# The Martingale Convergence Theorem

Ishaan Sharma

Euler Circle

July 10, 2025

# Why Study Martingale Convergence?

- **Martingales** model fair games and stochastic processes.
- Convergence theorems answer: *Do martingales settle down?*
- Central in:
  - Probability theory (e.g., Strong Law of Large Numbers)
  - Stochastic processes (e.g., random walks, Brownian motion)
  - Mathematical finance (e.g., risk-neutral pricing)
- Goal: Build familiarity around martingales and introduce the MCT.

# Foundations of Probability Theory

# Measurable Spaces and Measures

- A $\sigma$-**algebra** $\mathcal{F}$ on a set $\Omega$ is a collection of subsets:
  - Closed under countable unions and complements
  - Contains $\Omega$
- A **measure** $\mu$ assigns non-negative numbers to sets in $\mathcal{F}$:

$$\mu : \mathcal{F} \to [0, \infty], \quad \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

- A **measure space** is a triple $(\Omega, \mathcal{F}, \mu)$.

# From Measure Theory to Probability

- A **probability space** is a measure space with $\mu(\Omega) = 1$.
- A **random variable** is a measurable function $X : \Omega \to \mathbb{R}$.
- The **expectation** of $X$ is defined as:

$$\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P}$$

- This formalism links probability with Lebesgue integration.

# Conditional Expectation

- Suppose $X$ is a random variable and $\mathcal{G}$ represents some information we know (a sub-$\sigma$-algebra).
- The **conditional expectation** $\mathbb{E}[X \mid \mathcal{G}]$ is:
  - A new random variable that only depends on the information in $\mathcal{G}$,
  - The best estimate of $X$ given what we know from $\mathcal{G}$,
  - Averages out to the same value as $X$ over events in $\mathcal{G}$.
- **Key Idea:** $\mathbb{E}[X \mid \mathcal{G}]$ is not just a number—it's a random variable that reflects our updated knowledge.

# Stochastic Processes

# Stochastic Processes

- A **stochastic process** is a collection $\{X_t\}_{t \in T}$ of random variables indexed by time.
- Typical choices:
  - Discrete-time: $T = \mathbb{N}$
  - Continuous-time: $T = \mathbb{R}_{\geq 0}$
- Describes the evolution of a random system over time.
- Martingales form a subclass of stochastic processes that generalize fair games.

# Example: Symmetric Random Walk

- Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables with

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}.$$

- Define the partial sums

$$S_n = \sum_{i=1}^{n} X_i.$$

- Then $\{S_n\}_{n \in \mathbb{N}}$ is called the **symmetric random walk**, a discrete-time stochastic process.
- It models a particle that moves one unit left or right with equal probability at each time step.

# Visualising the Random Walk

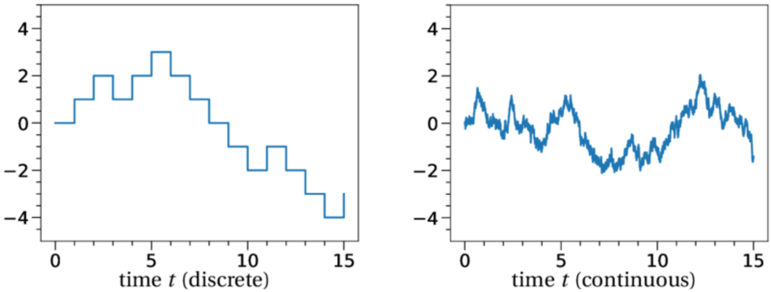


Figure 1: *

**Figure 1:** Sample path of a random walk in discrete then continuous time.

# Martingales

# Definition of Martingales

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n)_{n \geq 0}$ a filtration. A stochastic process $(X_n)_{n \geq 0}$ adapted to $(\mathcal{F}_n)$ is called:

- a **martingale** if $\mathbb{E}[|X_n|] < \infty$ for all $n$, and

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n \quad \text{a.s.}$$

- a **submartingale** if

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \geq X_n \quad \text{a.s.}$$

- a **supermartingale** if

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \leq X_n \quad \text{a.s.}$$

# Example: Symmetric Random Walk

- Let $S_n = \sum_{i=1}^{n} X_i$ denote the simple symmetric random walk, and let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ be the natural filtration. Then:

$$\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[S_n + X_{n+1} \mid \mathcal{F}_n] = S_n + \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = S_n$$

- since $X_{n+1}$ is independent of $\mathcal{F}_n$ and has mean zero.

- Thus, $(S_n, \mathcal{F}_n)$ is a martingale.

# Martingale Convergence Theorem

# Martingale Convergence Theorem

Theorem 1 (Martingale Convergence Theorem)

*Let* $(X_n)$ *be a submartingale (or martingale) such that*

$$\sup_n \mathbb{E}[|X_n|] < \infty$$

*Then* $X_n \to X_\infty$ *a.s. for some finite, integrable random variable* $X_\infty$*.*

# Example: Doob's Martingale

Let $X \in L^1$ be an integrable random variable and let $\mathcal{F}_n$ be an increasing filtration. Define:

$$X_n = \mathbb{E}[X \mid \mathcal{F}_n]$$

- $(X_n)$ is a martingale.
- $\mathbb{E}[|X_n|] \leq \mathbb{E}[|X|] < \infty$, so the martingale is uniformly integrable.
- By the Martingale Convergence Theorem: $X_n \to X_\infty$ a.s. and in $L^1$, where

$$X_\infty = \mathbb{E}[X \mid \mathcal{F}_\infty]$$

This is a key result in the foundation of modern probability and stochastic processes.

# Application to the SLLN

# Application: Strong Law via Martingales

**Theorem 2 (Strong Law of Large Numbers)**

*Let* $(X_n)_{n\geq 1}$ *be a sequence of i.i.d. random variables with* $\mathbb{E}[X_1] = \mu$ *and* $\mathrm{Var}(X_1) < \infty$. *Then:*

$$S_n = \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{a.s.} \mu.$$

*That is,*

$$\mathbb{P}\left(\lim_{n\to\infty} S_n = \mu\right) = 1.$$

# Application: Strong Law via Martingales

- Define centered variables: $Y_k = X_k - \mu$ so that $\mathbb{E}[Y_k] = 0$.
- Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ be the natural filtration.
- Define the partial sums:

$$M_n = \sum_{k=1}^{n} Y_k.$$

- Then $(M_n, \mathcal{F}_n)$ is a martingale:

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n + \mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = M_n.$$

- Additionally, since $\mathrm{Var}(X_1) < \infty$, the martingale $(M_n)$ has bounded $L^2$ norms.

# Application: Strong Law via Martingales

- By the Martingale Convergence Theorem, we have:

$$M_n \xrightarrow{\text{a.s.}} M_\infty \in \mathbb{R}.$$

- Since $M_\infty$ is finite almost surely, dividing by $n$ yields:

$$\frac{M_n}{n} \xrightarrow{\text{a.s.}} 0.$$

- Hence, the average can be written as:

$$\frac{1}{n}\sum_{k=1}^{n} X_k = \mu + \frac{M_n}{n} \xrightarrow{\text{a.s.}} \mu.$$

- Thus the SLLN follows as a direct example of martingale convergence.