Permutohedra and Associahedra

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Introduction

“Permutohedra” and “Associahedra” are two special types of polytopes. The most exciting aspect of these geometrical objects is their ability to visualize mathematical operations involving permutations and combinations, one of the most fundamental concepts in mathematics. Moreover, both structures possess rich geometric and algebraic properties and are found in surprising areas of mathematics and science.

A permutohedron is an n-dimensional polytope whose vertices represent each permutation of a group of the first n positive integers and whose edges represent the shortest way to reach from one vertex to another, through swapping places of two elements in the permutation. It is deeply connected to the symmetric group and the concept of orderings, such as weak and partial orders on permutations.

On the other hand, the associahedron is a polytope whose vertices correspond to ways to fully parenthesize a group of n symbols. For instance, for four symbols, there are five valid ways to insert parentheses, and the corresponding associahedron is a pentagon. The structures also relate to binary trees, Tamari lattice, and Dyck paths. The number of vertices of the associahedron is calculated by the Catalan numbers.

While permutohedra present a neat and symmetrical appearance, associahedra seem relatively disorganized. However, the connection between these two types of polytopes is closer than expected. This paper explores the definitions, structural properties, and real-world applications of these two families of polytopes. While some advanced topics exist, this paper focuses on the fundamentals, aiming to build a general understanding of the subject and show how science benefits from the further applications of these combinatorial structures.

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Acknowledgements

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Preliminaries

Definition 0.1 (Hyperplane)**.** A hyperplane is an $n-1$ dimensional geometrical shape described in an n dimensional space. For example, the convex hull of the points $\{(4,0,0),$ $(4,4,0),(4,0,4),(0,4,4)\}$ forms a rectangular two-dimensional shape, a hyperplane.

Definition 0.2 (Convex hull)**.** Assuming we have a set of finite points in space, a convex hull is the smallest geometrical shape that contains all of these points within itself with flat faces and straight edges.

Definition 0.3 (Polytope)**.** If $p_1, p_2, \cdots, p_n$ are points in $\mathbb{R}^n$, a polytope is the convex hull of these points, where

$$P = \left\{ \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_n p_n \,\middle|\, {}^{\lambda_1+\lambda_2+\cdots,\lambda_n=1}_{\lambda_1,\lambda_2,\cdots,\lambda_n \geq 0} \right\}.$$

An n dimensional polytope is referred to as an "n-polytope."

Definition 0.4 (Symmetric group)**.** A Symmetric group consisting of n elements is shown with S_n notation. S_n is the group of all possible permutations of n elements.

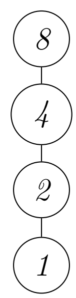
Definition 0.5 (Poset)**.** A partial order defines a way to arrange elements within a set so that certain pairs have a defined order—one element that is considered to come before another. The term *partial* highlights that this ordering does not necessarily apply to every pair of elements. A partial order is a binary relation on a set that satisfies three properties: it is *reflexive* (every element relates to itself), *antisymmetric* (no two distinct elements precede each other mutually), and *transitive* (the order is consistent across chained comparisons). A *partially ordered set*, or *poset*, is then defined as a pair

$$P = (X, \leq),$$

where X is the underlying set (often called the *ground set*) and $\leq$ is a partial order relation on X. In many cases, when the partial order is understood from context, the set X alone may be referred to as a poset.

Definition 0.6 (Hasse diagram)**.** A Hasse diagram is a graphical representation of a finite poset $(P, \leq)$. Each element of P is represented as a vertex, and edges are drawn between elements to indicate the covering relations in the poset. Specifically, there is an edge from x to y if and only if $x < y$ and there exists no $z \in P$ such that $x < z < y$; in this case, we say that y covers x, and write $x \lessdot y$. The diagram is drawn so that if $x \lessdot y$, then y is placed higher than x on the page, and no arrowheads are used, as the vertical positioning conveys the direction of the order. Transitive and reflexive relations are omitted for clarity, making the diagram a minimal visualization of the ordering structure.

