PRICING FX BARRIER OPTIONS VIA STOCHASTIC SIMULATION

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Abstract. We study the pricing of foreign exchange (FX) barrier options using stochastic simulation techniques, with a focus on improving estimator efficiency and accuracy. Barrier options are path-dependent derivatives whose valuation is highly sensitive to the monitoring frequency and volatility structure of the underlying asset. We begin by modeling the FX spot rate using a geometric Brownian motion (GBM) under the risk-neutral measure and simulate option payoffs using Monte Carlo methods. To address the high variance and discretization bias inherent in barrier options, we implement antithetic variates and control variates, which are two classical variance reduction techniques, alongside Brownian bridge interpolation to correct for missed barrier crossings. We then extend our analysis to the Heston stochastic volatility model, which introduces volatility clustering and leverage effects that better capture real-world FX dynamics. Empirical evaluations compare estimator performance under both GBM and Heston settings, highlighting tradeoffs between accuracy, computational cost, and model realism.

1. Introduction

Financial derivatives are contracts whose value depends on the behavior of an underlying asset, such as a stock price, interest rate, or foreign exchange (FX) rate. Among the many types of derivatives, options are some of the most widely traded and studied. An option gives its holder the right, but not the obligation, to buy or sell an asset at a specified price before or at a predetermined date. While some options depend only on the asset's value at expiration, others are path-dependent, meaning their payoff depends on the entire trajectory of the asset price over time.

Barrier options are a class of path-dependent options whose existence or value is contingent on whether the asset breaches a preset barrier level during its lifetime. These products are especially prevalent in FX markets, where they are often used by institutions seeking cost-effective ways to hedge currency exposure. A common example is a *down-and-out call option*, which gives the right to buy a currency at maturity only if the exchange rate has not fallen below a certain threshold during the option's term. Although these structures are useful, their path dependence makes pricing them far more challenging than vanilla options.

Unlike vanilla options, barrier options do not always admit closed-form pricing formulas, particularly when the barrier is monitored at discrete intervals or when the underlying asset follows a complex stochastic process. As a result, practitioners often turn to numerical methods such as *Monte Carlo simulation*, which estimates expected payoffs by generating many synthetic price paths and averaging over them. While flexible, this approach can be computationally expensive and statistically inefficient. This is the case especially for barrier options, where many paths may knock out early and contribute zero payoff, inflating the variance of the estimator.

This paper explores the use of Monte Carlo methods to price down-and-out FX barrier options, with a focus on improving simulation accuracy and efficiency. We begin by modeling the FX spot rate using geometric Brownian motion (GBM) under a risk-neutral framework. To address high variance and discretization bias, we implement two classical variance reduction techniques: *antithetic variates*, which reduce noise by averaging over symmetric paths, and *control variates*, which use analytically-priced options to adjust the estimate. We also apply *Brownian bridge interpolation*, a probabilistic correction that reduces bias from missed barrier crossings between discrete monitoring dates.

Recognizing that real-world FX dynamics often exhibit features not captured by constant volatility models, such as volatility clustering and leverage effects, we extend our analysis to the *Heston stochastic volatility model*, which treats volatility as a random process coupled to the underlying asset. While computationally more complex, this model better reflects market behavior and leads to more realistic pricing outcomes.

Our empirical evaluation compares Monte Carlo estimators across both models, highlighting how simulation accuracy and variance reduction effectiveness change under different assumptions. These results illustrate the tradeoffs between model realism, computational cost, and statistical efficiency in the valuation of exotic options, and provide guidance for practitioners and researchers interested in accurate simulation-based pricing of complex financial derivatives.

While the paper explains key financial structures and modeling choices in detail, it assumes a foundational background in probability theory. In particular, the reader is expected to be familiar with rudimentary concepts such as random variables and statistical terminology.

2. Financial Framework

We define foundational concepts from financial mathematics that will be used throughout the paper.

Definition 2.1 (Financial Asset)**.** A *financial asset* is a tradable instrument representing either ownership of a real or financial claim. In our setting, the underlying asset is the FX spot rate S_t, representing the domestic currency price of one unit of foreign currency.

Definition 2.2 (Risk-Free Asset)**.** A *risk-free asset* is one whose return is known with certainty. In this paper, we assume the existence of two risk-free assets: one denominated in the domestic currency with rate r_d, and another in the foreign currency with rate r_f.

Definition 2.3 (Market)**.** A *market* is a system or venue through which assets are exchanged. We assume an idealized setting with continuous trading, no transaction costs, and no arbitrage opportunities.

Definition 2.4 (Arbitrage)**.** An *arbitrage opportunity* is a self-financing trading strategy that requires zero initial capital, incurs no risk, and yields a strictly positive profit with positive probability. A financial market is said to be *arbitrage-free* if no such opportunities exist.

Definition 2.5 (Option)**.** An *option* is a financial derivative contract that gives the holder the right, but not the obligation, to buy (call) or sell (put) an underlying asset at a specified price (the strike) at or before a specified expiration date.

Definition 2.6 (Barrier Option)**.** A *barrier option* is a path-dependent derivative that is activated (knock-in) or extinguished (knock-out) if the underlying asset breaches a specified barrier level during the option's life. This paper focuses on *down-and-out* call options, which cease to exist if the asset falls below a lower barrier.

Definition 2.7 (Foreign Exchange (FX) Spot Rate)**.** The *FX spot rate* S_t denotes the price, in domestic currency, of one unit of foreign currency at time t. It is the underlying asset for the barrier options considered in this study.

3. Preliminaries

We work on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where:

- Ω is the sample space of possible outcomes,
- $\mathcal{F}$ is a sigma-algebra of events,
- $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration representing the evolution of information over time,
- $\mathbb{P}$ is the real-world (physical) probability measure.

We assume all stochastic processes are adapted to this filtration, and the usual regularity conditions (completeness and right-continuity) hold under this filtration.

Definition 3.1 (Filtration)**.** A *filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ is a non-decreasing family of sigma-algebras such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $0 \leq s \leq t$. It represents the evolution of information over time and formalizes the notion that future events cannot influence present decisions.

Definition 3.2 (Adapted Process)**.** Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. A stochastic process $\{X_t\}_{t\geq 0}$ is said to be *adapted* to $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$. This formalizes the idea that X_t only depends on information available up to time t.

