PDEs in Physics and the Green's Function

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Partial Differential Equations (PDEs)

What is a PDE?

In ODEs (ordinary differential equations), the function depends on a single variable, whereas in PDEs we focus on multiple independent variables. We call these multiple independent variables $x, y, z, ...$, and there is a dependent variable that we write as $u(x, y, z, ...)$. Here, the dependent variable u represents a scalar field like temperature or potential.

Definition

A PDE is an identity where the independent variables, the dependent variable, and the partial derivatives are all related. Using the variables we listed above, the PDE can be written as:

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0$$

We can write the solution to a PDE as a function $u(x, y, z, ...)$ such that it satisfies the equation identically, or at least in some region of the $x, y, z, ...$ variables. "Solving" a PDE is not isolating u on one side of the equation, but rather it is about finding a function that makes the PDE valid across some domain.

Waves and Diffusion

Simple Transport

Say that there is water flowing at a constant rate c in a pipe with a fixed cross section in the positive x direction, and there is a pollutant in the water. Let $u(x,t)$ be the concentration in grams/centimeter at time t.
We can recognize that the amount of pollutant (in grams) in the interval $[0,b]$ at time t is

$$M = \int_0^b u(x,t)\ dx.$$

This integral defines the total mass of pollutant in the interval at a given time. Since the concentration is in grams per centimeter, integrating over a length gives total grams. This mass should be conserved as the pollutant moves, assuming there's no source, sink, or diffusion.
At some later time $t+h$ the same pollutant molecules have moved right by $c \cdot h$ centimeters, which means that

$$M = \int_0^b u(x,t)\ dx = \int_{ch}^{b+ch} u(x,t+h)\ dx.$$

Simple Transport (Cont.)

Theorem

After more calculations and manipulations, the resulting PDE is $u_t + cu_x = 0$*. The equation means that the rate of change* u_t *of concentration is proportional to the gradient* u_x*, and we assume diffusion to be negligible. When we solve the equation, we find the concentration is a function of* $(x - ct)$ *only. This is because the general solution of* $u_t + cu_x = 0$ *is any function in the form*
$u(x, t) = f(x - ct)$*.*
This means that the substance moves right at a fixed speed c*, which means that each individual particle moves right at that same speed* c*. The solution is constant along lines* $x = ct +$ *constant, so the pollutant shifts to the right with velocity* c*.*

Vibrating String

Say that we have a string, like a guitar or violin string, that is either a flexible, elastic homogenous string or a string that undergoes small transverse vibrations. At some time t, the string may look like the one in Figure 1.
Say that $u(x,t)$ is the displacement from the equilibrium at time t and position x.
The tension is directed tangentially along the string, as seen in Figure 2.

Figure 1. Example: vibrating string

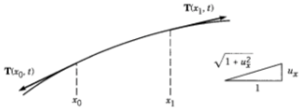

Figure 2. Force of tension directed tangentially along the string

Vibrating String (Cont.)

Let us call $T(x,t)$ the magnitude of the tension vector and let ρ be the density of the string (the density being constant since the string is homogenous). We will use Newton's law on part of the string between two arbitrary points x_0 and x_1.

Definition

$$\text{Longitudinal: } \frac{T}{\sqrt{1+u_x^2}}\Big|_{x_0}^{x_1} = 0$$

$$\text{Transverse: } \frac{Tu_x}{\sqrt{1+u_x^2}}\Big|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt} dx$$

Theorem

Using the definitions above along with the functions we used to model the vibrating string, we end up with $u_{tt} = c^2 u_{xx}$ where $c = \sqrt{\frac{T}{\rho}}$, where c is the wave speed, and the speed depends on the tension T and the mass density ρ.

Vibrating String (Cont.)