Example 0.1. *Consider the poset* $(\{1, 2, 4, 8\}, |)$, *where* $|$ *denotes divisibility. The Hasse diagram represents the covering relations between these elements.*



Definition 0.7 (Lattice)**.** A lattice is a partially ordered set $(L, \leq)$ in which every pair of elements $a, b \in L$ has both a least upper bound (called the *join* and denoted by $a \vee b$) and a greatest lower bound (called the *meet* and denoted by $a \wedge b$). Formally, for all $a, b \in L$, the elements

$$a \vee b = \inf\{x \in L \mid a \leq x \text{ and } b \leq x\}$$

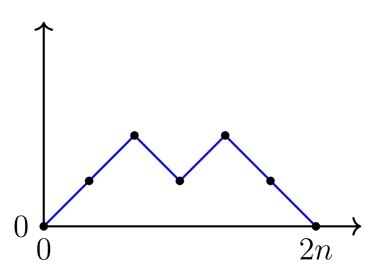
and

$$a \wedge b = \sup\{x \in L \mid x \leq a \text{ and } x \leq b\}$$

exist in L.

Definition 0.8 (Dyck Path)**.** A *Dyck path* of length $2n$ is a lattice path in the Cartesian plane from $(0, 0)$ to $(2n, 0)$ that consists of n up-steps $U = (1, 1)$ and n down-steps $D = (1, -1)$, and that never passes below the horizontal axis. That is, for every prefix of the path, the number of up-steps is greater than or equal to the number of down-steps.

Equivalently, a Dyck path can be viewed as a balanced sequence of n opening and n closing parentheses such that no initial segment of the sequence contains more closing than opening parentheses.



An example Dyck path of length 6: the step sequence is $UUDUDD$.

1 Permutohedra

Permutohedra are special types of polytopes that represent the concept of permutations in a spatial form. Given a set of first n natural numbers, the **permutohedron** P_n is the convex hull of all points obtained by permuting the coordinates of the vector $(1, 2, \ldots, n)$ in $\mathbb{R}^n$. The symmetric group of this set defines all the vertices of this shape, and each vertex corresponds to one of the permutations. Formally,

$$P_n := \operatorname{conv}\big\{(\sigma(1), \sigma(2), \ldots, \sigma(n)) \mid \sigma \in S_n\big\},$$

where S_n is the symmetric group on n elements.

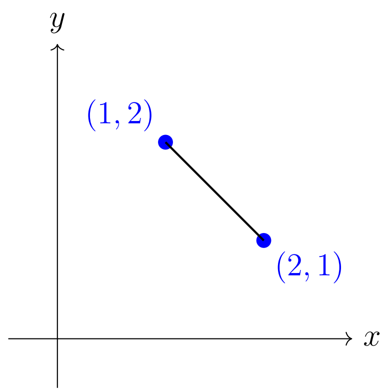


Figure 1: P_2

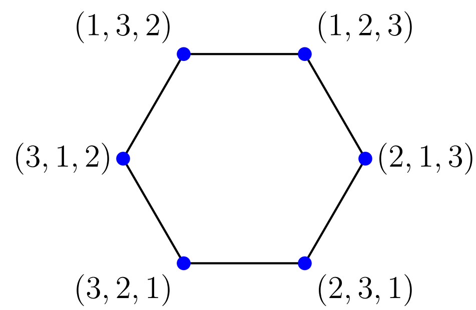


Figure 2: 2D projection of P_3

As seen in Figure 1 and Figure 2, each vertex of the permutohedra represents one particular permutation of their set. Each permutohedron forms a hyperplane. This property of permutohedra is explained in the following theorem.

Theorem 1.0 The permutohedron P_n is an $(n-1)$-dimensional polytope contained in the hyperplane

$$H := \{x \in \mathbb{R}^n : x_1 + x_2 + \cdots + x_n = \tfrac{n(n+1)}{2}\}.$$

Proof. Since each vertex of P_n is a permutation of $(1, 2, \ldots, n)$, the sum of coordinates is always

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Thus all points lie in H, and P_n is contained in this hyperplane. Moreover, the dimension is $n-1$ because this hyperplane reduces the dimension by one. □

1.1 P_4: The Permutohedron of S_4

The permutohedron P_4 is the convex polytope in $\mathbb{R}^4$ defined as the convex hull of all permutations of the vector $(1, 2, 3, 4)$. Explicitly, let

$$P_4 := \operatorname{conv}\left\{(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \in \mathbb{R}^4 \mid \sigma \in S_4\right\},$$

where S_4 denotes the symmetric group on four elements. Since $|S_4| = 24$, the polytope P_4 has 24 vertices. Each vertex lies in the affine hyperplane

$$H := \left\{ x \in \mathbb{R}^4 \;\middle|\; \sum_{i=1}^{4} x_i = 10 \right\},$$

and thus P_4 lies entirely within a 3-dimensional affine subspace of $\mathbb{R}^4$. Therefore, P_4 is a 3-dimensional polytope, see Figure 3.