Definition 3.3 (Standard Brownian Motion)**.** A stochastic process $\{W_t\}_{t\geq 0}$ is called a standard Brownian motion (or Wiener process) with respect to the filtration $\{\mathcal{F}_t\}$ and probability measure $\mathbb{P}$ if:

- $W_0 = 0$,
- W_t has independent increments,
- $W_t - W_s \sim \mathcal{N}(0, t-s)$ for $0 \leq s < t$,
- The paths of W_t are continuous.

Definition 3.4 (Martingale)**.** Let $\{X_t\}_{t\geq 0}$ be an adapted stochastic process. It is a *martingale* with respect to $\{\mathcal{F}_t\}$ and measure $\mathbb{P}$ if:

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s \quad \text{a.s. for all } 0 \leq s \leq t,$$

and $\mathbb{E}[|X_t|] < \infty$ for all $t \geq 0$.

Intuitively, a martingale represents a "fair game" where the expected future value, conditional on current information, equals the present value.

In the context of option pricing, we replace the physical measure $\mathbb{P}$ with a new probability measure $\mathbb{Q}$ under which all tradable asset prices become martingales when properly discounted with respect to a risk-free rate. This leads to the following key concept:

Definition 3.5 (Equivalent Martingale Measure)**.** A probability measure $\mathbb{Q} \sim \mathbb{P}$ is called an *equivalent martingale measure (EMM)* if the discounted price process $\{e^{-rt}S_t\}$ is a martingale under $\mathbb{Q}$, i.e.,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}S_T \mid \mathcal{F}_t\right] = e^{-rt}S_t \quad \text{for all } 0 \leq t \leq T.$$

Remark 3.1. The real-world probability measure $\mathbb{P}$ reflects observed outcomes and investor preferences, whereas the risk-neutral measure $\mathbb{Q}$ is a mathematical construct used for pricing. Under $\mathbb{Q}$, all tradable assets earn the risk-free rate, and arbitrage opportunities are eliminated. This shift is central to modern asset pricing theory and underpins the use of Monte Carlo simulation in this paper.

4. Itô's Lemma

To model asset prices governed by randomness, we must extend traditional calculus to stochastic processes such as Brownian motion, whose paths are continuous but nowhere differentiable. *Stochastic calculus* provides the framework to define and manipulate such processes. Central to this theory is the concept of a *stochastic differential equation* (SDE), which describes the evolution of a process in the form

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t,$$

where W_t is standard Brownian motion, and μ and σ describe the drift and volatility of the process. A process that satisfies such an equation is called an *Itô process*.

Itô's Lemma is the stochastic analog of the chain rule from classical calculus and is used to compute differentials of functions of Itô processes. This result is foundational to continuous-time finance and the derivation of option pricing models.

Definition 4.1 (Itô Process)**.** A stochastic process X_t is called an *Itô process* if it satisfies:

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t,$$

where μ and σ are measurable functions and W_t is a standard Brownian motion.

Lemma 4.1 (Itô's Lemma in One Dimension)**.** *Let X_t be an Itô process and let $f(t,x) \in C^{1,2}([0,T] \times \mathbb{R})$. Then $Y_t = f(t, X_t)$ satisfies:*

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma \frac{\partial f}{\partial x} dW_t.$$

Proof. We expand $f(t, X_t)$ using a stochastic Taylor expansion:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX_t)^2.$$

Substitute $dX_t = \mu dt + \sigma dW_t$, and use the identities $(dW_t)^2 = dt$, $dt^2 = dW_t dt = 0$ [Shr04]. With this, we get

$$(dX_t)^2 = \sigma^2 dt.$$

Hence:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} (\mu dt + \sigma dW_t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma^2 dt,$$

which simplifies to

$$df = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma \frac{\partial f}{\partial x} dW_t.$$

■

5. Geometric Brownian Motion

Geometric Brownian Motion (GBM) is one of the most widely used models for asset prices in quantitative finance. It captures the essential features of asset dynamics; namely, continuous paths, proportional returns, and randomness driven by Brownian motion, all while remaining mathematically tractable. In this section, we define GBM, solve its stochastic differential equation, and explain its behavior under both the real-world and risk-neutral measures.

Definition 5.1 (Geometric Brownian Motion (GBM))**.** A process S_t follows geometric Brownian motion under measure $\mathbb{P}$ if it satisfies the stochastic differential equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t,$$

with initial condition $S_0 > 0$, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants.

Proposition 5.1 (Solution under Physical Measure)**.** *The explicit solution to the GBM SDE is:*

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right].$$

Proof. Define $X_t = \log S_t$, so $dX_t = \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2$. Applying Itô's Lemma:

$$dX_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t.$$

Integrating:

$$X_t = \log S_t = \log S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t.$$

Exponentiate both sides:

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right].$$

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Proposition 5.2 (Solution under Risk-Neutral Measure)**.** *Under the risk-neutral measure $\mathbb{Q}$, the SDE becomes:*

$$dS_t = (r_d - r_f) S_t \, dt + \sigma S_t \, dW_t^{\mathbb{Q}},$$

and the solution is:

$$S_t = S_0 \exp\left[\left(r_d - r_f - \frac{1}{2}\sigma^2\right) t + \sigma W_t^{\mathbb{Q}}\right].$$

Proof. Same proof as above with drift μ replaced by $r_d - r_f$ and Brownian motion W_t replaced by $W_t^{\mathbb{Q}}$. ■

