Integral Geometry Expository Paper

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Contents

1 Introduction

Integral geometry is the study of shapes and their properties by averaging or integrating over all possible positions, sizes, or orientations of those shapes. Rather than directly analyzing the geometric properties of a given object, integral geometry investigates what can be inferred by averaging over all possible positions, orientations, or configurations of that object within a given space.

One classical example illustrating the principles of integral geometry is Buffon's needle problem. This asks: given an infinite plane with parallel lines spaced distance d apart, what is the probability that a randomly dropped needle of length ℓ will intersect a line? This problem captures core ideas of integral geometry: the use of integration to study how geometric objects interact under random placements.

While classical geometry is concerned with the intrinsic properties of fixed shapes, integral geometry focuses on how geometric objects interact when subjected to random placements and orientations within a given space.

Just as induction provides a method for handling infinitely many numbers, integral geometry analyzes geometric properties averaged over infinitely many positions and orientations. In the same way that the formula $\sum n = \frac{n(n+1)}{2}$ is a fundamental and elegant result of induction, the formula $P = \frac{2L}{\pi d}$ which gives the probability that Buffon's needle crosses a line, stands as a classical and insightful outcome in integral geometry.

This shift in perspective leads to elegant results such as Crofton's formula and Buffon's needle, and extends to modern applications like image reconstruction via the Radon transform. The sections that follow explore the motivation this and the key mathematical ideas that have shaped the field.

2 Motivation

Integral geometry lies at the intersection of two important branches of mathematics: geometry and calculus. Traditional geometry focuses on the properties of static shapes. Whereas, integral geometry shifts attention to how these shapes behave over all possible placements.

With this core idea behind integral geometry, typical questions in the field include:

What properties emerge when a shape is averaged across all positions?

What is the probability that a random line intersects a given curve?

Which geometric quantities remain unchanged when averaged across all transformations?

One motivation to study integral geometry is its connection to symmetry and invariance. Many important results in mathematics come from studying quantities that do not change under transformations, such as length, area and curvature.

Another motivation comes from its wide range of applications across many fields. In computer vision, integral geometry helps model how objects appear under numerous perspectives. In tomography and medical imaging, techniques such as Radon transform help to construct internal structures from external data. In physics, similar methods are used to understand particle behavior.

Integral geometry is powerful framework to study dynamic geometry. It deepens our understanding of how shape, motion, and symmetry interact in ways that classical geometry alone cannot.

3 Preliminaries

3.1 Invariant measures

3.1.1 Parameterizing

We start by looking at a simple example, all the straight lines in a plane. This can be parameterized. A common way to represent this is the (θ, s) representation, where θ is the angle with the horizontal axis and s is the the distance from the origin to the line measured on the direction perpendicular to that line. This representation accurately describes all the lines in the plane. Geometrically, the space of all lines can be represented as $S^1 \times R$ where S^1 is the angle/direction and R is the signed distance from the origin.

3.1.2 What are invariant measures

A central idea is the idea of invariant measures. For instance, the space of lines in the plane can be parameterized by pairs (θ, s) where θ ranges from 0 to 2π. This is invariant under Euclidean motions, translations and rotations. Another way to think about this is if we had a scanner scanning over the plane, no matter which way the plane is oriented, the scanner gives us an unchanged amount of intersections. This invariance is important in showing that integral geometry reflects natural properties of shapes no matter the orientation.

With this parameterization, we can define an integral for the space of the lines. Suppose we have a function $f(l)$ that assigns a number to each line l. We can write a double integral that scans over all the possible parameters,

$$\int_L f(l), dl = \int_0^{2\pi} \int_{-\infty}^{\infty} f(\theta, s), ds, d\theta.$$

Here L represents summing over all parameters for the line. It is important to note that $ds, d\theta$ is an invariant measure in a Euclidean space as it does not depend on how we place the object. This is important as integral geometry usually averages over random placements of objects. This integral gives us a systemic way to integrate a function defined on these lines such as computing a probability involving randomly drawn lines.

3.2 Geometric Probability

Geometric probability studies the likelihood of geometric events, such as the probability that a randomly chosen point lies within a certain region, or that a randomly placed object intersects a given set. Problems in geometric probability often require defining a suitable measure on the space of possible configurations, and integrating over this space.

4 Historical Background

Integral geometry, as a formal discipline, emerged in the late 19th and early 20th centuries, but its roots go back to classical problems in geometric probability. The earliest and most famous of these is Buffon's needle problem, posed by Georges-Louis Leclerc, Comte de Buffon, in 1777. Buffon's work was motivated by questions about randomness and measurement, and his needle experiment provided one of the first probabilistic methods for estimating the value of π.