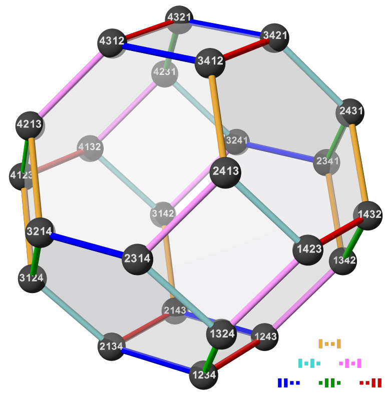


Figure 3: Image of P_4 by [10]

The permutohedra of S_n with $n > 4$ are not properly visualizable since the resulting geometric shapes exist in spaces of dimension ≥ 4.Therefore, P_4 remains as an important object to observe. Lets break-down the combinatorial structure of P_4:

- P_4 has 24 vertices, each representing a different permutation of the set $\{1, 2, 3, 4\}$.
- Two vertices are connected by an edge if you can go from one permutation to the other by swapping two adjacent elements. For example, (1 2 3 4) $\leftrightarrow$ (2 1 3 4) involves swapping the first two elements.
- There are 36 edges in total, each corresponding to one such adjacent swap between two permutations.
- P_4 contains 14 two-dimensional faces, which are flat surfaces bounded by edges. These faces come in two types:
  - **Square faces** appear when two adjacent swaps involve non-overlapping positions. For instance, swapping elements in positions 1 and 2, and separately in positions 3 and 4, can be done in either order with the same result. This symmetry forms a square.
  - **Hexagonal faces** appear when three swaps involve overlapping positions, such as swapping in positions 2 and 3, then 3 and 4, then 2 and 3 again. These sequences follow a pattern known as the *braid relation*, and the result is a cycle of six permutations forming a hexagon.

- The entire structure forms a three-dimensional polytope, which we call the permutohedron P_4.Although embedded in $\mathbb{R}^4$, the geometric realization of P_4 as a 3-polytope allows for visualization via projections. In such projections, the vertices can be layered according to their number of inversions, i.e., the length of the permutation, making it possible to observe the underlying weak order on S_4.

1.2 Face Lattice and Order Structure of , P_4

The structure of the faces of the permutohedron P_4 can be organized into a hierarchy called a *face lattice*. This lattice captures how faces of different dimensions (vertices, edges, 2D faces, and so on) fit together.

Interestingly, this face lattice corresponds exactly to a particular ordering of permutations known as the *weak Bruhat order* on the symmetric group S_4. In simpler terms, the weak Bruhat order is a way to arrange all permutations of four elements in a sequence where one permutation is considered "less than" another if it can be obtained by performing a series of adjacent swaps in a specific, minimal way.

Each step in this order, called a *covering relation*, corresponds precisely to swapping two adjacent elements in a permutation — the same operation that defines the edges of P_4. Thus, the permutohedron P_4 provides a geometric shape that visually represents this abstract ordering of permutations.

This connection between the geometric object P_4 and the weak Bruhat order is significant in algebraic combinatorics because it allows us to study algebraic and order-theoretic properties through the geometry of the permutohedron, linking combinatorial structures with spatial intuition. The face lattice of P_4 is isomorphic to the weak Bruhat order on S_4, where covering relations correspond to adjacent transpositions. This makes P_4 a geometric realization of a well-studied poset in algebraic combinatorics.

1.3 Volume of Permutohedra

The volume of the permutohedron P_n is an interesting geometric quantity that has been studied in combinatorics and geometry. A well-known formula for the volume of the permutohedron P_n, normalized concerning the standard lattice in the hyperplane

$$x_1 + x_2 + \cdots + x_n = \frac{n(n+1)}{2},$$

is given by

$$\mathrm{Vol}(P_n) = n^{n-2}.$$

This remarkable formula links the volume to the number of spanning trees on a complete graph with n vertices, highlighting deep combinatorial connections. The fact that the volume equals n^{n-2} suggests that the permutohedron's geometry encodes combinatorial structures such as trees. This volume is computed in the $(n-1)$-dimensional affine subspace containing P_n.

1.3.1 More Advanced Considerations

The permutohedron is an example of a *generalized permutohedron*, a class of polytopes whose volumes and face structures encode rich combinatorial information. Volumes of generalized permutohedra can often be expressed in terms of *mixed volumes*, *Stanley's volume polynomials*, or *valuations* related to matroids. Connections to *tropical geometry* and *submodular functions* also arise in the study of these volumes. Computing volumes explicitly for certain deformations or Minkowski sums of permutohedra remains an active area of research.

