On the Riesz Representation Theorem

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Abstract

This paper intends to explain some core aspects of measure theory, including algebras and σ-algebras, measures, and a definition of the integral with respect to a measure. After covering these topics, the paper will then explore some of the properties and theorems of locally compact Hausdorff spaces, most notably the Riesz Representation Theorem, on which this paper focuses.

1 Introduction

The Riesz Representation Theorem is a core result of 20th century functional analysis. Riesz originally proved it in the paper [Rie09] in 1909. Riesz originally proved the theorem for the space $C[0,1]$, the space of continuous real-valued functions on $[0,1]$. The theorem has since been generalized to all continuous functions.

The theorem itself has applications in various fields, notably in functional analysis (fairly obviously), probability theory, and the study of L^p spaces. In functional analysis the theorem allows us to represent a linear functional on the space of continuous functions with compact support as an integral. In probability theory, we can represent the expectation of a random variable as an integral with respect to some probability measure. With L^p spaces, we can use the theorem to represent an integral on an L^p space as an integral with respect to a measure. This paper focuses only on the functional analysis and measure theory. Measure theory is a field in which certain structures called *σ-algebras* and *measures* on those σ-algebras are studied.

The main sources for this paper are [Coh13] and [Fol99].

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2 Preliminaries

Definition 2.1. The *extended real numbers*, written as $[-\infty, +\infty]$, consist of the real numbers combined with $+\infty$ and $-\infty$. For all $x \in [-\infty, +\infty], -\infty < x$, and $x \in [-\infty, +\infty], +\infty > x$. For arithmetic operations, we define them quite intuitively, leaving the sums

$$(+\infty) + (-\infty)$$

and

$$(-\infty) + (+\infty)$$

undefined. We also define the products

$$0 \cdot (+\infty), 0 \cdot (-\infty), (+\infty) \cdot 0, (-\infty) \cdot 0$$

as equal to 0.

Definition 2.2. Let X be an arbitrary set. Then let $\mathcal{T}$ be a collection of subsets of X with the following properties.

1. Both $\varnothing$ and X are elements of $\mathcal{T}$.
2. Let A be an arbitrary set. Whenever $\{X_\alpha : \alpha \in \mathcal{T}\} \subseteq \mathcal{T}$, the union $\bigcup_{\alpha \in A} X_\alpha$ is an element of $\mathcal{T}$.
3. Whenever $Y, Z \in \mathcal{T}$, $Y \cap Z$ is in $\mathcal{T}$ as well.

The collection $\mathcal{T}$ is called a *topology* on X, and the pair $(X, \mathcal{T})$, often just referred to as X, is called a *topological space*.

Definition 2.3. Let X be a topological space. Elements of the topology $\mathcal{T}$ on X are referred to as *open* sets. The complement of an open set is called *closed*. A set that is both open and closed is called *clopen*.

Definition 2.4. Let X be a topological space, and let A be a subset. The closure of A, denoted as $\bar{A}$, is the intersection of all closed subsets of X containing A.

Definition 2.5. Let X be a topological space, and let a be a point contained in X. An open set containing a is called a *neighborhood* of a.

3 Measure Spaces

We will start by discussing measure spaces, which are spaces with a notion of "measure" attached to a certain collection of their subsets.

3.1 Algebras and σ-algebras

Before we can do something involving a measure, we need to know what we can measure. Specifically, we will be measuring elements of σ-algebras, but it is nice to know about algebras and the distinction between algebras and σ-algebras.

Definition 3.1. Let X be an arbitrary set. An *algebra* $\mathscr{A}$ on X is a collection of subsets of X such that the following properties hold.

1. Both $\varnothing, X \in \mathscr{A}$.
2. If $A \in \mathscr{A}$, then $A^c \in \mathscr{A}$.
3. For each finite sequence $A_1, \ldots, A_n$ of sets belonging to $\mathscr{A}$, the set $\cup_{i=1}^n A_i$ belongs to $\mathscr{A}$.
4. For each finite sequence $A_1, \ldots, A_n$ of sets belonging to $\mathscr{A}$, the set $\cap_{i=1}^n A_i$ belongs to $\mathscr{A}$.