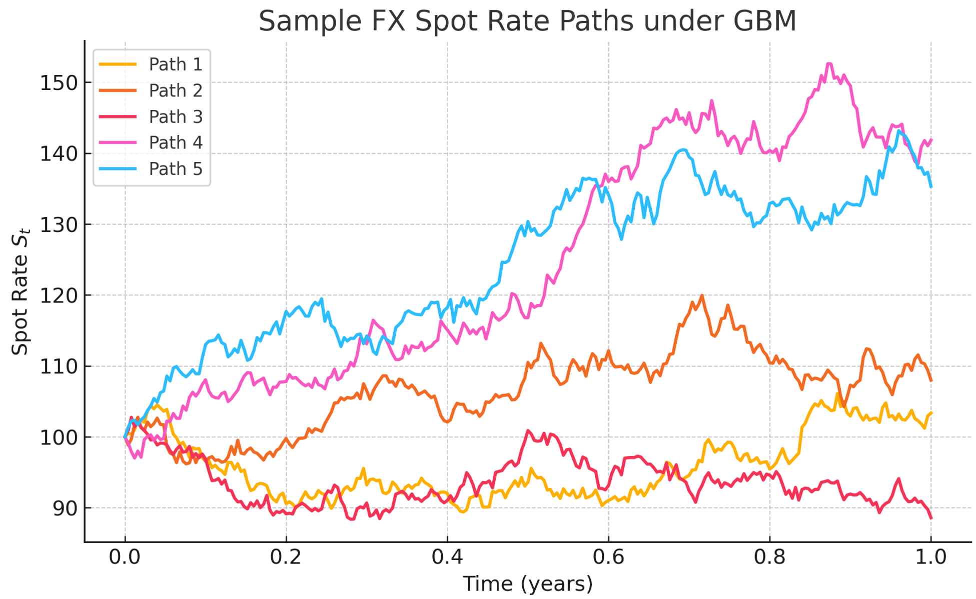

Figure 1. Sample FX spot rate paths simulated under geometric Brownian motion (GBM). Each line represents a possible trajectory of the exchange rate over a one-year horizon.

6. Introduction to Monte Carlo Methods

Monte Carlo simulation is a powerful numerical technique for approximating expectations by repeated random sampling [Gla04]. In financial engineering, it is commonly used to estimate the value of derivatives when analytical pricing formulas are unavailable or difficult to apply.

The central idea is to simulate a large number of possible future paths for an underlying asset, compute the payoff of a derivative along each path, and take the discounted average to approximate the expected value under a risk-neutral measure. Formally, for a contingent claim with payoff $\Phi(ST)$ at maturity T, the arbitrage-free price is given by:

$$V_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\Phi(ST)\right],$$

where $\mathbb{Q}$ is an equivalent martingale measure and r is the domestic risk-free rate.

Remark 6.1. We price under $\mathbb{Q}$ rather than the real-world measure $\mathbb{P}$ because, under $\mathbb{Q}$, the discounted prices of tradable assets are martingales. Because of this, all assets earn the risk-free rate in expectation, which eliminates arbitrage and lets us compute fair prices as risk-neutral expectations.

Monte Carlo methods are particularly attractive in high-dimensional or path-dependent settings, such as barrier options or Asian options, where closed-form solutions are rare. Unlike tree-based methods or finite difference schemes, Monte Carlo simulation can handle arbitrary payoff structures and stochastic processes with relative ease.

However, Monte Carlo simulation has its limitations. Its convergence rate is $\mathcal{O}(1/\sqrt{M})$, meaning that improving accuracy requires a quadratic increase in the number of simulations M. Furthermore, in the context of path-dependent derivatives like barrier options, the presence of binary payoffs and discrete monitoring can lead to high variance and subtle biases.

For these reasons, Monte Carlo methods are often paired with variance reduction techniques and pathwise corrections to improve efficiency and accuracy. This paper focuses on using Monte Carlo simulation to price FX barrier options, incorporating such techniques to produce reliable estimates.

In the following section, we introduce the mathematical model used to describe the underlying FX dynamics and define the structure of the barrier option payoff.

7. Mathematical Model for FX Barrier Options

This section introduces the modeling framework used to describe and price barrier options in the context of foreign exchange (FX) markets. Barrier options are contracts whose payoff depends not only on the value of the underlying asset at maturity, but also on whether the asset price has crossed a specified level, known as the barrier, at any point during the contract's life. These products are especially popular in FX markets, where they serve as tailored instruments for hedging and speculation [Hul18]. Their lower cost relative to vanilla options, combined with their ability to express specific directional or volatility views, makes them attractive in institutional trading. However, the path-dependence introduced by the barrier condition complicates their mathematical treatment and often rules out closed-form pricing formulas.

Let S_t denote the FX spot rate at time t, representing the domestic currency price of one unit of foreign currency. Under the domestic risk-neutral measure $\mathbb{Q}$, the spot rate is modeled as a geometric Brownian motion satisfying the stochastic differential equation

$$dS_t = (r_d - r_f)S_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}}, \tag{7.1}$$

where r_d and r_f are the domestic and foreign continuously compounded interest rates, respectively; $\sigma > 0$ is the volatility of the exchange rate; and $W_t^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$ [Shr04]. This model reflects the forward rate parity condition and ensures that the discounted spot rate process $e^{-r_d t}S_t$ is a $\mathbb{Q}$-martingale, in accordance with the no-arbitrage principle. The solution to (7.1) is given explicitly by

$$S_t = S_0 \exp\left(\left(r_d - r_f - \frac{1}{2}\sigma^2\right)t + \sigma W_t^{\mathbb{Q}}\right),$$

which implies that the logarithm of S_t is normally distributed.

The derivative contracts of interest are single-barrier call options with European-style exercise. These options grant the right to buy the underlying currency at a fixed strike price K at time T, but only if the spot rate has not breached a specified barrier level B during the option's life. We focus on the down-and-out call option, which becomes void if the spot rate ever falls to or below the barrier B, where $B < S_0$. If the barrier is not breached, the option behaves like a vanilla European call. The payoff at maturity is therefore defined by

$$\Phi(S_T) = \mathbb{1}_{\{\min_{0\le t\le T} S_t > B\}} \cdot \max(S_T - K, 0),$$

where the indicator function enforces the knockout condition. Other variants, such as knock-in, up-and-out, or double-barrier options, follow similar logic and can be handled with extensions of the same framework.

Under this model, the arbitrage-free price of the option at time zero is given by the risk-neutral valuation formula:

$$V_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-r_d T}\Phi(S_T)\right],$$

where the expectation is taken under the domestic risk-neutral measure. For a narrow class of barrier options with constant parameters and continuous monitoring, this expectation can be computed analytically using reflection techniques [RR91]. However, in most practical settings, such closed-form solutions are no longer valid. In particular, monitoring typically occurs at discrete times, such as daily market closes or fixings, rather than continuously.