1.4 Permutohedron and Weak Orders

A **partial order** (poset) on a set X is a binary relation $\leq$ that is reflexive, antisymmetric, and transitive. The permutohedron can be related to posets via **weak orders**, which are special types of partial orders that correspond to ways of arranging elements with possible ties.

A **weak order** on a set X is a total preorder: it is transitive and total but allows equivalences. Formally, for $x, y \in X$, either $x \preceq y$ or $y \preceq x$ holds, and $x \sim y$ means x and y are equivalent (tie). Weak orders can be viewed as equivalence classes of linear orders.

The faces of the permutohedron P_n correspond bijectively to weak orders on $[n] := \{1, 2, \ldots, n\}$. The vertices correspond to total orders (i.e., permutations), and higher-dimensional faces correspond to coarser weak orders. The **left weak order** and **right weak order** are partial orders defined on the symmetric group S_n based on covering relations induced by simple transpositions.

- **Left weak order**: For $\sigma, \tau \in S_n$, we say $\sigma \leq_L \tau$ if τ can be obtained from σ by multiplying by a sequence of simple transpositions s_i on the *left*, increasing the length (number of inversions). Equivalently, it orders permutations by inclusion of their inversion sets considering left multiplication.
- **Right weak order**: Similarly, $\sigma \leq_R \tau$ if τ can be obtained from σ by multiplying on the *right* by simple transpositions increasing length.

These orders make S_n into graded posets, with the rank given by the number of inversions in the permutation. To illustrate, consider P_3, the permutohedron for $n = 3$. Each vertex corresponds to a permutation of $(1, 2, 3)$:

$$(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1).$$

Edges correspond to adjacent transpositions, representing covering relations in the weak orders.

This graph corresponds to the Hasse diagram of the weak order on S_3. The poset structure encodes the covering relations where one permutation can be transformed into another by swapping adjacent elements. In higher dimensions, P_n generalizes to an $(n-1)$-dimensional polytope, and the combinatorial structure of weak orders and posets becomes richer, encoding subtle order relations between permutations.

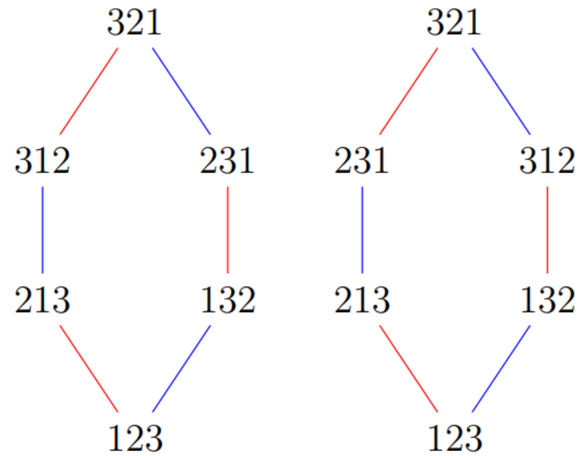


Figure 4: The left and right weak orders on S_3 with s_1 in red and s_2 in blue, by [9].

1.5 Connection to the Associahedron

While it is not strictly true that deleting a vertex of the permutohedron directly yields an associahedron, there is a deep and elegant connection between these two objects. Specifically, the *associahedron* (also known as the Stasheff polytope) can be obtained from the permutohedron through certain geometric operations such as:

- Taking specific **subdivisions** of the permutohedron,
- Projecting the permutohedron along certain directions,
- Or truncating the permutohedron in a way that retains only part of its face structure.

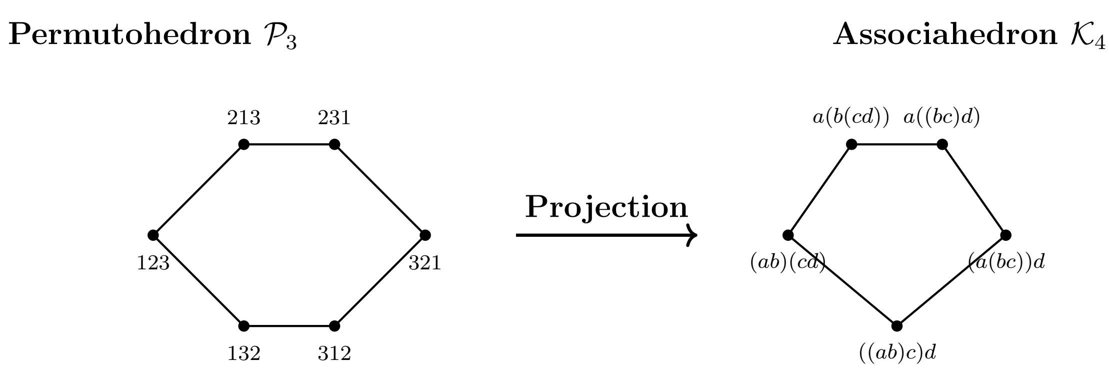


Figure 5: A projection from the permutohedron $\mathcal{P}_3$ to the associahedron $\mathcal{K}_4$.

