Permutohedra and Associahedra

Irmak Zelal Cengiz
iramk.zelal.cengiz2@gmail.com

Euler Circle

July 2025

Permutohedra

What is a Permutohedron?

Permutohedra are special types of polytopes that geometrically represent permutations.

Given the vector $(1, 2, \ldots, n)$ in $\mathbb{R}^n$, the **permutohedron** $\mathcal{P}_n$ is the convex hull of all vectors obtained by permuting its coordinates.

Definition 1.1

$$\mathcal{P}_n := \mathrm{conv}\big\{(\sigma(1), \sigma(2), \ldots, \sigma(n)) \mid \sigma \in S_n\big\}$$

where S_n is the symmetric group on n elements.

What is a Permutohedron?

The vertices of the permutohedron $\mathcal{P}_n$ are all permutations of the vector $(1, 2, \ldots, n)$. Since permutations preserve the sum of entries, every vertex lies on the same hyperplane.

Theorem 1.2

The permutohedron $\mathcal{P}_n$ is an $(n-1)$-dimensional polytope contained in the hyperplane

$$H := \left\{ x \in \mathbb{R}^n \;\middle|\; x_1 + x_2 + \cdots + x_n = \frac{n(n+1)}{2} \right\}.$$

What is a Permutohedron?

Proof.

Every vertex of $\mathcal{P}_n$ is a permutation of $(1, 2, \ldots, n)$. The sum of the coordinates is always

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

So all vertices lie on the hyperplane H. Since H is a single linear constraint in $\mathbb{R}^n$, it reduces the dimension by 1. Therefore, $\mathcal{P}_n$ lies entirely in an $(n-1)$-dimensional space.

□ ■

$\mathcal{P}_2$ and $\mathcal{P}_3$

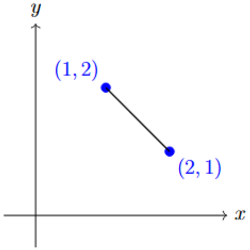


Figure 1: P_2

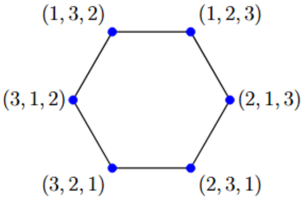
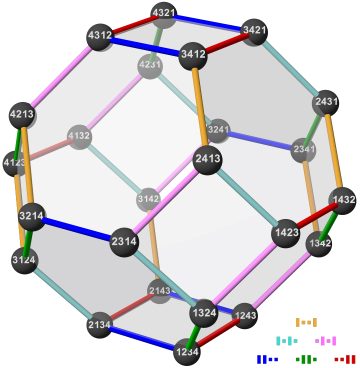


Figure 2: 2D projection of P_3

- $\mathcal{P}_2$: 1D segment
- $\mathcal{P}_3$: hexagon

$\mathcal{P}_4$: Truncated Octahedron

- 24 vertices, 36 edges
- 14 faces: squares and hexagons
- Lies in $\mathbb{R}^4$ but is 3D



Volume of the Permutohedron

Theorem 1.3

The volume of $\mathcal{P}_n$ (normalized so that the smallest simplex has volume 1) is

$$Vol(\mathcal{P}_n) = (n-1)! \cdot \textit{number of trees on } n \textit{ vertices} = (n-1)! \cdot n^{n-2}.$$

- This result connects $\mathcal{P}_n$ to **Cayley's formula**, which states that the number of labeled trees on n vertices is n^{n-2}.
- The geometry of the permutohedron encodes rich combinatorial structures, including trees and networks.

What is a Permutohedron?

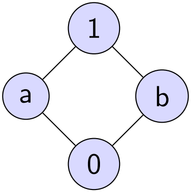
High-dimensional Permutohedra:
For $n > 4$, the permutohedron $\mathcal{P}_n$ exists in $(n-1)$-dimensional space, making it difficult to visualize directly.

Why $\mathcal{P}_n$ Matters:

- $\mathcal{P}_n$ lives in 3D space (a hyperplane in $\mathbb{R}^4$), making it the highest-dimensional permutohedron that can be meaningfully visualized.
- It serves as a crucial example to understand the geometry and combinatorics of permutohedra.

What is a Lattice?

- A **lattice** is a partially ordered set where every pair of elements has:
  - a **least upper bound** (called the **join** $\vee$)
  - a **greatest lower bound** (called the **meet** $\wedge$)
- Lattices appear in algebra, geometry, and combinatorics, especially when studying polytopes like permutohedra.