Notice that only one of the third or fourth properties is actually necessary. For a sequence of sets $A_1, \ldots, A_n$ in $\mathscr{A}$, the union $\cup_{i=1}^n A_i^c$ is in $\mathscr{A}$. Since $\mathscr{A}$ is closed under complementation, $\cap_{i=1}^n A_i$, which complements $\cup_{i=1}^n A_i^c$, is in $\mathscr{A}$. We can also quickly check the converse. For a sequence of sets $A_1, \ldots, A_n$ in $\mathscr{A}$, the intersection $\cap_{i=1}^n A_i^c$ is in $\mathscr{A}$. Since $\mathscr{A}$ is closed under complementation, $\cup_{i=1}^n A_i$, the complement of $\cap_{i=1}^n A_i^c$, is in $\mathscr{A}$. Next, we will define the σ-algebra, the structure that we will actually measure.

Definition 3.2. Let X be an arbitrary set. An σ-*algebra* $\mathscr{A}$ on X is a collection of subsets of X such that the following properties hold.

1. Both $\varnothing, X \in \mathscr{A}$.
2. If $A \in \mathscr{A}$, then $A^c \in \mathscr{A}$.
3. For each countable sequence $A_1, \ldots, A_n$ of sets belonging to $\mathscr{A}$, the set $\cup_{i=1}^{n} A_i$ belongs to $\mathscr{A}$.
4. For each countable sequence $A_1, \ldots, A_n$ of sets belonging to $\mathscr{A}$, the set $\cap_{i=1}^{n} A_i$ belongs to $\mathscr{A}$.

Here we have a few examples of collections that are σ-algebras on a set X.

Example. The set

$$\{\varnothing, X\}$$

is a σ-algebra on X.

Example. The power set $\mathcal{P}(X)$, which contains every subset of X is a σ-algebra on X.

Consider that all σ-algebras are algebras, as a finite union of sets $\cup_{i=1}^{n} A_i$ can be made into the countable union $\cup_{i=1}^{\infty} A_i, A_i : i > n = \varnothing$, which is equivalent. Similarly, a finite intersection of sets $\cap_{i=1}^{n} A_i$ can be made into the countable union $\cup_{i=1}^{\infty} A_i, A_i : i > n = X$, such that $\mathscr{A}$ is a σ-algebra on X.

Definition 3.3. Let X be a set, and let $\mathscr{A}$ be a σ-algebra on X. Then the pair $(X, \mathscr{A})$ is called a *measurable space.*

Now, we can actually generate a σ-algebra from a collection of subsets of a set. To do that, we need the following statement.

Proposition 3.4. *Let X be a set. Then the intersection of a nonempty collection of σ-algebras on X is a σ-algebra on X.*

Proof. Let $\mathcal{C}$ be a nonempty collection of σ-algebras on X, and let $\mathscr{A}$ be the intersection of the σ-algebras belonging to $\mathcal{C}$. Then, by what we have shown above, to show that $\mathscr{A}$ is a σ-algebra, we only need to show that it contains $\varnothing, X$, is closed under complementation, and is closed under formation of countable unions. First, $\varnothing$ and X are both in $\mathscr{A}$, as they are in all σ-algebras and thus in all $C \in \mathcal{C}$. Then $\mathscr{A}$ is also closed under complementation, as for any $A \in \mathscr{A}$, A is in all σ-algebras in $\mathcal{C}$. Then A^c is in all σ-algebras in $\mathcal{C}$ and so is in $\mathscr{A}$. Finally, suppose (A_i) is a sequence of sets in $\mathscr{A}$ and thus in all σ-algebras in $\mathcal{C}$. Then $\cup_{i=1}^{\infty} A_i$ is in all $\mathcal{C}$ and so in $\mathscr{A}$. □

Proposition 3.5. *For an arbitrary collection of subsets $\mathscr{C}$ of some set X, there exists a unique smallest σ-algebra on X containing $\mathscr{C}$.*

Proof. In this context, the smallest σ-algebra on X containing $\mathscr{C}$ is the one such that $\mathscr{A}$ is a σ-algebra on X containing $\mathscr{C}$, and every σ-algebra on X containing $\mathscr{C}$ also contains $\mathscr{A}$. Let $\mathcal{C}$ be the collection of all σ-algebras on X containing $\mathscr{C}$. Then $\mathscr{C}$ is nonempty, since it contains the σ-algebra containing every subset of X. The intersection of all elements of $\mathcal{C}$ is a σ-algebra containing $\mathscr{C}$ that is included in all σ-algebras containing $\mathscr{C}$. This σ-algebra is unique, as if $\mathscr{A}_1$ and $\mathscr{A}_2$ are both smallest σ-algebras containing $\mathscr{C}$, then $\mathscr{A}_1 \subseteq \mathscr{A}_2$ and $\mathscr{A}_2 \subseteq \mathscr{A}_1$, so $\mathscr{A}_1 = \mathscr{A}_2$. □

We say that this smallest σ-algebra is the σ-algebra generated by $\mathscr{C}$. Using this, we can define an important type of σ-algebra.