When the barrier is monitored at a finite set of times $\{t_0, t_1, \ldots, t_n\} \subset [0, T]$, the effective payoff becomes

$$\Phi^{\text{disc}}(S_T) = \mathbb{1}_{\{\min\{S_{t_0}, S_{t_1}, \ldots, S_{t_n}\} > B\}} \cdot \max(S_T - K, 0).$$

This modification changes the distribution of payoffs and introduces a systematic upward bias in the price relative to the continuous-monitoring case, since paths that breach the barrier between monitoring points are not detected. Several studies have quantified this effect and proposed correction terms, though in practice, such adjustments are often handled numerically [BGK97].

Some barrier contracts also include fixed amounts paid immediately upon barrier breach, known as *rebates*. These payments may be contingent on how and when the barrier is hit, and they affect both the valuation and the optimal hedging strategy. In this paper, we restrict attention to the case of zero-rebate options to simplify exposition and focus on the core path-dependent pricing challenge. Additionally, we assume the FX spot rate follows constant volatility GBM throughout, though more realistic models could include stochastic volatility or jumps.

For convenience, we summarize the main parameters used in the modeling framework in Table 1 below. These variables will be used throughout the remainder of the paper in the development of simulation methods and analysis of numerical results.

Table 1. Model Parameters for FX Barrier Option

Symbol	Description
S_t	FX spot rate at time t
S_0	Initial FX spot rate
K	Strike price
B	Barrier level
T	Maturity (years)
r_d, r_f	Domestic and foreign interest rates
σ	Volatility of FX spot rate
$\Phi(S_T)$	Payoff function
$W_t^{\mathbb{Q}}$	Brownian motion under risk-neutral measure

8. Numerical Methods: Monte Carlo Simulation

As shown in the previous section, pricing a barrier option requires computing the expected discounted payoff under a risk-neutral measure. For European-style barrier options with simple conditions and continuous monitoring, closed-form solutions are available. However, these formulas rely on idealized assumptions, such as constant volatility, continuous observation, and lognormality, that often break down in practical settings. When barriers are monitored at discrete times, or when the payoff structure is path-dependent or lacks symmetry, closed-form pricing formulas are generally unavailable. Even when such formulas exist, they may rely on idealized assumptions (e.g., continuous monitoring or constant volatility) that limit their practical accuracy. In these cases, simulation-based numerical methods provide flexible, model-agnostic alternatives that adapt more readily to realistic market settings.

To apply Monte Carlo methods to the pricing problem at hand, we begin by discretizing the stochastic process S_t defined in equation (7.1). For each simulated path, we generate a sequence of asset prices $\{S_{t_0}, S_{t_1}, \ldots, S_{t_n}\}$ over a uniform grid of times $0 = t_0 < t_1 < \cdots < t_n = T$, where the number of time steps n is chosen large enough to capture the possibility of barrier crossing between steps. We use a uniform time grid with n steps, yielding a constant time increment $\Delta t = T/n$.

Given the dynamics of the spot rate under the risk-neutral measure,

$$dS_t = (r_d - r_f)S_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}},$$

we simulate each path using the exact solution to this stochastic differential equation. That is, for each step we update the spot rate using the formula

$$S_{t_{k+1}} = S_{t_k} \cdot \exp\left[\left(r_d - r_f - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t} \cdot Z_k\right],$$

where $Z_k \sim \mathcal{N}(0,1)$ are independent standard normal random variables. This scheme preserves the exact distributional properties of GBM at discrete time points, avoiding the discretization bias that arises in first-order approximations.

After simulating a large number M of independent paths $\{S_t^{(m)}\}_{t\in\{t_0,\ldots,t_n\}}$, we compute the payoff for each path according to the discretely monitored down-and-out call structure:

$$\Phi^{(m)} = \mathbb{1}_{\left\{\min\{S_{t_0}^{(m)},\ldots,S_{t_n}^{(m)}\}>B\right\}} \cdot \max(S_T^{(m)} - K, 0).$$

The final Monte Carlo estimate for the option price is then given by the average discounted payoff:

$$\hat{V}_0 = \frac{1}{M}\sum_{m=1}^{M} e^{-r_d T} \cdot \Phi^{(m)}.$$

This estimator is unbiased under the discrete monitoring framework and converges almost surely to the true value as $M \to \infty$. This estimator is unbiased under the discrete monitoring framework and converges almost surely to the true value as $M \to \infty$. To improve efficiency, we later incorporate variance reduction techniques that reduce estimator variance without significantly increasing computational cost.

Remark 8.1. While the Monte Carlo estimator is unbiased with respect to the *discretely* monitored model, it overestimates the *true* price under continuous monitoring due to undetected barrier crossings between time steps. Hence, it is structurally upward-biased when compared to the ideal continuous-monitoring case.

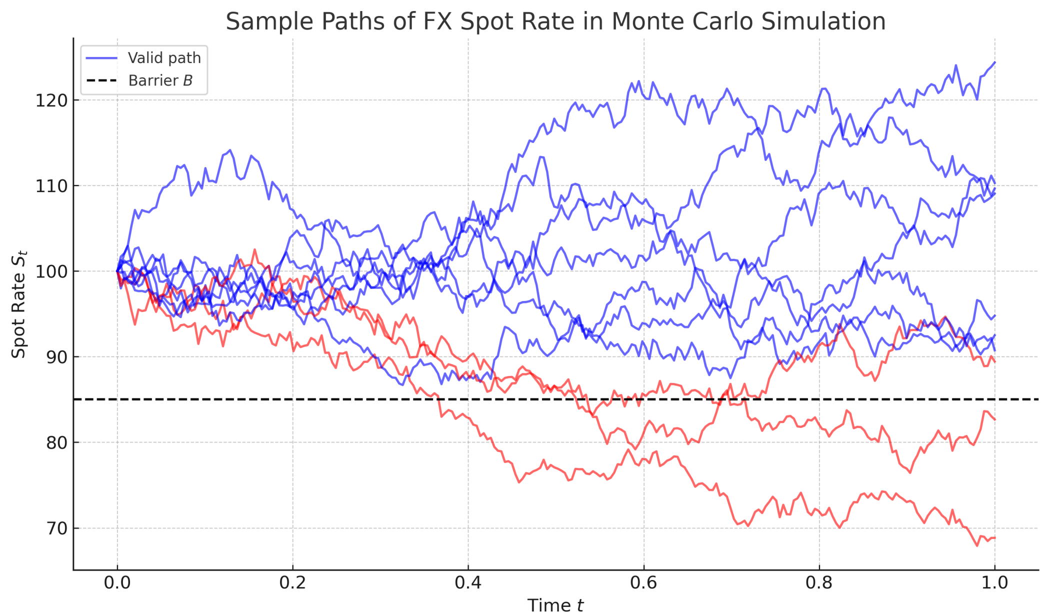

Figure 2. Sample paths of the FX spot rate under geometric Brownian motion used in Monte Carlo simulation. Paths that cross the barrier level $B = 85$ are shown in red and contribute no payoff, while valid paths are shown in blue. The dashed line indicates the barrier. This illustrates the path-dependence of the down-and-out option payoff.

