i=1}^{k} C_i$ and a generic subspace L of dimension $n - k + 1 - r$ will intersect C in a line. Since C decomposes as a sum of C_i, the line can be rewritten as the span of the vector $v = u_1 \oplus \ldots \oplus u_k$ with $u_i \in \mathbb{C}^n/\{0\}$. Since we are considering the subspaces $V \subset C$ that intersect L in at least a line, it follows that $v \in V$. Writing $V = \bigoplus_{i=1}^{k}(V \cap C_i)$ we see that $u_i \in V$ for all i, so $V = \langle u_1, \ldots, u_k \rangle$. It is important to notice that this point is unique, thus the intersection $\Omega_\lambda(F) \cap \Omega_\mu(E) \cap \Omega_{(r)}(L)$ contains a single point. ☐

Now that we have proved the Piere rule holds for cohomology classes, we can combine this with the corresponding results for Schur functions, and fully establish that there does indeed exist a ring isomorphism between the cohomology ring and Schur functions in some ambient rectangle. Now, in order to solve intersection problems, we require one more construction, that being the Littlewood Richardson tableau. We first introduce some important terminologies and definitions that we will use to describe these tableaux.

Definition 34. A word $w_1 w_2 w_3 \cdots w_n$ where each $w_i \in \{1, 2, \ldots\}$ is Yamanouchi, also lattice or ballot, if every suffix $w_k w_{k+1} \cdots w_n$ contains at least as many letters equal to i as $i+1$ for all i.

Definition 35. A Littlewood Richardson Tableau is a semistandard Young tableau whose reading word is Yamanouchi

The following is an important construction and will be helpful for future computations.

Definition 36. A sequence of skew tableaux $T_1, T_2, \ldots$ form a chain if their shapes do not overlap and

$$T_1 \cup T_2 \cup \cdots \cup T_i$$

is a partition shape for all i.

Going forwards, we will utilize an important property of Schur functions which is known as the Littlewood Richardson rule.

Theorem 37. *Let $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)}$, the product of their corresponding Schur functions can be written in the basis of Schur functions via the formula*

$$s_{\lambda^{(1)}} s_{\lambda^{(2)}} \cdots s_{\lambda^{(m)}} = \sum_{\upsilon} c^{\upsilon}_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(m)}} s_{\upsilon}$$

where $c_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(m)}}$ is the number of chains of Littlewood Richardson tableaux of contents $\lambda^{(i)}$ with total shape υ.

While we will not prove the general theorem given that such a proof is quite involved, we will direct the reader to the following proofs. [1] contains a combinatorial approach while [5] opts for a different method that involves some extra setup that we have not covered in this paper. We will however, prove a commonly used corollary using this theorem. This treatment follows what is presented in [2].

Corollary 38. *Given partitions λ and μ, the product of the corresponding Schur functions can be written in the basis of Schur functions via the formula*

$$s_\lambda s_\mu = \sum_{\upsilon} c^{\upsilon}_{\lambda,\mu} s_{\upsilon}$$

where $c^{\upsilon}_{\lambda,\mu}$ is the number of Littlewood-Richardson tableaux of skew shape υ/λ and content μ

Proof. By the above theorem, $c^{\upsilon}_{\lambda\mu}$ is the number of chains of two Littlewood Richardson tableaux of content λ and μ with total shape υ. The first tableau of content λ is a straight shape tableau, so by the Yamanouchi reading word condition and the semistandard condition, the top row can only contain 1's. Continuing this reasoning inductively, it has only i's in its i-th row for each i. Therefore the first tableau in the chain is the unique tableau of shape λ and content λ. Thus the second tableau is the Littlewood Richardson tableau of shape υ/λ and content μ, which leads to our result. □

Now we can finally relate this all back to cohomology classes and intersections of varieties. First we use our results about the relationship between the cohomology ring and Schur functions to obtain some properties about multiplication of elements in the cohomology ring.

Theorem 39. *In $H^*(Gr(n,k))$ we have*

$$\sigma_{\lambda^{(1)}} \cdots \sigma_{\lambda^{(m)}} = \sum_{\upsilon} c^{\upsilon}_{\lambda^{(1)},\ldots,\lambda^{(m)}} \sigma_{\upsilon}$$

where the sum is restricted to partitions υ fitting in the ambient rectangle.