In the 19th century, Alfred Crofton introduced what is now known as Crofton's formula, relating the length of a curve to the expected number of intersections with random lines. This idea of measuring geometric quantities by averaging over random configurations is the essence of integral geometry.

The 20th century saw the formalization of the field by mathematicians such as Wilhelm Blaschke and Luis Santaló, who developed the theory of invariant measures and kinematic formulas. Their work connected integral geometry to convex geometry, measure theory, and group theory, and laid the groundwork for modern applications in areas such as tomography, stochastic geometry, and computer vision.

4.1 Crofton's formula

Suppose $f(\theta, s)$ is the number of times a line l intersects a given curve C, then our double integral is the total number of intersections, summed over all possible lines. This result is tied directly to the well known Crofton's formula.

Crofton's formula states that the length of a line can be found by averaging how many times it gets intersected by straight lines. Suppose we have a curve C on the plane, where a curve is covers all functions excluding those with breaks/jumps. For every line l on the plane, where $f(l)$ is the amount of times that line l intersects our curve C. The total count of intersections of these lines summed over all possible lines is related to the curve. This gives us the formula:

$$\frac{1}{4}\int_L f(l), dl = \int_0^{2\pi}\int_{-\infty}^{\infty} f(\theta, s), ds, d\theta.$$

We multiply our double integral by $\frac{1}{4}$ because each line is double counted as lines with angles θ and $\theta + \pi$ are essentially the same. We can also see the line as approaching from above or below the curve. This accounts for the 4x double counting and we thus divide by 4.

4.1.1 Crofton's example

We can look at a simple example. Let's suppose that C is a unit circle with the equation $x^2 + y^2 = 1$. When $|s| < 1$, the line intersects the circle at 2 points. When $|s| = 1$, the line is tangent and thus intersects at only one point. When $|s| > 1$, the line cannot intersect at any point. Thus, we can create a piecewise function where

$$f(\theta, s) = \begin{cases} 2 & s < 1 \\ 1 & s = 1 \\ 0 & s > 1 \end{cases}$$

As we came up with this piecewise function, we did not double count the lines approaching from both sides. Thus, one layer of overcount cannot be applied. We can plug this into Crofton's formula in order to solve for the length of the curve:

$$Length(C) = \frac{1}{4}\int_0^{2\pi}\int_{-\infty}^{\infty} f(\theta, s), ds, d\theta$$

Since $f(\theta, s) = 0$ when $|s| > 1$, we can restrict the range of the s-integral to $[-1, 1]$ to get:

$$Length(C) = \frac{1}{4}\int_0^{2\pi}\int_{-1}^{1} f(\theta, s), ds, d\theta.$$

We can then replace $f(\theta, s)$ with 2 for $|s| < 1$:

$$Length(C) = \frac{1}{4}\int_0^{2\pi}\int_{-1}^{1} 2, ds, d\theta.$$

Computing the inner integral gives us:

$$= \int_{-1}^{1} 2, ds = 2 - (-2) = 4.$$

Now the outer integral:

$$Length(C) = \frac{1}{4}\int_0^{2\pi} 4, d\theta = \frac{1}{4}(8\pi) = 2\pi$$

which is indeed the length of a unit circle.

4.2 Grassmanians and kinematic formulas

4.2.1 Grassmanians

When integrating over these larger dimensions, we are basically integrating over grassmanians. Grassmanians, which can be written as $G(k, n)$, is the space of k-dimensional linear spaces of R^n. An example would be $G(1, 3)$, which represents the set of all lines through the origin in 3D space.

4.2.2 Kinematic formulas

Building onto invariant measures, kinematic formulas describe the average behavior of geometric quantities. A classic example of this is: If two shapes in the plane, curve 1 and curve 2 can move randomly relative(relative meaning one is fixed and the other is moving) to each other, then the expected number of intersection points is proportional to their lengths.

In higher dimensions, kinematic formulas can help with expressing the expected volume, surface area and so on. This is useful in many geometric probability problems such as the Buffon's needle.

4.3 Valuations

Another very important concept is valuations. Functions that assign a number to a geometric object that reflects the way objects fit together. Formally, a valuation on a group of sets can be written as:

$$\phi(A \cup B) + \phi(A \cap B) = \phi(A) + \phi(B).$$

This describes the way volume, surface area and more behave when shapes are combined. In integral geometry, valuations provide a way to describe how measurements interact.