One of the most accessible variance reduction methods is *antithetic variates*, which we describe in detail in Section 10. Briefly, it involves simulating a second path using the negated random draws of each Brownian increment to reduce variance through cancellation of symmetric fluctuations.

We also explore the *control variate* method in Section 10, which leverages a correlated variable with known expected value (e.g., a vanilla European option) to reduce variance through analytical adjustment.

Beyond these classical techniques, more sophisticated methods such as conditional Monte Carlo and importance sampling have also been applied to barrier option pricing [Gla04] [Bla21]. However, these methods often require nontrivial mathematical setup and problem-specific tuning. In this paper, we focus on plain Monte Carlo simulation as well as the two aforementioned variance reduction methods: antithetic and control variates, which are both accessible and effective in practice.

A crucial consideration when implementing Monte Carlo for barrier options is the accurate detection of barrier crossing. Since the true asset path is continuous but simulations are discretized, a barrier may be crossed between two time steps without being detected. This leads to overestimation of the option value. To mitigate this, one can increase the number of monitoring points, apply Brownian bridge interpolation techniques, or use correction factors derived from analytic asymptotics. In this work, we address the issue by choosing a sufficiently fine time grid to make the approximation error negligible relative to the sampling error from the simulation.

Finally, to assess the reliability of our pricing estimates, we compute confidence intervals based on the central limit theorem.

Assumption 8.1. The confidence interval formula assumes that the number of simulations M is sufficiently large for the central limit theorem to apply. In practice, choosing $M \geq 10^4$ typically ensures a good approximation to normality.

Given the standard deviation s of the sample payoffs $\{\Phi^{(m)}\}$, the 95% confidence interval for $\hat{V}_0$ is given by the open interval

$$\left(\hat{V}_0 - 1.96 \cdot \frac{s}{\sqrt{M}},\ \hat{V}_0 + 1.96 \cdot \frac{s}{\sqrt{M}}\right).$$

This allows us to quantify the statistical error and evaluate the efficiency gains achieved through variance reduction.

By combining model-based path simulation with careful statistical techniques, we are able to compute accurate estimates for a wide range of contract configurations and parameter values. The next section presents numerical results from our implementation and analyzes the effectiveness of each variance reduction strategy.

9. Correcting Discrete Monitoring Bias via Brownian Bridge Interpolation

A well-known issue in pricing discretely monitored barrier options is upward bias: simulated paths may cross the barrier between monitoring dates without detection. When only discrete prices are checked, these breaches go unnoticed, leading to falsely retained paths and overestimated option prices. This problem worsens with coarser monitoring grids or tighter barrier levels.

To mitigate this, we implement a correction based on Brownian bridge interpolation. A *Brownian bridge* is a Brownian motion conditioned to start and end at specified values over a fixed time interval. In our setting, we model the asset path between time steps as a Brownian bridge conditioned on the simulated prices at those endpoints. Rather than immediately discarding a path when all discrete prices lie above the barrier, we scale its payoff by the conditional survival probability that the path remained above the barrier between observations.

Proposition 9.1. *Let $\log S_t$ follow Brownian motion under the risk-neutral measure. Then the probability that the process stays above a log-barrier $\log B$ on $[t_{i-1}, t_i]$, conditional on $\log S_{t_{i-1}}$ and $\log S_{t_i}$, is given by:*

$$P_i = 1 - \exp\left(-\frac{2(\log S_{t_{i-1}} - \log B)(\log S_{t_i} - \log B)}{\sigma^2 \Delta t}\right).$$

We define the corrected payoff as

$$\hat{\Phi}^{BB} = \max(S_T - K, 0) \cdot \prod_{i=1}^{n} P_i.$$

This formulation preserves the binary path-dependence structure while accounting for missed barrier crossings probabilistically.

A derivation of this formula can be found in [Gla04] and [Shr04].

Remark 9.1. This adjustment is inexpensive to compute, involving only a product of closed-form expressions, and adds negligible overhead even for large numbers of paths. The method becomes especially valuable when simulation constraints limit the number of time steps.

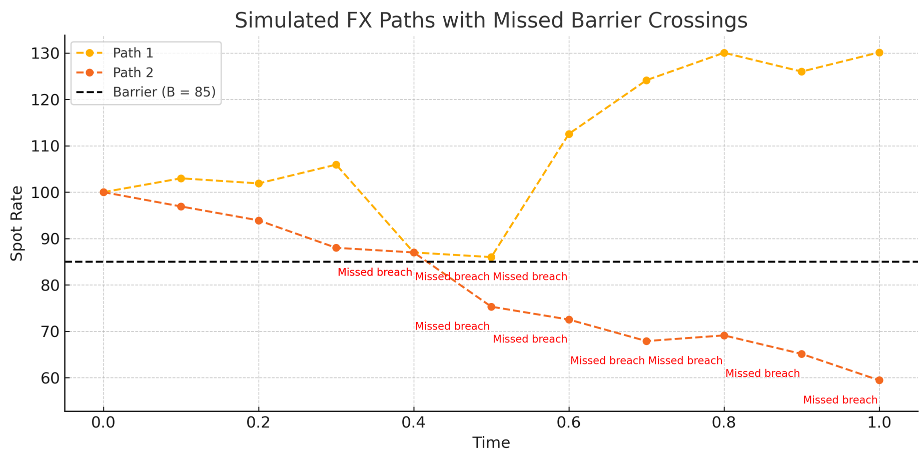

Figure 3. Two FX paths (dashed) that survive discrete monitoring yet likely breach the barrier when interpolated. Brownian bridge adjustment penalizes these cases.

We now investigate how this correction interacts with the monitoring frequency. Table 2 shows the option price under both naive and Brownian bridge-corrected methods as the number of monitoring dates increases. The corrected price remains stable, while the naive estimate converges toward it from above.