Proof. By the general Pieri formula for Schur functions,

$$s_{\lambda^{(1)}} s_{\lambda^{(2)}} \cdots s_{\lambda^{(m)}} = \sum_{\upsilon} c^{\upsilon}_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(m)}} s_{\upsilon}$$

where $c^{\upsilon}_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(m)}}$ is the number of chains of Littlewood Richardson tableaux of contents $\lambda^{(i)}$ with total shape υ. Because of the isomorphism between Schur functions and the cohomology ring, we get our result. □

And finally, we introduce the theorem that will let us calculate the answer to some of our linear intersection problems, know as the Zero-Dimensional Littlewood Richardson Rule.

Theorem 40. *(Zero-Dimensional Littlewood Richardson Rule) Let B be the $k \times (n-k)$ ambient rectangle and let $\lambda^{(1)}, \ldots, \lambda^{(m)}$ be partitions fitting inside B such that $|B| = k(n-k) = \sum_i |\lambda_i|$. Also let $F^{(1)}, \ldots, F^{(m)}$ be any m generic flags. Then*

$$c^{B}_{\lambda^{(1)},\ldots,\lambda^{(m)}} := |\Omega_{\lambda^{(1)}}(F^{(1)}), \ldots, \Omega_{\lambda^{(m)}}(F^{(m)})|$$

is equal to the number of chains of Littlewood Richardson tableaux of contents $\lambda^{(1)}, \ldots, \lambda^{(m)}$ with total shape equal to B.

In this last section, we will bring together all that we have covered so far to approach the following problem. We want to know that how many lines intersect 4 given lines in 3-dimensional space. The Schubert variety $\Omega_{(1,0)}(F) \subset Gr(4,2)$ consists of the 2-dimensional subspaces V of $\mathbb{C}^4$ for which $\dim(V \cap F_2) \geqslant 1$. Under the quotient map from $\mathbb{C}^4 \to \mathbb{P}^3$, we can see that this is equivalent to the space of all lines that intersect a given line in at least a point, which is what we want. Using this, we can reduce the problem to finding out properties of the intersection

$$\Omega_{(1)}(F^{(1)}) \cap \Omega_{(1)}(F^{(2)}) \cap \Omega_{(1)}(F^{(3)}) \cap \Omega_{(1)}(F^{(4)})$$

where $F^{(1)}$, $F^{(2)}$, $F^{(3)}$, and $F^{(4)}$ are distinct flags. Using our results on the cohomology ring of the Grassmannian, we can restate this problem again as

$$\sigma_{(1)} \cdot \sigma_{(1)} \cdot \sigma_{(1)} \cdot \sigma_{(1)} = c \cdot \sigma_{(2,2)}$$

where the value of c is our answer. Finally we use our results about Schur functions and the Littlewood Richardson rule. We find that $c = c^{(2,2)}_{(1),(1),(1),(1)}$ which is the number of ways to fill a 2×2 rectangle using a chain of Littlewood Richardson tableaux each consisting of a single box. Since each tableau of the chain must contain a single 1 as its entry, we label them with subscripts to indicate their step in the chain. From this, we find that there are exactly two chains satisfying these conditions

1_1	1_2
1_3	1_4

1_1	1_3
1_2	1_4

.

Therefore $c^{(2,2)}_{(1),(1),(1),(1)} = 2$, thus there are 2 lines intersecting 4 generic lines in $\mathbb{P}^3$.

In fact, by introducing a couple more tools, we can prove a more complicated result: How many $k-1$ dimensional subspaces of $\mathbb{P}^{n-1}$ intersect $k(n-k)$ given subspaces of dimension $n-k-1$ non-trivially?

We first introduce the notions of a standard Young tableau and of hook length.

Definition 41. A standard Young tableau of shape λ where $|\lambda| = n$ is a semistandard Young tableau numbered by the integers $1, 2, \ldots, n$.

Definition 42. For a square s in a Young diagram, the hook length is defined to be the sum of the number of squares strictly below s, plus the number of squares strictly to the right of s, plus 1 for s itself. We denote this sum to be $hook(s)$

We can in fact use hook length to find the number of standard Young tableaux of a particular shape.

Theorem 43. *(Hook Length Formula) The number of standard Young tableaux of shape λ is*

$$\frac{|\lambda|!}{\prod_{s\in\lambda} hook(s)}$$

It is beyond the scope of this paper to prove this. We direct the reader to [7] for a full proof.