If there is a significant air resistance factor r, then there is an extra proportional term to the speed u_t, so we can rewrite the equation as

$$u_{tt} - c^2 u_x x + r u_t = 0 \text{ where } r > 0.$$

This is because the term ru_t represents a resistive force, like air drag, which lessens kinetic energy. This equation is a damped wave equation which shows decaying oscillations rather than constant waves.
If there is a transverse elastic force (like in a coiled spring), there is an extra proportional term to the displacement u, resulting in the equation

$$u_{tt} - c^2 u_{xx} + ku = 0 \text{ where k>0.}$$

This is because the term ku is a Hookean restoring force which leads to solutions that have standing waves or harmonic motion.
If there is an external force applied, there is an extra term which results in in the equation:

$$u_{tt} - c^2 u_{xx} = f(x, t)$$

making the equation inhomogeneous. This is because $f(x, t)$ introduces energy into a system and affects nonhomogeneous behavior.

Vibrating Drumhead

A drumhead is an elastic, flexible, homogenous two-dimensional string, which is to say a membrane/blob like in Figure 3. The drumhead lies in the xy plane where $u(x, y, t)$ has vertical displacement and there is no horizontal movement.

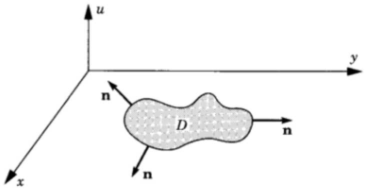

Figure 3. The two-dimensional string: a "drumhead"

Vibrating Drumhead (Cont.)

Say that D is any domain in the xy plane. We define the boundary curve as bdy D, we can say that the vertical component approximately gives:

$$F = \int_{\text{bdy } D} T \frac{\partial u}{\partial n} ds = \int \int_D \rho u_{tt} \, dx \, dy = ma.$$

Using Green's theorem, we can rewrite this as

$$\int \int_D \nabla \cdot (T \nabla u) \, dx \, dy = \int \int_D \rho u_{tt} \, dx \, dy.$$

Vibrating Drumhead (Cont.)

Theorem

After the application of Green's theorem and the divergence theorem, we are left with

$$u_{tt} = c^2 \nabla \cdot (\nabla u) \equiv c^2 (u_{xx} + u_{yy}),$$

where $c = \sqrt{\frac{T}{\rho}}$. *This is the two-dimensional wave equation, like in the wave equation before and* $\nabla \cdot (\nabla u) =$ *div grad u* $= u_{xx} + u_{yy}$ *is the two-dimensional Laplacian.*

Diffusion

Given a motionless liquid filling a straight pipe and some chemical like a dye, diffusing through the liquid, we can define simple diffusion using the following logic:
The dye moves from regions of high concentration to low concentration, and the rate of this motion is proportional to the concentration gradient; this is known as the Fick's law of diffusion. If $u(x,t)$ is the concentration of the dye at position x and time t, then in the part of the pipe that spans from x_0 to x_1, the mass of the dye is written as:

$$M(t) = \int_{x_0}^{x_1} u(x,t)\ dx \to \frac{dM}{dt} = \int_{x_0}^{x_1} u_t(x,t)\ dx.$$

Diffusion (Cont.)

Using Fick's law:

$\frac{dM}{dt} = \text{flows into} - \text{flows out} = ku_x(x_1, t) - ku_x(x_0, t)$ where k is the proportionality

Fick's law tells us that the diffusive flux at any point is $-ku_x$, but since we are calculating the net flux into the interval, we write it as a difference of flux terms at the boundaries.

Theorem

After more manipulation, we are left with $u_t = ku_{xx}$; this is known as the diffusion equation. In three dimensions, we write

$$\int\int\int_D ut\ dx\ dy\ dz = \int\int_{bdy\ D} k(\mathbf{n} \cdot \nabla u)\ dS,$$

where D is any solid domain and bdy D is its bounding surface.

Heat Flow

Say that $u(x, y, z, t)$ is the temperature and say that $H(t)$ is the amount of heat contained in a region D. We can say that

$$H(t) = \int\int\int_D c\rho u \; dx \; dy \; dz$$

where c is the specific heat of the material and ρ is the density.
The change in heat is

$$\frac{dH}{dt} = \int\int\int_D c\rho u_t \; dx \; dy \; dz.$$

Fourier's law tells us that the heat flux vector $\overrightarrow{q} = -\kappa \nabla u$ where $\kappa > 0$ is thermal conductivity. The negative sign makes sure that the heat flows from higher to lower temperatures, and since there are no internal sources/sinks, heat can only leave D through its boundary.