Figure 5 illustrates a conceptual projection from the permutohedron $\mathcal{P}_3$ to the associahedron $\mathcal{K}_3$. In this projection, certain permutations that correspond to the same associativity structure are identified and merged. For example, permutations that maintain the relative order of a, b, and c but differ by adjacent swaps map to the same associativity class. The result is a collapse of the six permutation vertices into five associativity classes, demonstrating how the associahedron can be seen as a simplified, structured "shadow" of the permutohedron. This projection captures the essence of associativity by ignoring symmetric reorderings that do not change the bracketing structure.

A particularly striking connection appears in the work of Loday and others, who construct the associahedron as a certain *quotient* or *face* of the permutohedron by collapsing together permutations that correspond to the same parenthesization structure [8].

In this sense, the associahedron can be viewed as a combinatorially simpler shadow of the permutohedron, encoding associativity relations (instead of adjacent transpositions) in a lower-dimensional but still highly structured way.

This relationship plays a key role in areas like category theory, operads, and homotopy theory, where both polytopes serve as models for associativity and symmetry.

2 Associahedra

The associahedron, also known as the Stasheff polytope, is a convex polytope whose vertices correspond to the different ways of inserting parentheses in a product of n terms. It arises naturally in algebraic topology, combinatorics, and category theory, particularly in the study of A_∞-spaces and homotopy associativity. Let K_n denote the $n-2$-dimensional associahedron. Then, the vertices of K_n correspond to the n-fold parenthesizations of n symbols (i.e., all ways of fully parenthesizing n variables using binary operations); the edges correspond to single applications of the associativity rule; and the face poset of K_n is isomorphic to the poset of partial bracketings under refinement.

The number of vertices of K_n is given by the $(n-1)$-th Catalan number:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Theorem 2.0 The n-dimensional associahedron K_n can be realized as a convex polytope in $\mathbb{R}^{n-2}$ whose face lattice is isomorphic to the poset of planar rooted trees with $n+2$ leaves ordered by edge contractions.

Proof 2.0 Consider the set of all rooted planar binary trees with $n+2$ leaves. Each such tree corresponds to a fully parenthesized expression of $n+2$ variables, since each internal node represents a binary operation. The covering relations in the poset are given by local associativity moves — i.e., replacing a subtree $(a(bc))$ with $((ab)c)$.

There exists an explicit construction of K_n as a convex polytope in $\mathbb{R}^{n-2}$, due to Loday and others, using Minkowski sums of simplices or tropical geometry. The resulting polytope has faces corresponding to partially parenthesized products, and its face lattice matches the refinement poset of binary trees under edge contractions. Since every such poset is known to be graded and bounded with unique minimal and maximal elements, and the polytope realization preserves these relations, the face lattice of K_n matches that of the desired poset. Faces of K_n correspond to partial parenthesizations, where some operations remain grouped, and others are left flexible. The face lattice is the **Tamari lattice**, a partially ordered set that organizes all parenthesizations by associativity rotations.

Computing the volume of associahedra is more complex than for permutohedra, as associahedra are not zonotopes. They can be realized as Minkowski sums of simplices or as secondary polytopes associated to polygon triangulations.

Exact volume formulas are known only for small n, but the volume relates closely to Catalan combinatorics and polygon dissections.

2.1 The Tamari Lattice and Associahedra

The Tamari lattice is a foundational structure in algebraic and combinatorial contexts that captures the partial order on parenthesizations (or equivalently, binary trees) of a sequence of elements. It plays a critical role in the combinatorial and geometric understanding of the associahedron.Let us consider all full parenthesizations of a product of n elements. These can be naturally represented by binary trees with n leaves and $n-1$ internal nodes. The Tamari lattice $\mathcal{T}_n$ is a partially ordered set (poset) whose elements are these parenthesizations (or trees), with the order defined by a sequence of right rotations on binary trees. Equivalently, a cover relation in the Tamari lattice corresponds to replacing a subexpression of the form $(A \cdot (B \cdot C))$ with $((A \cdot B) \cdot C)$, moving parentheses to the left (called an associativity move or rotation).