Now we can approach our question. Similarly to our first example, we can reduce the problem to computing the coefficients in the expansion

$$\sigma_1 \cdot \cdots \cdot \sigma_1 = c^B_{1,\ldots,1}\sigma_B.$$

This is the number of ways to full the ambient rectangle with content $(1, 1, \ldots, 1)$. Notice that this is the same as finding the number of standard Young tableau of shape $(k(n-k))^k$, which is what we get by applying the hook length formula to the ambient rectangle. In the table, the integer in each box denotes its hook length.

$n-1$	$n-2$	$\cdots$	$k+1$	k
$n-2$	$n-3$		k	$k+1$
$\vdots$			$\vdots$	
$k+1$	k	$\cdots$	3	2
k	$k+1$		2	1

Now, to apply the Hook length formula, we notice that the produce of the hook lengths along the top and right sides of the box, starting from the first box in row i to the $n-k+1-i$-th box in row i, and then down the $n-k+1-i$-th column is given by the expression

$$\frac{(n-i)!}{(i-1)!}.$$

So when considering the number of standard Young tableaux in the ambient rectangle, i.e considering all i, we have the following:

$$\prod_{s\in B} hook(s) = \frac{(n-1)!(n-2)!\cdots(n-k+1)!(n-k)!}{1!2!\cdots(k-2)!(k-1)!}.$$

Thus, we can apply the hook length formula to find that

$$c^B_{1,\ldots,1} = \frac{|B|!}{\prod_{s\in B} hook(s)} = \frac{((k(n-k))!}{\frac{(n-1)!(n-2)!\cdots(n-k+1)!(n-k)!}{1!2!\cdots(k-2)!(k-1)!}} = \frac{(k(n-k))!(k-1)!(k-2)!\cdots 2!1!}{(n-k)!(n-k+1)!\cdots(n-2)!(n-1)!}.$$

This is the number of $k-1$ dimensional subspaces of $\mathbb{P}^{n-1}$ which will intersect each of the $k(n-k)$ fixed subspaces of dimension $n-k-1$ non-trivially.

From these two example, we can see that it is possible to solve problems using the framework we have established and some simple reasoning, or more complicated problems if we are willing to work with a bit more machinery.

5 Conclusion

The rigorous foundation we have presented for solving these intersection problems reveals many interesting connections between different constructions in math. We have only covered a couple of these connections, and have alluded to many more. For those interested in the subject, *Young Tableaux* by William Fulton [1] is a more advanced text that covers these topics in greater detail and in more general settings, while also presenting more connections to combinatorics and representation theory that we have not delved into here.

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References

[1] William Fulton. *Young tableaux. With applications to representation theory and geometry.* English. Vol. 35. Lond. Math. Soc. Stud. Texts. Cambridge: Cambridge University Press, 1997. ISBN: 0-521-56724-6.

[2] Maria Gillespie. "Variations on a theme of Schubert calculus". English. In: *Recent trends in algebraic combinatorics.* Cham: Springer, 2019, pp. 115–158. ISBN: 978-3-030-05140-2; 978-3-030-05141-9. DOI: `10.1007/978-3-030-05141-9_4`.

[3] Allen Hatcher. *Algebraic topology.* English. Cambridge: Cambridge University Press, 2002. ISBN: 0-521-79540-0.

[4] Drew A. Hudec. *The Grassmannian as a Projective Variety.* VIGRE Research Experience for Undergraduates, University of Chicago. Undergraduate research paper. PDF available at `https://www.math.uchicago.edu/~may/VIGRE/VIGRE2007/REUPapers/FINALFULL/Hudec.pdf`. 2007.

[5] John Naughton. *Schubert calculus.* REU Research Paper. Mentored by Pallav Goyal. University of Chicago Mathematics REU, 2021.

[6] Felice Ronga. *Schubert Calculus according to Schubert.* Preprint, arXiv:math/0608784 [math.AG] (2006). 2006. URL: `https://arxiv.org/abs/math/0608784`.

[7] R. Stanley. *Enumerative Combinatorics: Volume 2.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2023. ISBN: 9781009262514. URL: `https://books.google.com/books?id=Y5LOEAAAQBAJ`.

[8] Tuomas Tajakka. "Cohomology of the Grassmannian". Master's thesis. Aalto University School of Science, 2015. URL: `https://aaltodoc.aalto.fi/server/api/core/bitstreams/0c1e813a-5939-4a0d-8e84-e46be4da7dfc/content`.