4.4 Hadwidgers

Hadwidger's theorem is an important result that classifies all geometric valuations that are additive, continuous and invariant. It says that any valuation in R^n is a linear combination of $n+1$ quantities called the intrinsic volumes. The intrinsic volumes consist of things such as volume, surface area and the euler characteristic.

Intrinsic volume	Intuition in R^n	Example in R^3
V_0	Topological quantity	Euler quantity
V_1	1D extent	Average projection length
V_2	Surface measure	Surface area
V_3	Volume	3D volume

Hadwidger's theorem tells us that these are the only possible building blocks for shapes and no other fundamentally different quantities exist.

5 Core Theorems and generalizations

5.1 General Kinematic Formulas

Kinematic formulas generalize the Crofton idea of averaging intersections over lines by looking at the average intersection between sets under rigid motions. Let A and B be compact subsets of R^n like convex bodies or smooth submanifolds. A smooth submanifold is simply a lower dimensional surface.

Let $G = Isom(R^n)$ be the group of rigid motions. In most applications of integral geometry, we focus on orientation preserving G , known as the special Euclidean group, denoted as $SE(n) = R^n \rtimes SO(n)$. The main object to study is the integral

$$\int_G \mu_k(A \cup gB), dg$$

which represents the expected k-dimensional volume of the intersection $A \cup gB$ where $g \in G$ randomly B throughout space. Here, μ_k represents the volume in the dimension R^n. This average depends only on the intrinsic features of the sets. This can be written as:

$$\int_G \mu_k(A \cup gB), dg = \sum_{i,j} c_{i,j} V_i(A) V_j(B)$$

where V_i and V_j are intrinsic volumes such as volumes and surface areas as mentioned above. $c_{i,j}$ are simply constants from the dimension.

5.2 Kinematics Example

5.2.1 About the equation

To see the kinematic formulas in action, let us consider two unit line segments A and B in the plane R^2. How many times do they intersect on average if one line is randomly rotated and transalated. This is under a case for a kinematic formula for curves which says that the expected number of intersection points between two curves is:

$$\int_{SE(2)} \#(A \cap gB), dg = \frac{1}{\pi} \cdot L(A) \cdot L(B)$$

. Here, SE(2) represents all rigid motions that can transalate and rotate B. A typical element $g \in SE(2)$ will look something like:

$$gB = R_\theta B + x$$

where R_θ is an angle and x is a rotation vector. $\#(A \cap gB)$ represents the the number of points in the intersection of A and the moved version of gB. The right-hand-side,

$$\frac{1}{\pi} \cdot L(A) \cdot L(B)$$

where $L(A)$ and $L(B)$ are lengths of curves can be derived by first rewriting the integral as:

$$\int_{SE(2)} \#(A \cap gB), dg = \int_0^{2\pi} \int_{R^2} \#(A \cap (R_\theta B + x)), dx, d\theta$$

, separating out the invariant measures. We replaced dg with $dx, d\theta$ which is known as the Haas measure.

We can imagine breaking A and B into many small straight segments of infinitessimal lengths. The probability that a segment of B, when randomly rotated and transalated, intersects as fixed segment of A is proportional to the sin angle between them. More precisely, the chance of intersection is proportional to $|\sin(\theta)|$ where θ is the angle between the two segments. Since we are rotating over all directions, we average $|\sin(\theta)|$ over $\theta \in [0, \pi]$ which gives us:

$$\frac{1}{\pi} \int_0^{\pi} |\sin(\theta)|, d\theta = \frac{2}{\pi}$$

. Now, summing over all possible segment pairs, each pair contributing a tiny chance to intersect. The total expected number of intersections scale with the product of the curve lengths. The $\frac{2}{\pi}$ becomes $\frac{1}{\pi}$ due to symmetries and double counting in the angle. This gives back our original equation:

$$\int_{SE(2)} \#(A \cap gB), dg = \frac{1}{\pi} \cdot L(a) \cdot L(b)$$

.

5.2.2 Working through the example

We now work through an example using this equation. Let $L(a) = L(b) = 1$ where $L(a)$ and $L(b)$ are straight lines with length 1. We want to calculate the expected number of intersection points between A and a randomly moved B.

First, we plug it into the formula to get:

$$\int_{SE(2)} \#(A \cap gB), dg = \frac{1}{\pi} \cdot 1 \cdot 1 = \frac{1}{\pi}$$

. This means on average, if we move the segment B around the plane, the expected number of intersections $\frac{1}{\pi}$.