A simple lattice: $a \wedge b = 0$, $a \vee b = 1$

Face Lattice of $\mathcal{P}_4$ and the Weak Bruhat Order

Hierarchical Structure of $\mathcal{P}_4$:

- The faces of $\mathcal{P}_n$ (vertices, edges, 2D faces, etc.) are organized into a *face lattice*.
- This lattice reflects how lower-dimensional faces are nested within higher-dimensional ones.

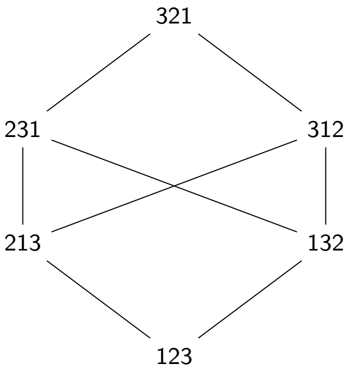
Connection to Weak Bruhat Order:

- The face lattice of P_4 corresponds to the **weak Bruhat order** on the symmetric group S_4.
- In this order, one permutation is "less than" another if it can be reached by a sequence of *adjacent transpositions*.
- A *covering relation* represents a single adjacent swap:

$$(1\,2\,3\,4) \prec (2\,1\,3\,4)$$

Weak Bruhat Order on S_3

- The weak Bruhat order is a partial order on permutations where:
  - A permutation covers another if it is obtained by an adjacent transposition.
- Below is the Hasse diagram of the weak Bruhat order on S_3:



Why is the Bruhat Order Important?

- **Sorting Networks:**
  - The **shortest paths** represent **optimal sorting strategies**, minimizing steps.
- **Real-life Applications:**
  - **Parallel computing:** Sorting networks optimize data flow across processors.
  - **Data analysis**
  - **Economics and decision theory:** Models of preference rankings use Bruhat-like orders.

Why is the Bruhat Order Important?

Significance in Algebraic Combinatorics:

- The edges of P_4 represent covering relations.
- Therefore, P_4 is a *geometric realization* of the weak Bruhat poset on S_4.
- This allows algebraic and order-theoretic properties to be studied via geometry.

Associahedra

What is an Associahedron?

The **associahedron** (*Stasheff polytope*) is a convex polytope whose:

- **Vertices** correspond to all distinct full parenthesizations of a product of $n+2$ elements using binary operations,
- **Edges** connect parenthesizations that differ by a single associativity move (e.g., $(ab)c \leftrightarrow a(bc)$),
- **Faces** of higher dimension correspond to partial associativity relations.

What is an Associahedron?

Formal Properties

Let K_n denote the associahedron of dimension $n-2$. Then:

- $\dim(K_n) = n$,
- It corresponds to parenthesizing n elements,
- It has $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$ vertices (the $(n-1)^{\text{th}}$ Catalan number).

Applications

Appears in:

- Homotopy theory and loop spaces,
- Category theory (coherence laws),
- Algebraic and combinatorial structures (operads, triangulations),
- Syntax trees and expression evaluation in computer science.

Permutohedra to Associahedra

Associahedra can be obtained from the permutohedra through certain geometric operations such as:

- Taking specific **subdivisions** of the permutohedron,
- Projecting the permutohedron along certain directions,
- Or truncating the permutohedron in a way that retains only part of its face structure.

Permutohedra to Associahedra

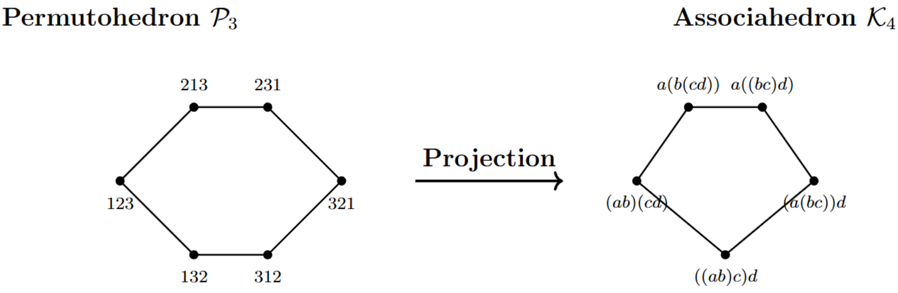


Figure 5: A projection from the permutohedron $\mathcal{P}_3$ to the associahedron $\mathcal{K}_4$.

Why Permutohedra and Associahedra

- Investigating their properties and relationships reveals deep insights into:
  - Symmetry and ordering (permutohedra)
  - Associativity and parenthesization patterns
  - Linking abstract mathematics with practical problems across science and engineering.
- Their interplay helps us understand complex structures such as:
  - Sorting algorithms and optimization
  - Higher-dimensional category theory and homotopy
  - Network theory and data structures
  - Computational biology
  - Robotics and motion planning

Thank you for listening!