Pricing FX Barrier Options via Stochastic Simulation

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Definition: Financial Asset

A **financial asset** is a tradable instrument representing ownership of a real or financial claim.
In our setting: the underlying asset is the FX spot rate S_t, the domestic price of one unit of foreign currency.

Definition: Risk-Free Asset

A **risk-free asset** is one with a known return.
We assume two:

- Domestic currency risk-free rate: r_d
- Foreign currency risk-free rate: r_f

Definition: Market

A **market** is a system through which assets are exchanged.
We assume:

- Frictionless trading
- Continuous time
- No transaction costs
- No arbitrage

Definition: Option

An **option** is a financial derivative giving the holder the right (but not the obligation) to:

- Buy (call) or
- Sell (put)

an asset at a specified **strike price** K at or before expiration.

Definition: Barrier Option

A **barrier option** is path-dependent:

- Activated (*knock-in*) or
- Terminated (*knock-out*)

based on whether the underlying asset breaches a **barrier level** B during its life.

We study: **Down-and-Out Call**, knocked out if $S_t < B$.

Definition: FX Spot Rate

The **FX spot rate** S_t is the price, in domestic currency, of one unit of foreign currency at time t.

Definition: Standard Brownian Motion

A process $\{W_t\}$ is a **standard Brownian motion** if:

- $W_0 = 0$
- Independent increments
- $W_t - W_s \sim \mathcal{N}(0, t - s)$
- Continuous paths

Definition: Itô Process

A process $\{X_t\}$ is an **Itô process** if:

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$$

where W_t is Brownian motion and μ, σ are measurable functions.

Itô's Lemma (One Dimension)

If X_t is an Itô process and $f(t,x) \in C^{1,2}$, then:

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma \frac{\partial f}{\partial x} dW_t$$

Definition: Geometric Brownian Motion (GBM)

S_t follows GBM if:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

Solution:

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right]$$

Risk-Neutral GBM

Under risk-neutral measure Q, GBM becomes:

$$dS_t = (r_d - r_f)S_t\, dt + \sigma S_t\, dW_t^Q$$

Solution:

$$S_t = S_0 \exp\left[\left(r_d - r_f - \frac{1}{2}\sigma^2\right) t + \sigma W_t^Q\right]$$

Motivation

- FX barrier options are popular derivatives in currency markets.
- Path-dependent: payoff depends on whether price hits a barrier.
- Challenging to price $\rightarrow$ no closed-form solutions in most cases.

What is a Barrier Option?

- Example: **Down-and-Out Call**
  - Right to buy at strike K if spot rate never drops below barrier B.
  - If $S_t < B$ at any time: **knocked out**, value $= 0$.
- Payoff: $\Phi(S_T) = \mathbf{1}_{\min S_t > B} \cdot \max(S_T - K, 0)$

Why is Pricing Hard?

- Options are priced using expectation under risk-neutral measure.
- Standard options: depend only on S_T.
- Barrier options: depend on full path $\{S_t\}_{t \in [0,T]}$.
- Discretized simulations miss barrier crossings $\Rightarrow$ biased estimates.

Modeling the FX Spot Rate

- Under risk-neutral measure:

$$dS_t = (r_d - r_f)S_t\,dt + \sigma S_t\,dW_t$$

- Simulated via geometric Brownian motion (GBM).
- Discretized using exact solution:

$$S_{t+\Delta t} = S_t \cdot \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}Z\right]$$

Monte Carlo Simulation

- Generate many FX paths under GBM.
- Compute payoff for each:

$$\Phi^{(m)} = \mathbf{1}_{\min S_t^{(m)} > B} \cdot \max(S_T^{(m)} - K, 0)$$

- Take average (discounted): $\hat{V}_0 = \frac{1}{M} \sum e^{-r_d T} \Phi^{(m)}$

Problem: Discretization Bias

- Barrier crossings between time steps may go undetected.
- Leads to systematic overpricing.
- Especially bad when monitoring frequency is low.

Brownian Bridge Correction

- Idea: interpolate between steps using Brownian bridge.
- Estimate crossing probability over $[t_i, t_{i+1}]$:

$$P_i = 1 - \exp\left(-\frac{2(\log S_i - \log B)(\log S_{i+1} - \log B)}{\sigma^2 \Delta t}\right)$$

- Adjust payoff:

$$\hat{\Phi}_{BB} = \max(S_T - K, 0) \cdot \prod P_i$$

Variance Reduction Techniques

- Monte Carlo is slow: error $\sim \frac{1}{\sqrt{M}}$
- Two techniques:
  - **Antithetic Variates**: Use Z and $-Z$ together
  - **Control Variates**: Use correlated known-value variable (e.g., vanilla call)

Empirical Results (GBM)

- Plain MC: large variance, wide confidence interval.
- Antithetic: $\sim$70% reduction.
- Control variates: $>$99% reduction.

Estimated price: \$6.20–\$6.32
CI width: 0.12 (plain) $\rightarrow$ 0.01 (control variates)

Brownian Bridge Improves Bias

- Naive MC overestimates true price:

$$\text{Bias:} \quad +2.65\% \text{ (50 steps)}, +10.3\% \text{ (10 steps)}$$

- Brownian bridge brings price closer to continuous-monitoring truth.

Summary

- Barrier options are hard to price due to path dependence.
- Monte Carlo simulation + Brownian bridge reduces bias.
- Variance reduction techniques make pricing far more efficient.