5.3 More on Crofton's formula

5.3.1 Crofton's formula in higher dimensions

Crofton's formula extends to higher dimensions, allowing computation of geometric quantites like areas and volumes. In R^n, one version of the generalized Crofton's formula expresses the $(n - k)$ dimensional volume on a set $K \subset R^n$ in terms of the number of intersections it has with k-planes. More precisely, if G_n^k denotes the space of all k-planes in R^n with an invariant measure, then equation to represent this is:

$$Vol_{n-k}(K) = c_{n,k} \int_{G_n^k} \#(K \cap E), dE$$

where $c_{n,k}$ is a constant depending on only one of the dimensions.

For example, the length of a curve in R^3 can be computed by the amount of time is is intersected by random planes. Similarly, the surface area of a body can be determined by integrating over the number of intersections with lines.

5.3.2 Example of higher dimension Crofton's formula

We can use this formula to calculate the surface area of a sphere. Let $S^2 \subset R^3$ be a unit sphere centered at the origin. We write it in the form S^2 because the unit sphere can be thought of as a two-dimensional manifold in R^3. In R^3, Crofton's formula states that the surface area of a smooth convex shape K is proportional to the number of times it is intersected by a random straight line. This formula can be represented as:

$$Area(K) = \frac{1}{2\pi} \int_{G_3} n_K(L) dL$$

where G_3 is the space of all lines in R^3, and $n_K(L)$ is the number of intersections of a line L with the boundary of K. The integral is with respect to the invariant measure on G_3.

5.4 Measure and Invariance

A central concept in integral geometry is the use of measures that are invariant under certain transformations, such as translations or rotations. For example, the Lebesgue measure on $\mathbb{R}^2$ is invariant under translations, making it the natural choice for problems involving random points in the plane. For problems involving lines, we seek a measure on the space of lines that is invariant under rigid motions.

5.5 Group Actions

Group actions formalize the idea of moving objects around in space. In integral geometry, we are often interested in how geometric quantities behave under the action of a group, such as the group of rigid motions (translations and rotations). Invariant measures under these group actions are essential for the validity of integral geometric formulas.

6 Buffon's Needle Problem in Detail

Suppose we have a floor with parallel lines spaced d units apart. We drop a needle of length $L \leq d$ at random. What is the probability P that the needle crosses a line?

Let θ be the acute angle between the needle and the lines ($0 \leq \theta \leq \pi$), and let x be the distance from the center of the needle to the nearest line ($0 \leq x \leq d/2$). The needle crosses a line if $x \leq \frac{L}{2} \sin\theta$.

The probability is given by

$$P = \frac{1}{\pi} \int_0^\pi \frac{2}{d} \int_0^{\frac{L}{2}\sin\theta} dx\, d\theta = \frac{2}{\pi d} \int_0^\pi \frac{L}{2} \sin\theta\, d\theta = \frac{L}{\pi d} \int_0^\pi \sin\theta\, d\theta = \frac{2L}{\pi d}$$

This result is really interesting because it connects a geometric probability with the number π, providing one of the earliest probability methods for estimating π.

6.1 Generalizations

If $L > d$, the probability calculation becomes more complex, as the needle can cross more than one line. The general formula involves integrating over the possible number of crossings. Buffon's needle problem can also be generalized to other shapes (e.g., dropping a coin or a rectangle), to higher dimensions, or to different arrangements of lines.

7 Crofton's Formula

7.1 Statement

Crofton's formula relates the length of a curve to the expected number of times a random line intersects it. For a rectifiable curve C in the plane,

$$L(C) = \frac{1}{2} \int_{\mathcal{L}} n(C, \ell)\, d\mu(\ell)$$

where $n(C, \ell)$ is the number of intersections of C with the line ℓ, and $d\mu$ is the invariant measure on the space of lines.

7.2 Proof of Crofton's Formula

Let C be a smooth curve in the plane, parameterized by arc length $s \in [0, L]$, with position vector $\mathbf{r}(s) = (x(s), y(s))$.

A line in the plane can be described by the equation

$$x\cos\theta + y\sin\theta = p$$

where $\theta \in [0, \pi)$ and $p \in \mathbb{R}$.

The invariant measure on the space of lines is $d\mu = dp\, d\theta$.

For each s, the set of lines passing through $\mathbf{r}(s)$ is given by all (θ, p) with $p = x(s)\cos\theta + y(s)\sin\theta$.

