Rational Billiards

How to Win Every Pentagonal Pool Game

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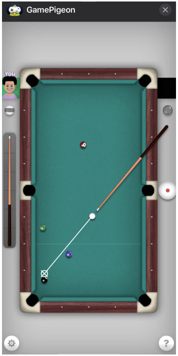
Pool Tables! (Billiards)



GamePigeon

Elliptical table

Pentagonal table???

What Are Billiards?

Definition

A **billiard system** is: A frictionless ball moving in a straight line reflecting off the sides of a polygon.

Rational Billiards

If all angles in the polygon are **rational multiples of** π, the system is called a **rational billiard**.

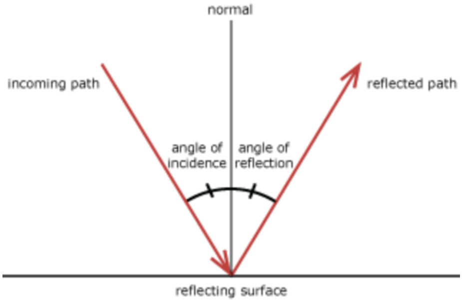
Properties of Billiards

Key Rules

The ball travels at a constant speed

It reflects with the **law of reflection** angle in = angle out.

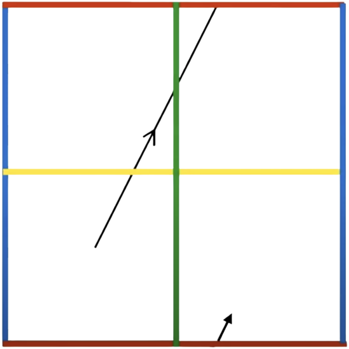
Some trajectories are periodic as the ball returns to where it started.



Unfolding

Unfolding

Instead of reflecting the ball, we reflect the polygon (unfolding it) and let the ball continue on a straight path.



Square Torus Translational Surface

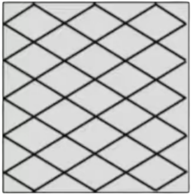
Square Billiards

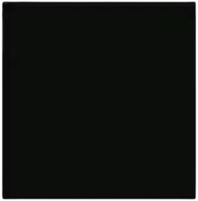
Trajectories on Square Billiards

If the ball's direction has a **rational slope**, the path is periodic. If the slope is **irrational**, the path fills the table.

Theorem (Periodicity in Square Billiards)

Let a billiard ball start at a point in a square table and travel in a direction with slope $\frac{p}{q}$, where p and q are integers with no common factors.
Then the billiard trajectory is periodic, and it returns to its starting point after bouncing exactly $2(p+q)$ times.

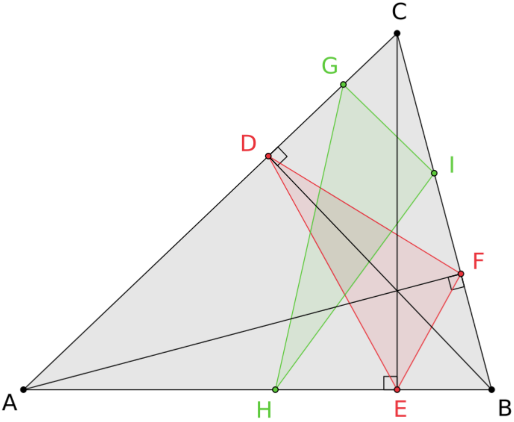




Fagnano Orbit in Acute Triangles

Fagnano Orbit

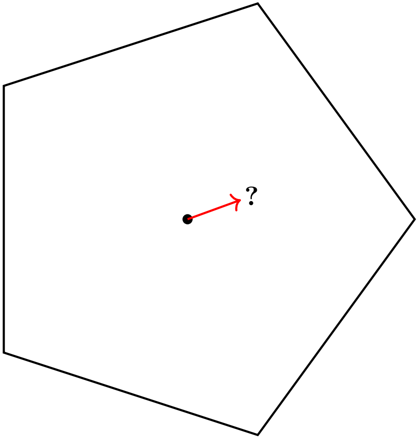
In an **acute triangle** there's a 3-bounce periodic orbit called the **Fagnano triangle** which connects the feet of the altitudes.



The Pentagon Problem

What happens when a ball bounces inside a regular pentagon?

Interior angles: $3\pi/5$; does not tile the plane well like a square



From Pentagon to Golden L

Reflect the pentagon to unfold a pentagon ring which can be simplified into a double pentagon

Apply a shear and cut/paste to obtain the Golden L

Golden L to Double Pentagon Shear

$$P = \begin{bmatrix} 1 & \cos\left(\frac{\pi}{5}\right) \\ 0 & \sin\left(\frac{\pi}{5}\right) \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & \cos\left(\frac{\pi}{5}\right) \\ 0 & \sin\left(\frac{\pi}{5}\right) \end{bmatrix}^{-1}$$

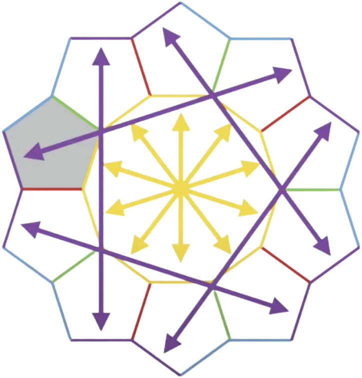
Example:

Example: Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then:

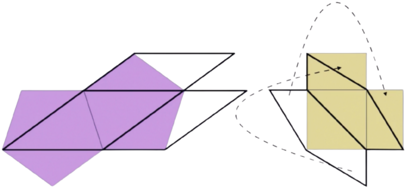
$$P \cdot \vec{v} = \begin{bmatrix} 1 & \cos\left(\frac{\pi}{5}\right) \\ 0 & \sin\left(\frac{\pi}{5}\right) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + \cos\left(\frac{\pi}{5}\right) \\ \sin\left(\frac{\pi}{5}\right) \end{bmatrix}$$

From Pentagon to Golden L

Pentagon Ring



Golden L

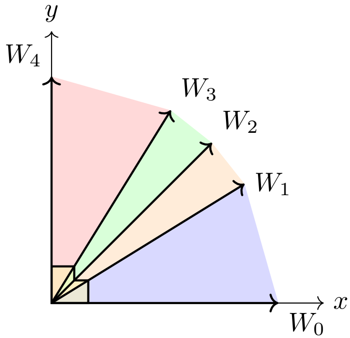


The Shears

Golden L separates first quadrant into 4 sections
Each section has a shear which maps the quadrant into the section

Section Shears

$$\sigma_0 = \begin{bmatrix} 1 & \phi \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 1 & 0 \\ \phi & 1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} \phi & 1 \\ \phi & \phi \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & \phi \\ 0 & 1 \end{bmatrix}$$



Building the Tree of Directions

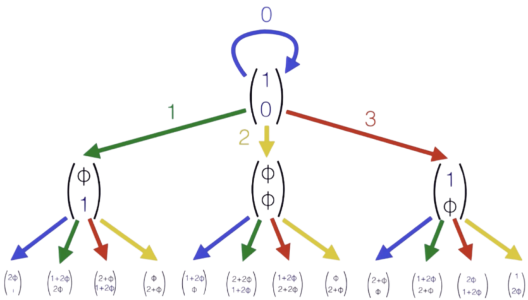
Construction of Tree

Start at $[1, 0]$, a known periodic direction

Apply one of four shears

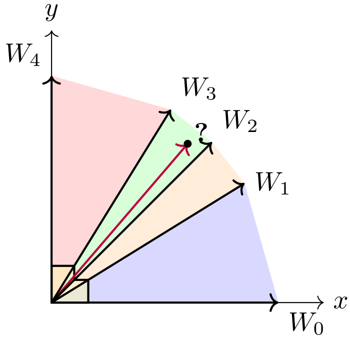
Repeat to generate an infinite tree

Tree word (e.g. 0213) encodes the path taken



In Reverse!

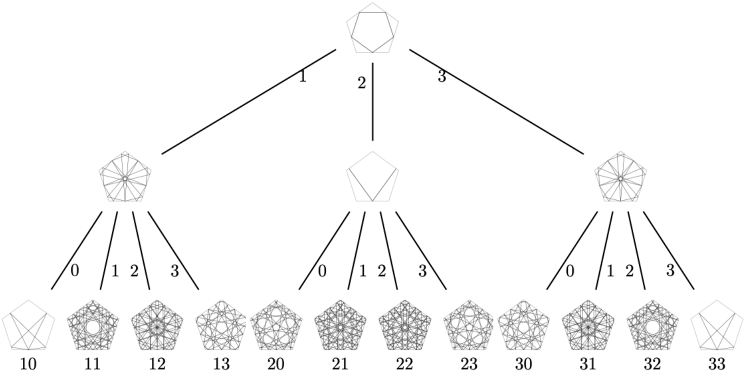
We can reverse engineer the periodic direction from a given point in the first quadrant



Tree of Periodic Directions Visualized

Theorem (Periodic Directions)

Every periodic billiard trajectory on the regular pentagon arises from a direction in the tree of periodic directions.



Combinatorial Period

Theorem (Form of Periodic Directions)

Each tree word $\to$ vector of the form $[a + b\phi,\ c + d\phi]$

Theorem (Combinatorial Period on the Double Pentagon)

Given a tree word corresponding to the vector $[a + b\phi,\ c + d\phi]$, the combinatorial period on the double pentagon is:

$$P_{\text{double}} = 2(a + b + c + d)$$

Theorem (Adjustment for Regular Pentagon)

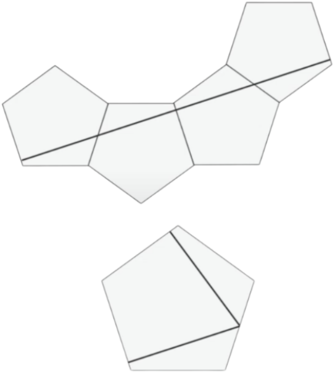
If the corresponding path on the double pentagon is:

Asymmetric, then $P_{\text{pentagon}} = P_{\text{double}}$

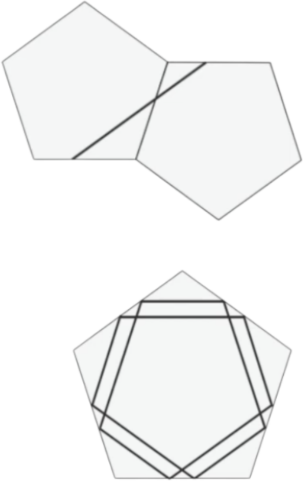
Symmetric, then $P_{\text{pentagon}} = 5P_{\text{double}}$

Combinatorial Period

Asymmetric



Symmetric



Wrapping Up

Summary

- Cool billiard tables
- A tree that captures every periodic direction in the regular pentagon
- A formula for periodic trajectories in the regular pentagon

Maybe we can win some games of pool now

Thanks for listening!