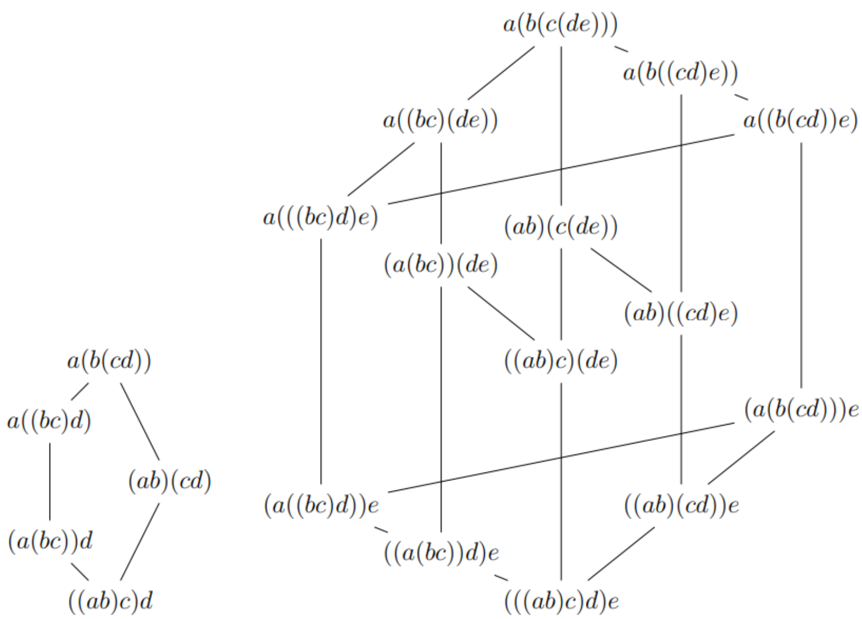


Figure 6: Tamari lattice by [9]

The 1-skeleton (the graph formed by the vertices and edges) of the associahedron encodes the Tamari lattice: two vertices are connected by an edge if and only if they are related by a single rotation, and the direction of the rotation gives the orientation in the Tamari poset. More generally, the face poset of the associahedron is isomorphic to the poset of partial bracketings or planar trees with fewer internal edges, ordered by refinement. The Hasse diagram of this poset visually represents the Tamari lattice when restricted to its 1-skeleton and oriented appropriately.

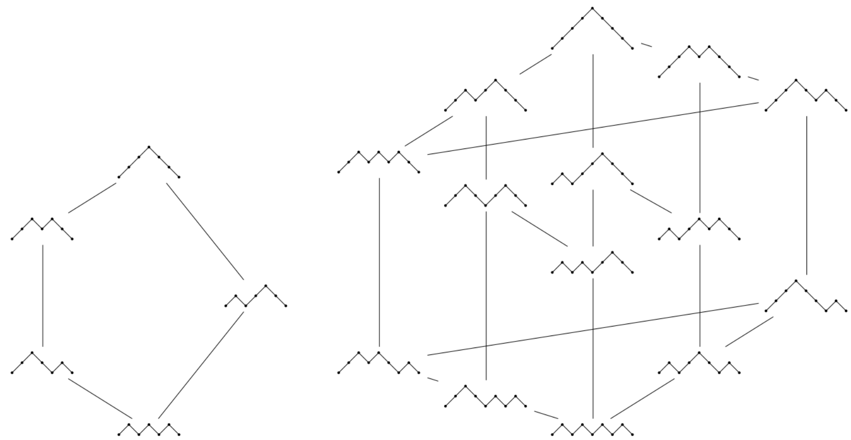


Figure 7: Tamari lattice on Dyck paths by [9]

Consider the case $n = 3$, where we are parenthesizing three elements a, b, c. There are two full binary trees or parenthesizations:

$$(a \cdot b) \cdot c \quad \text{and} \quad a \cdot (b \cdot c)$$

There is a single associativity move (rotation) that transforms one into the other. Hence, the Tamari lattice $\mathcal{T}_3$ is a chain of length one, and the associahedron K_3 is a 1-dimensional edge reflecting this relation.