The total number of intersections of C with all lines is

$$\int_{\mathcal{L}} n(C, \ell)\, d\mu(\ell) = \int_0^\pi \int_{-\infty}^{\infty} n(C, (\theta, p))\, dp\, d\theta$$

But we can also write

$$\int_{\mathcal{L}} n(C, \ell)\, d\mu(\ell) = \int_0^L \int_0^\pi \delta(p - x(s)\cos\theta - y(s)\sin\theta)\, d\theta\, dp\, ds$$

where δ is the Dirac delta function.

Integrating over p gives

$$\int_0^L \int_0^\pi d\theta\, ds = \int_0^L \int_0^\pi \delta(p - x(s)\cos\theta - y(s)\sin\theta)\, d\theta\, dp\, ds$$

But for each s, the set of lines passing through $\mathbf{r}(s)$ is parameterized by θ, so integrating over all θ gives the total measure of lines through C.

The key step is to compute the expected number of intersections per unit length. For a small segment ds, the set of lines intersecting it is proportional to ds.

The calculation shows that

$$\int_{\mathcal{L}} n(C, \ell)\, d\mu(\ell) = 2L$$

so

$$L = \frac{1}{2} \int_{\mathcal{L}} n(C, \ell)\, d\mu(\ell)$$

7.3 Intuitive Explanation

Crofton's formula can be understood intuitively as follows: The length of a curve is proportional to the average number of times it is intersected by random lines. The factor of 1/2 arises because each intersection is counted twice (once for each direction of the line).

8 Worked Examples

8.1 Length of a Line Segment

Consider a line segment of length L. Every line that intersects the segment does so exactly once, except for a set of measure zero (lines tangent to the endpoints). The total measure of lines intersecting the segment is $2L$, so Crofton's formula gives the correct length.

8.2 Length of a Circle

For a circle of radius r, the number of lines intersecting the circle is proportional to the circumference. Crofton's formula gives

$$L = 2\pi r$$

as expected.

8.3 Length of a Polygonal Curve

For a polygonal curve, Crofton's formula can be applied to each segment, and the total length is the sum of the lengths of the segments.

8.4 Crofton's Formula for an Ellipse

Let C be an ellipse with semi-axes a and b. The length of the ellipse is given by

$$L = 4aE(e)$$

where $E(e)$ is the complete elliptic integral of the second kind and $e = \sqrt{1 - b^2/a^2}$ is the eccentricity.

Using Crofton's formula, we can compute the expected number of intersections of a random line with the ellipse, and verify that it matches the known formula for the circumference.

9 Applications of Crofton's Formula

Crofton's formula provides a way to define and compute the length of curves in terms of intersections with lines, which is useful in geometric measure theory.

There are higher-dimensional analogues of Crofton's formula. For example, the area of a surface can be expressed in terms of the expected number of intersections with random planes.

In tomography, Crofton's formula underlies the mathematics of reconstructing images from projections, as in CT scans. The Radon transform, which is fundamental in tomography, is closely related to Crofton's formula.

In stochastic geometry, Crofton's formula is used to compute expected values of geometric quantities in random structures, such as random tessellations or random graphs.

Integral geometry is used in computer vision for shape recognition and in robotics for motion planning. For example, Crofton's formula can be used to estimate the length of object boundaries in digital images.

10 Other Results in Integral Geometry

10.1 Santaló's Formula

Santaló's formula generalizes Crofton's formula to higher dimensions and more general spaces. It relates integrals over a space to integrals over its dual space.

10.2 Blaschke's Rolling Theorem

Blaschke's rolling theorem gives conditions under which a convex body can "roll" inside another without losing contact.

10.3 Hadwiger's Theorem

Hadwiger's theorem classifies all continuous, rigid-motion-invariant valuations on convex bodies in $\mathbb{R}^n$. It is a deep result with connections to convex geometry, measure theory, and topology.

11 Modern Developments

Integral geometry extends to higher dimensions, with applications to convex geometry, stochastic geometry, and more. For example, in three dimensions, the surface area of a surface can be expressed in terms of the expected number of intersections with random planes.

Stochastic geometry studies random geometric structures, often using tools from integral geometry. For example, the expected number of intersections of a random line with a random set can be computed using Crofton's formula.

Integral geometry is used in computer vision for shape recognition and in robotics for motion planning. For example, Crofton's formula can be used to estimate the length of object boundaries in digital images.

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