Heat Flow (Cont.)

Theorem

After the application of more theorems and some manipulation, we get

$$c\rho \frac{\partial u}{\partial t} = \nabla \cdot (\kappa \nabla u).$$

This is the general form of the heat equation, and it states that the rate of change in temperature at any point is equal to the divergence of the heat flux vector. If κ *is constant, this becomes* $u_t = \frac{\kappa}{c\rho}\Delta u$ *which is the standard diffusion equation.*

Quick Exposition to Boundary Conditions

Boundary Conditions

PDEs usually have many solutions, and we try to find out a single solution by using additional conditions to narrow down the possible solutions. These conditions are usually either initial or boundary conditions. An initial condition tells us about the physical state at some time t_0.
For the diffusion equation, for some given function $\phi(x) = \phi(x, y, z)$, an initial condition is

$$u(\mathbf{x}, t_0) = \phi(\mathbf{x}).$$

$\phi(x)$ is the initial concentration for the diffusing substance, the initial temperature for heat flow, and the initial condition for the Schrödinger equation.
For the wave equation, there are two initial conditions where $\phi(x)$ is the initial position and $\psi(x)$ is the initial velocity:

$$u(\mathbf{x}, t_0) = \phi(\mathbf{x}),\ \frac{\partial u}{\partial t}(\mathbf{x}, t_0) = \psi(\mathbf{x}).$$

Boundary Conditions (Cont.)

In every problem we looked at in the previous section, there is a domain D where the PDE is valid. In the vibrating string, D is the interval $0 < x < l$ so D's boundary has only the two points $x = 0$ and $x = l$. In the drumhead, the domain is a plane and its boundary is a closed curve. For a diffusing substance, D is a container holding the liquid meaning that its boundary is a surface such that $S =$ bdy D.

Boundary Conditions (Cont.)

There are three major boundary condition types, where a is a function of x, y, z, and t:

1. (D) u is specified, known as the Dirichlet condition.
2. (N) $\frac{\partial u}{\partial n}$ is specified, known as the Neumann condition.
3. (R) $\frac{\partial u}{\partial n} + au$ is specified, known as the Robin condition.

Each of these conditions hold for all t and for $\mathbf{x} = (x, y, z)$ in bdy D. We write $(D), (N), (R)$ as equations, like when we write (N) as $\frac{\partial u}{\partial n} = g(\mathbf{x}, t)$ where g is a function that we call the boundary datum.

Green's Functions

The Formulas

To understand Green's functions, we need to consider Green's formulas and their applications.

Theorem

$$\textit{Green's First Formula:} \int_{\partial D} v \frac{\partial u}{\partial n} dS = \int_D \nabla v \cdot \nabla u d\mathbf{x} + \int_D v \Delta u d\mathbf{x},$$

$$\textit{Green's Second Formula:} \int\int\int_D (u\Delta v - v\Delta u) d\mathbf{x} = \int\int_{\partial D} (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}) dS,$$

valid across any function pair u *and* v *and across any solid region* D.

Green's Representation Theorem

Theorem

Using the two formulas in the previous slides, along with a LOT of mathematical manipulation, we get this equation that is known as Green's Representation Theorem: The representation theorem formula can represent any harmonic function as an integral over the boundary. If $\Delta u = 0$ in D, then

$$u(\mathbf{x}_0) = \int\int_{\partial D} [-u(\mathbf{x}) \frac{\partial}{\partial n} (\frac{1}{|\mathbf{x} - \mathbf{x}_0|} \frac{\partial u}{\partial n})] \frac{dS}{4\pi}.$$

Now, even though we did not directly use Green's formulas or theorems in the Waves and Diffusion section, many of the calculations we did not have time to go over featured the use of Green's theorems, and we explored the applications of these functions while talking about the examples throughout the slides.

Thank you! Are there any questions?