Monitoring Points (n)	Naive Estimate	BB-Corrected	Relative Bias (%)
10	7.0123	6.3588	+10.28
50	6.4972	6.3294	+2.65
100	6.3825	6.3241	+0.92
250	6.3418	6.3127	+0.46

Table 2. Effect of monitoring frequency on naive and Brownian bridge-corrected estimates.

All simulations in this table use the same random seed and baseline parameters specified in Section 11: $S_0 = 100$, $K = 100$, $B = 85$, $T = 1$, $\sigma = 0.15$, $r_d = 0.02$, and $r_f = 0.01$.

Remark 9.2. These results confirm that the naive Monte Carlo estimator systematically overstates the option's value when monitoring is coarse. While the Brownian bridge correction is itself an approximation, it more closely reflects the continuously monitored barrier condition and serves as a practical benchmark when closed-form solutions are unavailable. For some configurations (e.g., constant parameters and continuous sampling), closed-form solutions can be derived via reflection principles [RR91].

In our benchmark setting with 100,000 paths and 50 monitoring dates, the corrected price is approximately 2.6% lower than the naive estimate. This difference is economically meaningful within the context of institutional pricing.

Method	Estimate	Std. Dev	95% CI Width
Naive Monte Carlo (100k)	6.4972	0.8601	0.1142
Monte Carlo + Brownian Bridge	6.3294	0.8457	0.1218

Table 3. Barrier option prices with and without Brownian Bridge correction.

Remark 9.3. While both estimators exhibit similar variance, only the Brownian bridge method captures the bias due to missed barrier crossings. The standard deviation remains stable because both methods are applied to the same set of simulated paths.

Thus, Brownian bridge interpolation offers a principled and computationally cheap way to reduce pricing bias from discrete monitoring, enhancing the realism of simulation-based methods without sacrificing tractability.

10. Variance Reduction Techniques

While Monte Carlo simulation offers a flexible framework for pricing complex derivatives, its slow convergence rate often results in high computational cost, especially for path-dependent options with binary features like barrier conditions. As described in Section 6, our baseline estimator computes the expected discounted payoff by simulating a large number of FX spot rate paths under the risk-neutral measure and averaging the resulting payoffs. However, this approach can suffer from high variance, particularly when many paths are knocked out and yield zero payoff.

To mitigate this issue, we implement two classical variance reduction techniques called *antithetic variates* and *control variates*, which improve estimator efficiency without introducing bias.

The antithetic variates method reduces variance by exploiting symmetry in the underlying randomness. For each simulated Brownian path used to generate an asset price trajectory, we also simulate its antithetic counterpart by negating the Brownian increments. Let $X^{(i)}$ and $\tilde{X}^{(i)}$ denote the discounted payoffs from a pair of original and antithetic paths. The antithetic estimator is given by:

$$\hat{V}_{\text{anti}} = \frac{1}{2M} \sum_{i=1}^{M} \left(X^{(i)} + \tilde{X}^{(i)} \right).$$

This averaging reduces random fluctuations due to the anti-correlation between the path pairs. While it does not guarantee improvement in all cases, it is computationally cheap and generally effective for options with symmetric path dependencies.

The control variates method improves accuracy by introducing a second random variable Y, called the control variate, whose expected value $\mathbb{E}^{\mathbb{Q}}[Y]$ is known and which is highly correlated with the original payoff X. The adjusted estimator takes the form:

$$\hat{V}_{\text{ctrl}} = \bar{X} + \lambda \left(\mu_Y - \bar{Y} \right),$$

where $\bar{X}$ and $\bar{Y}$ are the sample means of X and Y over the simulated paths, μ_Y is the known expected value of Y, and λ is the optimal coefficient minimizing variance:

$$\lambda = \frac{\text{Cov}(X, Y)}{\text{Var}(Y)}.$$

In our implementation, we select as the control variate the terminal payoff of a European call option with the same strike, maturity, and underlying asset as the barrier option. Because this vanilla option has a closed-form Black-Scholes price under the same GBM dynamics, it provides a natural and highly correlated benchmark. The strength of this method depends on the correlation between the two payoffs: the closer their movements, the more effectively variability in the barrier option payoff can be canceled out by deviations in the vanilla option payoff.

These variance reduction techniques are simple to implement and significantly improve the efficiency of Monte Carlo pricing. Their empirical performance in the context of FX barrier options will be evaluated in the following section using simulated data under the geometric Brownian motion model.

11. Empirical Evaluation of Variance Reduction (GBM Model)

We now evaluate the effectiveness of the variance reduction techniques developed in previous sections under a geometric Brownian motion (GBM) model for the FX spot rate. This serves as a baseline for later comparison with stochastic volatility models. The option under consideration is a down-and-out European call with the following specifications: initial spot rate $S_0 = 100$, strike price $K = 100$, barrier level $B = 85$, maturity $T = 1$ year, domestic and foreign interest rates $r_d = 0.02$ and $r_f = 0.01$, and volatility $\sigma = 0.15$. The barrier is monitored daily, yielding 250 observation dates. For each method, we simulate $M = 100{,}000$ paths and report the standard deviation and 95% confidence interval width of the estimated option price.

Using plain Monte Carlo simulation, we obtain an unbiased estimate under discrete monitoring, but the resulting confidence interval is wide due to the binary nature of the barrier condition. Many paths are knocked out early and contribute zero payoff, inflating variance across the sample.

To address this, we first apply antithetic variates, which pairs each simulated path with a second path generated using negated Brownian increments. This approach reduces random fluctuations due to its symmetric construction and achieves a standard deviation reduction of over 70%, while preserving the unbiasedness of the estimator.

We then apply the control variate method, using the payoff of a vanilla European call option as the auxiliary variable, priced analytically using the Black-Scholes formula. Because this payoff is highly correlated with that of the barrier option (sample correlation exceeding 0.9), the adjusted estimator achieves dramatic gains in efficiency. In our baseline case, the variance is reduced by over 99%, and the confidence interval tightens by an order of magnitude.

Table 4 summarizes the numerical results for all three estimators. Each method yields a consistent price estimate, but with markedly different levels of statistical uncertainty.

