Burnside's Problem

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Preliminary Definitions

A **group** G is a set defined under an operation $(*)$ with:

- **associativity:** $a * (b * c) = (a * b) * c$
- **identity:** There exists an element $e \in G$ where $\forall : g \in G, e * g = g * e = g$
- **inverse:** For every $g \in G$, there exists g^{-1} where $g * g^{-1} = g^{-1} * g = e$.

What is the Burnside Problem?

In 1902, William Burnside asked the following question:

General Burnside Problem

If G is a finitely generated, periodic group, then is G necessarily finite?

For reference, a **finitely generated** group is a group G for which there exists some set of finite generators S such that every element of G can be written as a combination of elements in S and its inverses.

Furthermore, a **periodic group** is a group G such that for every $g \in G$, there exists some positive integer n where $g^n = e$.

Motivations!

- In group theory, we often ask: *What global properties follow from local constraints?*
- Finite groups are classifiable, which is a great way to understand lots of cool stuff!

Bounded Variant

In addition to the general case, Burnside also asked the following variants:

Bounded Burnside Problem (unsolved)

Let $B(n, m)$ be the **free Burnside Group,** which is a group with m unique generators and $g^n = e$ for every $g \in B(n, m)$. Then, for what n and m is $B(n, m)$ finite?

For some values of m, n some nice properties arise ((1,n), (m,2)). In Burnside's 1902 Paper, he showed that:

- $B(m, 2)$ is finite
- $B(m, 3)$ is finite
- $B(2, 4)$ is finite

Restricted Variant

In the 1930s, people proposed a new question:

Restricted Burnside Problem

Given some group G with m generators and exponent n, is the order of G dependent on m and n?

An equivalent question is: Are there only finitely many finite groups with m generators of exponent n, up to isomorphism?

Golod-Shafarevich Theorem (which my paper covers!)

Golod-Shafarevich Theorem (1964)

Let $A\langle x_1, \ldots, x_n\rangle$ be the **free algebra** over a field with $n = d + 1$ non-commuting variables (d is the dimension of the field). Let J be the **two-sided ideal** of A, generated by homogenous elements of degree ≥ 2 (note that there will be infinite of these). Let r_i be the number of elements with degree i. $B = A/J$ is the **graded algebra**. Let $b_j = \dim B_j$ (the dimension of the j-dimensional part of A after quotienting). Then,

$$b_j \geq nb_{j-1} - \sum_{i=2}^{j} b_{j-i} r_i$$

Notably, B has infinite dimension if $r_i \leq \frac{d^2}{4}$ for all positive integers i.

Definitions (Free Algebra)

For a commutative ring R, the **free algebra** is a free R-module (generalization of a vector space) with a basis consisting of all words over the alphabet $\{x_1, \ldots, x_n\}$. For example, we could consider $x_1x_2x_3$ to be a word, which isn't equal to $x_1x_3x_2$. Also, $x_1^2x_2 \neq x_1x_2x_1$. More formally: For an arbitrary set $X = \{X_i, i \in I\}$, the **free R-algebra** is

$$R\langle X\rangle := \bigoplus_{w \in X^*} Rw,$$

where X^* is the set of words, and Rw is the R-module over a single word w.

Definitions (2-sided Ideal)

For a given ring R, the **2-sided ideal** I is a subgroup of the additive group R where for every $r \in R$ and $x \in I$, then $rx \in I$ and $xr \in I$.

Definitions (Graded Algebra)

An associative algebra over a ring R is a **graded algebra** if it is graded as a ring. That is, it can be decomposed as the direct sum of many subrings for some monoid I (think of $\mathbb{R}$, for instance):

$$R = \bigoplus_{i \in I} R_i.$$

Furthermore, $R_i R_j \subseteq R_{i+j}$. We can think of these i's and j's as the degrees of a polynomial, for intuition's sake (x^3 has grading 3, x^2y has grading 2, and so on).

Novikov-Adian Theorem (1968)

Novikov-Adian Theorem

$B(m, n)$ is infinite for odd $n \geq 4381$.

This was later extended in 1975, when Sergei Adian proved it was infinite for odd $n \geq 665$.

This proof utilizes some theoretical computer science (which I'm not familiar with). Generally, they relate words to Turing Machines through some complex isomorphisms, and utilize the Halting Problem to show that for certain exponents, you cannot find an algorithm to solve the code (image of the words), and thus the group is infinite.

Restricted Burnside Problem: Solved

In 1989, Efim Zelmanov proved the positive of the Restricted Burnside Problem: There are finitely many groups with m generators and exponent n, up to isomorphism. He was awarded the Fields medal in 1994 for his contributions!

Further Implications and Open Questions

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Any questions?