Transcendental Number Theory

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Euler Circle

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Definitions

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A *transcendental number* is a number that is not a root of any polynomial with integer coefficients.

Existence of Transcendental Numbers

$$r_1 = 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \cdots$$

$$r_2 = 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$$

$$r_3 = 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \cdots$$

$$r_4 = 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0 \cdots$$

$$r_5 = 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1 \cdots$$

$$\vdots \qquad\qquad \ddots$$

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Theorem (Cantor)

The set of real numbers is uncountable.

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Assume $h = n + |a_1| + |a_2| + |a_3| + \cdots + |a_n|$ for some integer n and some sequence of integers a_i.
For each h, a finite number of polynomials with degree n and coefficients a_i can be constructed and each polynomial has finite roots, which means that a one-to-one correspondence of algebraic numbers with natural numbers is possible.

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Corollary (Cantor)

The set of transcendental numbers is uncountable.

Liouville's Theorem

Theorem (Liouville)

For every algebraic number α, there exists some c such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^n}$$

for all rationals $\frac{p}{q}$.

Definition

The *degree of* α is the degree of the lowest degree polynomial with integer coefficients that has α as a root.

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$$L = \sum_{n=1}^{\infty} 10^{-n!}$$

Transcendence of e

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e is transcendental.

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For any distinct algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ and any non-zero algebraic numbers $\beta_1, \beta_2, \ldots, \beta_n$, we have

$$\beta_1 e^{\alpha_1} + \beta_2 e^{\alpha_2} + \cdots + \beta_n e^{\alpha_n} \neq 0.$$

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This implies that e^{α} is transcendental for any algebraic $\alpha \neq 0$.

Since $e^{i\pi} = -1$ is not transcendental, $i\pi$ must not be algebraic, meaning π must also be transcendental.

Squaring the Circle

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It requires constructing $\sqrt{\pi}$. However, since π, and thus $\sqrt{\pi}$, is transcendental, this is impossible.

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It also further generalized previous work and laid the foundation for future theorems.

More Recent Results

Theorem (Baker)

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be non-zero algebraic numbers such that their natural logarithms are linearly independent over rational numbers. Then, for any non-zero algebraic numbers $\beta_1, \beta_2, \ldots, \beta_n$,

$$\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 + \cdots + \beta_n \log \alpha_n$$

is transcendental.

More Recent Results

Nesterenko is also known for his work on specifically algebraic independence. He proved that π and e^{π} are algebraically independent over the rationals.

Definition

A set of elements is *algebraically independent* over a field if there is no nontrivial polynomial relation among them with coefficients in that field.

Example

The sets $\{\pi\}$ and $\{\sqrt{2\pi + 1}\}$ are algebraically independent over the rationals while the set $\{\pi, \sqrt{2\pi + 1}\}$ is algebraically dependent over the rationals, as it is a solution to the equation $2x - y^2 + 1 = 0$.

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Both would contribute significantly to the field and there are also many other open problems in the area.

Thank you!

Thank you so much for your attention!