Riesz Representation Theorem

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Algebras and σ-algebras

For a set X, a σ-algebra on X is a collection of subsets of X such that the following properties hold:

1. $\varnothing, X$ are present.
2. If A is present, so is A^c.
3. If (A_1) is a countable sequence of sets in the σ-algebra, $\cup_{i=1}^{\infty} A_i$ is also in the σ-algebra.
4. If (A_1) is a countable sequence of sets in the σ-algebra, $\cap_{i=1}^{\infty} A_i$ is also in the σ-algebra.

The *Borel σ-algebra* for a topological space X is the σ-algebra generated by the open sets of X.

Measures

A measure μ is a function with a σ-algebra as its domain and the extended half-line $[0, +\infty]$ (or a subset of the extended half-line) as its range, and satisfies the following two properties.

1. $\mu(\varnothing) = 0$.
2. $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

If X is a set, $\mathscr{A}$ is a σ-algebra on X, and μ is a measure, then we call $(X, \mathscr{A})$ a *measurable space* and $(X, \mathscr{A}, \mu)$ a *measure space*.

A measure is *Borel* if it is a measure on the Borel σ-algebra for a topological space X.

Outer measure

We call the collection of all subsets of a set X the powerset of X, and denote it as $\mathcal{P}(X)$.

An *outer measure* on X is a function $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ such that

1. $\mu^*(\varnothing) = 0$,
2. if $A \subseteq B \subseteq X$, then $\mu^*(A) \leq \mu^*(B)$, and
3. if (A_n) is an infinite sequence of subsets of X, then $\mu^*(\cup_n A_n) \leq \sum_n \mu^*(A_n)$.

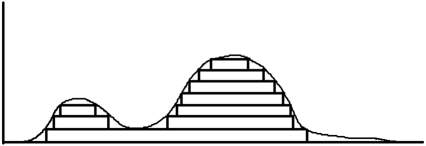
Measurable functions

Let $(X, \mathscr{A})$ be a measurable space, and let A be a subset of X that belongs to $\mathscr{A}$. For a function $f : A \to [-\infty, \infty]$, the function is called $\mathscr{A}$*-Measurable* if for each real number t the set $\{x \in A : f(x) \leq t\}$ belongs to $\mathscr{A}$.

Lebesgue Integration

The construction of an integral we can connect to a measure is necessary. As it's the focus of someone else's talk, I won't go into too much detail, but it follows this general formula.

1. Start by defining the integral for simple functions.
2. Continue by using a sequence of the integrals of simple functions to get the integral of a $[0, +\infty]$-valued function.
3. Subtract the integrals of the positive and negative parts of an extended-real-valued function to get its integral, defining integrals for functions on $[-\infty, \infty]$.



Locally compact Hausdorff spaces

A topological space X is called *Hausdorff* if for any two points a, b that are not equal, there exist disjoint neighborhoods of a and b.

If every point $x \in X$ has a neighborhood with a closure that is a compact subset of X, then X is called *locally compact*.

The Riesz Representation Theorem

The Riesz Representation Theorem itself was proven in 1909 by Riesz, for continuous real-valued functions on $[0, 1]$. It has since been generalized to all continuous functions.

The theorem statement is as follows: *Let X be a locally compact Hausdorff space, and let I be a positive linear functional on $\mathscr{K}(X)$. Then there is a unique Radon measure μ on X such that*

$$I(f) = \int f \, d\mu$$

holds for all f in $\mathscr{K}(X)$.

There also is a version involving Hilbert spaces, but that one I am admittedly not as familiar with.

Proving the Riesz Representation Theorem

The proof of the Riesz Representation Theorem involves 3 main steps:

1. Constructing the measure.
2. Verifying the construction.
3. Verifying that the measure is unique.

Questions, Answers, and a Duck

Pictured here is a duck, for asking questions to.