These results confirm that even simple variance reduction methods can dramatically improve simulation efficiency. Antithetic variates offer a nearly cost-free improvement, while control variates are highly effective when an analytically priced, strongly correlated reference is available. These gains reduce computational overhead and enable more precise pricing estimates with fewer simulations, which is important particularly in high-dimensional or time-sensitive settings.

Method	Estimate	Std. Dev	95% CI Width	Variance Reduction (%)
Plain Monte Carlo	6.2943	9.7629	0.1210	0.00
Antithetic Variates	6.3278	5.2937	0.0656	70.60
Control Variates	6.1997	0.8493	0.0105	99.24

Table 4. Monte Carlo pricing results for down-and-out FX barrier option under GBM.

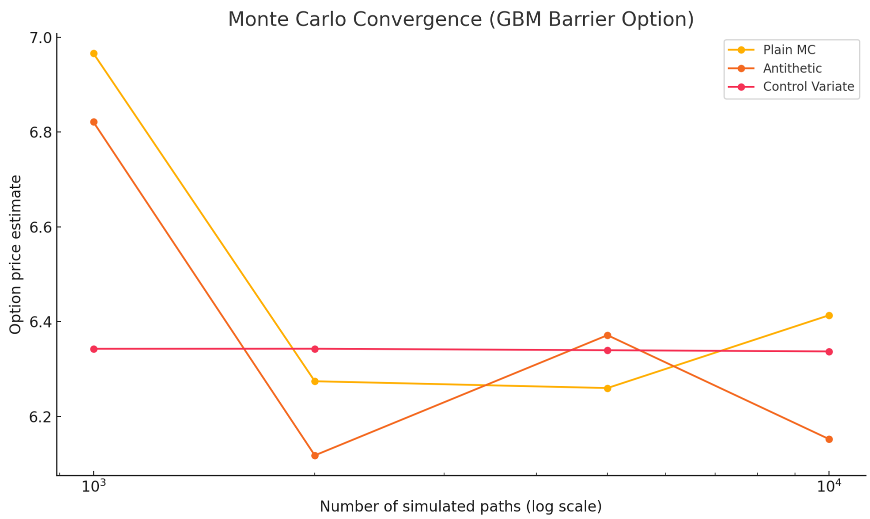

Figure 4. Convergence of Monte Carlo price estimates for the down-and-out FX barrier option under GBM. Control variates deliver the flattest curve, indicating the strongest variance reduction.

12. Heston Stochastic Volatility Model

The assumption of constant volatility in geometric Brownian motion is a significant simplification. In foreign exchange markets, volatility exhibits empirical features such as clustering, mean reversion, and correlation with asset price movements. To incorporate more realistic volatility dynamics observed in FX markets, we adopt the Heston stochastic volatility model, in which the asset price and variance evolve jointly under a coupled system of stochastic differential equations.

The Heston model modifies the asset dynamics by introducing a second stochastic process for variance. The system of SDEs under the risk-neutral measure becomes:

$$dS_t = (r_d - r_f)S_t\,dt + \sqrt{V_t}S_t\,dW_t^S,$$
$$dV_t = \kappa(\theta - V_t)\,dt + \sigma_v\sqrt{V_t}\,dW_t^V,$$

where V_t is the instantaneous variance and W_t^S, W_t^V are Brownian motions with correlation ρ, so that $dW_t^S dW_t^V = \rho dt$. This formulation accounts for several observed properties of volatility: it tends to revert to a long-term level θ, fluctuates with a degree of uncertainty proportional to σ_v, and typically exhibits negative correlation with asset returns (leverage effect).

Parameter	Interpretation
κ	Speed of mean reversion toward θ
θ	Long-run average variance
σ_v	Volatility of variance ("vol of vol")
ρ	Correlation between asset and variance shocks
V_0	Initial variance

Table 5. Heston model parameters and their financial interpretations.

Since barrier options do not admit closed-form solutions under the Heston model, we rely on simulation-based methods for pricing. We discretize the system using Euler-Maruyama. At each time step Δt, we simulate:

$$V_{t+\Delta t} = V_t + \kappa(\theta - V_t)\Delta t + \sigma_v \sqrt{V_t \Delta t}(\rho Z_1 + \sqrt{1-\rho^2} Z_2),$$
$$S_{t+\Delta t} = S_t \cdot \exp\left[(r_d - r_f - \tfrac{1}{2}V_t)\Delta t + \sqrt{V_t \Delta t} Z_1\right],$$

where $Z_1, Z_2 \sim \mathcal{N}(0,1)$ are independent. The variance path must be kept non-negative. To ensure stability, we apply a reflection scheme and replace negative values with a small positive threshold, i.e., $V_{t+\Delta t} \leftarrow \max(V_{t+\Delta t}, \varepsilon)$, where $\varepsilon = 10^{-6}$.

Remark 12.1. This adjustment is necessary for numerical stability, especially when the Feller condition $2\kappa\theta \geq \sigma_v^2$ is violated. While this introduces a slight upward bias in price estimation, it is a practical necessity.

Simulating under the Heston model is computationally more intensive than the constant volatility case. Each path requires generating two correlated random sequences and tracking both asset and variance trajectories. However, the model captures phenomena that are critical for pricing path-dependent instruments. For instance, volatility clustering significantly affects the likelihood of barrier breaches, and high-volatility periods increase the probability of knockouts.

Remark 12.2. In Figure 5, we observe that the stochastic volatility path exhibits periods of elevated variance. These clusters of volatility increase the effective diffusion of the spot rate, amplifying the likelihood of hitting a barrier even when the path starts well above it.

The Heston framework enables a direct comparison of simulation performance and pricing outcomes under constant versus stochastic volatility. In particular, it provides a setting to assess the robustness of variance reduction strategies and examine how volatility clustering affects barrier breach probabilities and estimator variance. While more complex, the Heston framework aligns more closely with real FX market behavior and enables more robust risk assessments for exotic options.

13. Empirical Evaluation under the Heston Model

To complement our earlier analysis under the geometric Brownian motion (GBM) framework, we now simulate and evaluate the pricing of the same down-and-out FX barrier option under the Heston stochastic volatility model. This model incorporates time-varying volatility through a mean-reverting square-root process and is better suited to capturing volatility clustering and the leverage effect observed in real-world FX markets.