2.2 Dyck Paths and Tamari Intervals

There exists a well-known bijection between Dyck paths of length $2n$ and binary trees with $n+1$ leaves. Under this correspondence, the Tamari order can be interpreted geometrically: one Dyck path is below another in the Tamari lattice if it never goes above the other path when both are plotted in the plane from left to right. These so-called Tamari intervals correspond to intervals in the Tamari lattice and reveal deep structural properties. This viewpoint is particularly useful in higher algebra and operad theory, where Dyck paths and Tamari intervals form the basis for understanding cluster complexes, Stasheff polytopes, and higher associativity.

The Tamari lattice provides a rich combinatorial structure that is deeply intertwined with the geometry of the associahedron. Through binary trees, Dyck paths, and bracketings, it serves as both a combinatorial and topological guide to the internal organization of associativity spaces. Its order-theoretic and algebraic features make it indispensable in modern applications spanning algebraic topology, category theory, and mathematical physics.

3 Real-life Applications

This section aims to explore how permutohedra and associahedra are used beyond pure theory. While these polytopes are often studied in abstract algebra, combinatorics, and geometry, they also appear in a wide range of applied fields, including data analysis, computer science, optimization, and mathematical physics. Here, by reviewing several research articles, we will highlight concrete examples of how permutohedra and associahedra are used for real-world mathematical applications.

- Projection Geometry and Angle Analysis [4]

  This article focuses on how permutohedra behave under generic linear projections from high-dimensional space into lower-dimensional subspaces. The authors prove that the number of faces of each dimension in the image of a permutohedron remains invariant under almost all such projections. This result is significant in the study of dimensionality reduction, where high-dimensional geometric objects are simplified while retaining structural information. In particular, this property makes permutohedra useful in contexts such as compressed sensing and data visualization, where geometric projections play a central role.

  In addition to projection properties, the article also examines the so-called "angle sums" of permutohedra, which refer to the total measure of angles around the faces of the polytope. The authors derive exact formulas and asymptotic estimates for these sums, showing how they relate to the characteristic polynomial of the hyperplane arrangement associated with the permutohedron. These results provide tools for quantifying the curvature and complexity of the polytope's geometry, especially in high dimensions.

  By studying both the projection-invariant face counts and the geometric angle behavior, the paper highlights how permutohedra can model real-world systems where both combinatorial ordering and spatial constraints are important. Applications include areas such as high-dimensional data analysis, optimization over ordered structures, and geometric probability, where understanding how a structure changes—or doesn't—under projection is critical.

- Enumeration of Max-Pooling Responses[3]

  This work examines the combinatorial complexity of max-pooling layers—common components in convolutional neural networks—by analyzing how they partition input space into linear regions. The key observation is that the Newton polytope of a max-pooling layer is a Minkowski sum of standard simplices, a structure that places it within the class of generalized permutohedra. By characterizing these polytopes in terms of acyclic directed graphs and walks in graphs, the authors derive exact generating functions and closed formulas for the number of vertices and facets in one-dimensional max-pooling scenarios, with explicit results for window size and stride. They extend the enumeration of vertices to certain two-dimensional cases and identify a recurrence relation for face numbers, corroborated by computational data. These results link the geometry of generalized permutohedra directly to the behavior of neural network modules, providing precise counts of piecewise-linear regions—a useful measure for understanding expressivity and complexity in deep learning architectures.

- Color–Kinematics Duality [1]

This paper investigates the algebraic and combinatorial structure of BCJ numerators in tree-level scattering amplitudes, and connects them to permutohedra. The authors show that the complete formula for each BCJ numerator can be understood as a sum over all faces—of every dimension—of a permutohedron: for instance, the numerator involving two scalars and $n-2$ gluons corresponds precisely to the faces of a $(n-3)$-dimensional permutohedron. Each term in this sum is associated with a distinct boundary of the polytope, and comes with its own gauge-invariant form and spurious-pole structure. Moreover, they demonstrate that the Hopf-algebraic coproduct operation naturally mirrors the decomposition of the permutohedron into its sub-faces, leading to new recursion relations and factorization identities when the numerator is evaluated on a facet. These observations are extended to pure Yang–Mills amplitudes and to a heavy-mass effective field theory for two massive particles. By revealing that BCJ numerators literally "live" on the combinatorial skeleton of permutohedra, the article establishes a deep geometric foundation for color–kinematics duality and provides powerful new tools—recursion, factorization, and Hopf-structure—that simplify amplitude computations in both massless and heavy-particle contexts.

