Integral Geometry: Measuring by Averaging Intersections

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What Is Integral Geometry?

- Integral geometry studies how to compute **lengths, areas, volumes** by averaging over families of intersecting objects.
- Rather than measuring directly:
  - Ask: How often do lines or planes intersect the shape?
- Integrals are taken using **invariant measures**, unchanged by motion.
- Fundamental idea: translate measurement problems into intersection problems.
- Appears in convex geometry, probability, imaging, and tomography.

Crofton's Formula in $\mathbb{R}^2$

$$\text{Length}(C) = \frac{1}{4} \int \#(\ell \cap C)\, d\ell$$

- ℓ: oriented line in plane
- $d\ell$: measure over all directions and offsets
- $\#(\ell \cap C)$: number of intersection points
- $\frac{1}{4}$ accounts for orientation and direction (double counting)
- Used in stereology to estimate length from random line probes

Worked Example — Crofton on Circle

- Let $C = S^1$, the unit circle of radius 1.
- Lines are parameterized by $(\theta, p) \in [0, \pi) \times \mathbb{R}$
- A line intersects S^1 if $|p| < 1$, and intersects in 2 points
- We can plug in our values into the Crofton's formula to solve:

$$\int_0^\pi \int_{-1}^1 2\, dp\, d\theta = 8\pi$$

- Apply Crofton:

$$\text{Length}(S^1) = \frac{1}{4} \cdot 8\pi = 2\pi$$

- Confirms circumference using integral geometry

Line Space and Invariant Measure

- A line is determined by:
  - Direction angle $\theta \in [0, \pi)$
  - Signed perpendicular distance $p \in \mathbb{R}$
- So line space is $S^1 \times \mathbb{R}$
- Natural measure: $d\ell = dp\, d\theta$
- Invariance: translating or rotating the curve doesn't change the integral
- Essential for Crofton-type results to be geometric (coordinate-free)

Kinematic Formula for Curves

$$\int_{\mathrm{SE}(2)} \#(A \cap gB)\, dg = \frac{1}{\pi} \cdot L(A) \cdot L(B)$$

- A, B: curves in the plane
- $gB = R_\theta B + x$: rotated + translated version of B
- $SE(2) = \mathbb{R}^2 \rtimes SO(2)$: rigid motions in the plane
- Left-hand side: integrate over all positions + rotations of B
- Applications: average behavior in geometry, image analysis

Why the $\frac{1}{\pi}$?

- For small segments, probability of intersection depends on angle θ
- Contribution $\propto |\sin\theta|$
- Angular average:
$$\frac{1}{\pi}\int_0^{\pi} |\sin\theta|\, d\theta = \frac{2}{\pi}$$
- Normalization from uniform rotation leads to constant $\frac{1}{\pi}$
- Shows how motion symmetry controls average intersection behavior

Worked Example — Two Segments

- Let A, B be unit-length segments in the plane.
- Then:

$$\int_{\mathrm{SE}(2)} \#(A \cap gB)\, dg = \frac{1}{\pi}$$

- Proof sketch:
  - Break curves into infinitesimal segments.
  - Each pair contributes $\propto |\sin\theta|$.
  - Integrate over all directions + translations.
- More generally:

$$\text{Expected intersections} = \frac{1}{\pi} L(A) L(B)$$

- Example: if $L(A) = 2, L(B) = 3$, then expected intersections $= \frac{6}{\pi}$

Crofton in Higher Dimensions

$$\mathrm{Vol}_{n-k}(K) = c_{n,k} \int \#(K \cap E)\, dE$$

- Applies to volumes, areas, and lengths in $\mathbb{R}^n$
- E: affine k-dimensional plane
- Examples:
  - Length from plane slices in $\mathbb{R}^3$
  - Area from line probes
- $c_{n,k}$: dimension-dependent constant (often known explicitly)

Worked Example — Surface Area of Sphere

- Let $S^2 \subset \mathbb{R}^3$ be the unit sphere.
- Almost all lines intersect it in 2 points.
- Integrate over all such lines:

$$\int \#(S^2 \cap \ell)\, d\ell = 2 \cdot \mu(\text{lines hitting } S^2)$$

- Normalize to get:

$$\text{Area}(S^2) = 4\pi$$

- Works for convex bodies: surface area from counting line intersections

Grassmannians

- $G(k, n)$: space of all k-dimensional subspaces of $\mathbb{R}^n$
- Appears in Crofton-type integrals where we integrate over all planes of fixed dimension
- $\dim G(k, n) = k(n - k)$
- Natural invariant measure on $G(k, n)$: basis for uniform integration over orientations
- Used in Blaschke–Petkantschin formulas and stereological analysis

Summary

- Integral geometry = measure via intersection averages
- Crofton: lengths/areas via line or plane intersections
- Kinematic: intersections under motion — averages over $SE(n)$
- Grassmannians and invariant measures provide uniformity
- Applications: stereology, tomography, random shapes, convex geometry

Thank you! Questions?

Presented by Charvi Atreya, J

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