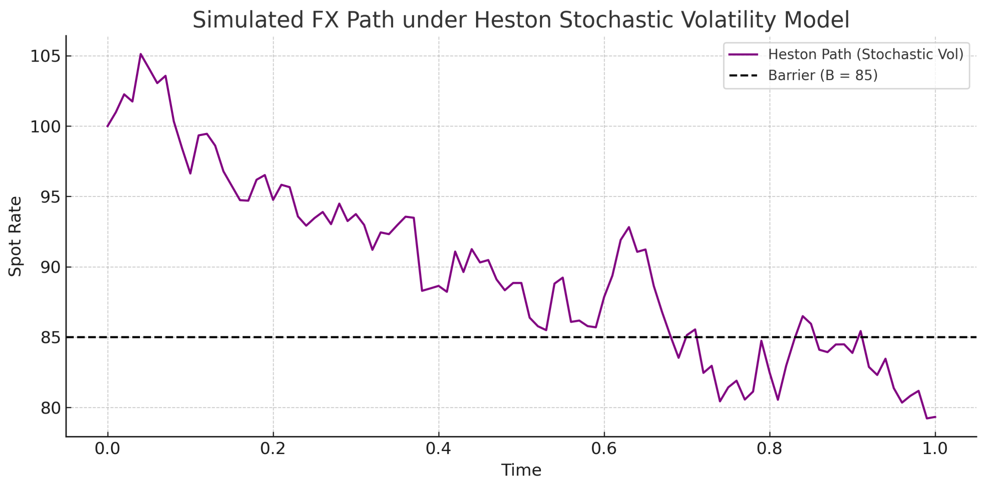

Figure 5. Illustrative comparison of GBM (dashed) and stochastic volatility (purple) FX paths. Volatility clustering increases path variability and affects barrier breach risk.

We simulate 100,000 FX spot paths using the full Heston model described in Section 12, with the following parameters:

- Initial variance $v_0 = 0.04$
- Mean reversion rate $\kappa = 2.5$
- Long-run variance $\theta = 0.04$
- Volatility of volatility $\sigma_v = 0.3$
- Correlation $\rho = -0.5$

The option contract setup is identical to the GBM experiment: spot rate $S_0 = 100$, strike $K = 100$, barrier $B = 85$, maturity $T = 1$ year, and 250 daily monitoring steps. We assume domestic and foreign risk-free rates $r_d = 0.02$ and $r_f = 0.01$.

Using plain Monte Carlo simulation without any variance reduction techniques, we obtain the following results:

Table 6. Barrier Option Price under Heston Model (100,000 Monte Carlo paths)

Method	**Estimate**	**Std. Dev**	**95% CI Width**
Plain Monte Carlo (Heston)	7.6666	11.7521	0.0728

These results show a noticeably higher price compared to the GBM benchmark. While the Heston model increases the probability of early barrier breach due to volatility spikes, it also introduces periods of elevated variance that increase the potential for large terminal payoffs, which can outweigh the additional knockouts. Although variance reduction methods were effective under GBM, their effectiveness under Heston is expected to decrease due to increased payoff variability and weakened correlation structures, especially for control variates derived from GBM assumptions.

Because our implementation uses only the plain Monte Carlo estimator for Heston, we leave a full variance reduction comparison under stochastic volatility for future work. However, this baseline result illustrates the impact of incorporating stochastic volatility into barrier option pricing and the need for careful calibration and computational strategy.

14. Discussion

This work investigated the pricing of down-and-out foreign exchange barrier options via Monte Carlo simulation under both constant and stochastic volatility models. Beginning with the geometric Brownian motion (GBM) framework, we implemented antithetic variates and control variates, and assessed their effectiveness through empirical experiments. Under GBM, variance reduction was substantial, with control variates achieving a reduction of over 99% in standard error when an appropriately chosen vanilla European option was used as the auxiliary variable. However, selecting an effective control variate is often nontrivial; its success depends on strong correlation with the target payoff and the availability of an exact or efficiently computed expected value.

We subsequently extended the model to incorporate stochastic volatility through the Heston framework, in which the spot rate evolves jointly with a mean-reverting square-root process for instantaneous variance. Simulation results under Heston indicated higher barrier option prices relative to the GBM benchmark. This increase can be attributed to volatility clustering and the broader distributional tails associated with the stochastic variance process. However, this realism comes at the cost of significantly increased estimator variance.

The results also highlight structural limitations of variance reduction under stochastic volatility. The control variate approach, while effective under GBM due to the availability of closed-form prices and strong payoff correlation, performs poorly under Heston unless the auxiliary variable is adapted to the stochastic structure. Furthermore, methods such as Brownian bridge interpolation, which improve discretization accuracy in the GBM case, are not directly transferable to Heston due to the lack of conditional tractability and Gaussian structure.

In summary, Monte Carlo methods remain a robust tool for pricing barrier options, particularly when analytical solutions are unavailable. Nevertheless, their practical effectiveness depends critically on the modeling framework and on the availability of variance reduction strategies that align with the structure of the underlying dynamics. The analysis here provides a benchmark for performance under both GBM and Heston models, and motivates further work on model-aware variance reduction techniques and more efficient simulation schemes under stochastic volatility.

15. Conclusion and Future Work

We investigated the pricing of down-and-out FX barrier options using Monte Carlo simulation under both constant and stochastic volatility models. In the constant volatility setting, we implemented antithetic variates, control variates, and Brownian bridge interpolation to address the high variance and discretization bias inherent in barrier option simulation. The control variate method, using the payoff of a European call option under the Black-Scholes framework, led to substantial variance reduction due to strong correlation between the auxiliary and target payoffs.

Under the Heston stochastic volatility model, we observed higher option prices and increased variance in the Monte Carlo estimator, reflecting the broader distribution of terminal payoffs induced by volatility clustering. Variance reduction methods designed for GBM were not directly transferable: while still applicable, their effectiveness was significantly reduced, particularly in the case of control variates.

Several extensions remain. First, constructing effective control variates under stochastic volatility, possibly through simulated vanilla payoffs with common variance paths, is a natural direction. Second, interpolation schemes adapted to non-Gaussian settings could improve bias correction in models such as Heston. Third, hybrid simulation methods that combine quasi-Monte Carlo techniques or multi-level methods with variance reduction may improve both convergence rate and computational efficiency.

This work provides a reference implementation of simulation-based pricing for barrier options and identifies structural limitations and methodological adjustments required in moving from GBM to more realistic dynamics.

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