- Stringy Canonical Forms [6]

  This paper constructs a broad family of integrals—called "stringy canonical forms"—for generalized permutohedra, which encompasses both associahedra and cyclohedra. The authors show that any generalized permutohedron formed as a Minkowski sum of coordinate simplices admits a "rigid" stringy integral. These integrals exhibit two key features:

  First, their configuration spaces are proven to be binary geometries: whenever one coordinate variable tends to zero, all variables incompatible with that facet converge to one. This property ensures that the integrals factorize cleanly along the faces of the polytope, mirroring the combinatorial structure given by its Minkowski decomposition, and generalizing known behavior for associahedra.

  Second, the authors derive explicit formulas (and asymptotics) for these integrals, including their "u-equations," and show that, in the limit of vanishing deformation parameter α', the integrals factor into products over lower-dimensional permutohedra or associahedra. In type-A and type-B cases (classical associahedra and cyclohedra), these formulas reduce to the well-known cluster-string integrals; in the more general setting, they extend this structure systematically. Through this construction, the paper highlights a deep link between the combinatorial geometry of these polytopes and the algebraic structure of string-like integrals, offering a unified framework that applies across a spectrum of polytopal types.

- Irreducibility of Generalized Permutohedra [5]

  The structure of generalized permutohedra is analyzed by studying how these polytopes can—or cannot—be expressed as Minkowski sums of simpler polytopes. They define an element to be *irreducible* if it cannot be non-trivially decomposed via Minkowski sums aside from trivial scaling or translation, and prove that every generalized permutohedron has a unique decomposition into irreducible building blocks.

The authors provide both upper and lower bounds on the number of these irreducible cases: they explicitly enumerate small-dimensional instances (e.g., for $n \leq 4$), and derive asymptotic estimates showing that their count grows doubly exponentially. In parallel, they analyze an equivalent problem on balanced multisets. These results enrich our understanding of the combinatorial geometry of generalized permutohedra, showing that their complexity stems fundamentally from these atomic, indecomposable elements. Such decomposition results are vital in applications where geometric objects are built from simpler ones—such as optimization, matroid theory, and polyhedral combinatorics—because they identify the essential "prime factors" underlying complex structures.

- Quivers and Knots [2]

  This doctoral thesis investigates deep connections between knot theory, supersymmetric gauge theories, and the combinatorial geometry of permutohedra through the framework known as the *knots-quivers correspondence*. The correspondence associates to each knot a quiver—a directed graph encoding interactions—and relates knot invariants to algebraic and geometric properties of the quiver. Central to this approach is the appearance of *permutohedral graphs*, which reflect the combinatorial structure of the symmetric group and serve as geometric tools to analyze the symmetry properties of knots and gauge theories.

  A key contribution of the thesis is the introduction and detailed study of quiver A-polynomials, an extension of the classical A-polynomial invariant from knot theory to the setting of quivers. These polynomials capture essential data about the moduli spaces of representations of the quivers, and their structure is intimately linked to permutohedral combinatorics. The work demonstrates that permutohedra arise naturally as polytopal objects encoding the symmetries and factorization properties of these A-polynomials.

  Furthermore, the thesis explores the role of permutohedra in understanding dualities and wall-crossing phenomena in supersymmetric gauge theories, where the geometry of the permutohedron governs transitions between different physical phases. The geometric realization of permutation symmetries via permutohedra also facilitates computations of knot invariants and offers new algebraic tools for studying their categorifications.

  Overall, this thesis exemplifies how advanced combinatorial and geometric concepts, such as permutohedra and their associated graphs, can illuminate intricate structures in mathematical physics and knot theory, bridging discrete geometry with continuous physical phenomena.

- Associahedra via Spines [7]

  This article develops a new combinatorial and geometric framework for understanding associahedra through the concept of *spines*, which are oriented and labeled dual trees associated with triangulations of polygons. By considering the spine as the dual graph of a triangulation enriched with additional structure, the authors create a link between the classical description of associahedra—often expressed in terms of binary trees or bracketings—and a more geometric viewpoint inspired by J.-L. Loday's construction.

The spine-based perspective offers several advantages: it provides clearer combinatorial descriptions of the faces of the associahedron and allows the authors to give streamlined, concise proofs of key properties such as the realization of associahedra as convex polytopes. This method simplifies previous approaches by making the relationship between triangulations and their corresponding associahedra more explicit and tangible.

Moreover, the paper explores how this viewpoint generalizes to related polytopes, giving insight into the structure of cluster algebras and other combinatorial geometries connected to associahedra. By grounding associahedra in the language of spines, the authors open pathways to better understand the rich combinatorial and geometric interplay underlying these fundamental objects, which have applications in algebraic geometry, mathematical physics, and beyond.

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