Definition 3.6. Let X be a topological space. Then the *Borel σ-algebra* $\mathscr{B}(X)$ is the σ-algebra generated by the open subsets of X. The *Borel subsets* of X are the subsets of X that are in $\mathscr{B}(X)$.

3.2 Measures

Now, we will define measures. There are actually two types of measures, *countably additive* and *finitely additive* measures. This paper will focus exclusively on countably additive measures.

Definition 3.7. Let $X, \mathscr{A}$ be a measurable space. A countably additive function μ is a function with domain $\mathscr{A}$ and a range of the extended half-line, or $[0, +\infty]$, and has the property that

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

is true for all countable sequences (A_i) of disjoint sets belonging to $\mathscr{A}$.

Definition 3.8. Let $X, \mathscr{A}$ be a measurable space. Then a measure μ is a function for which the following two properties are true.

1. $\mu(\varnothing) = 0$.

2. μ is countably additive.

This also gives us some intuition as to why we are using σ-algebras: a σ-algebra has the property that unions of countable sequences of elements of the σ-algebra are also elements of the σ-algebra, so if we take any countable sequence (A_i) of disjoint $\mathscr{A}$-measurable sets, we can measure each individual element, or we can measure any union of all these sets.

Definition 3.9. Let $(X, \mathscr{A})$ be a measurable space. Then if $A \in \mathscr{A}$, A is called *$\mathscr{A}$-measurable*.

Definition 3.10. If X is a set, $\mathscr{A}$ is a σ-algebra on X, and μ is a measure on $\mathscr{A}$, then $(X, \mathscr{A}, \mu)$ is called a *measure space*.

Definition 3.11. Let μ be a measure on a measurable space $(X, \mathscr{A})$. Then μ is a *finite* measure if $\mu(X) < +\infty$. If X is the union of a sequence $A_1, A_2, \ldots$ of sets in $\mathscr{A}$ where $\mu(A_i) < +\infty$ for all i, then μ is a *σ-finite* measure. The measure space is also called *finite* or *σ-finite* if μ is finite or σ-finite.

Definition 3.12. A measure on a measurable space $(X, \mathscr{B}(X))$ is called *Borel*.

Definition 3.13. Let X be a set, and let $\mathcal{P}(X)$ be the power set of X. An *outer measure* on X is a function $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ such that the following properties are true.

1. $\mu^*(\varnothing) = 0$.

2. If $A \subseteq B \subseteq X$, then $\mu^*(A) \leq \mu^*(B)$.

3. If (A_n) is an infinite sequence of subsets of X, then $\mu^*(\cup_n A_n) \leq \sum_n \mu^*(A_n)$.

A measure is not always an outer measure, and an outer measure is not always a measure.

Definition 3.14. Let X be a set, and let μ^* be an outer measure on X. A subset $B \subseteq X$ is *μ^*-measurable* or *measurable with respect to μ^** if

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$$

for all subsets A of X.

Proposition 3.15. *Let X be a set, let μ^* be an outer measure on X, and let $\mathscr{M}_{\mu^*}$ be the collection of all μ^*-measurable subsets of X. Then $\mathscr{M}_{\mu^*}$ is a σ-algebra, and the restriction of μ^* to $\mathscr{M}_{\mu^*}$ is a measure on $\mathscr{M}_{\mu^*}$.*

Proof. We begin by showing that $\mathscr{M}_{\mu^*}$ is an algebra of sets. First, we'll show that for all subsets $B \subseteq X$ such that $\mu^*(B) = 0$ or $\mu^*(B^c) = 0$, B is μ^*-measurable. The countable subadditivity of μ^* implies that for each subset A of X, $\mu^*(A) \leq \mu^*(A \cap B) + \mu^*(A \cap B^c)$. Then we need only check that each subset A of X has the property that

$$\mu^*(A) \geq \mu^*(A \cap B) + \mu^*(A \cap B^c)$$

. As one of $\mu^*(B)$ and $\mu^*(B^c)$ is zero, and μ^* is monotone, one of the terms on the right-hand side of the inequality vanishes, and the other has a measure that is at most $\mu^*(A)$. It follows that both $\varnothing$ and X are μ^*-measurable, so they are both in $\mathscr{M}_{\mu^*}$. Next, the equation

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$$

does not change if B and B^c are swapped, so $\mathscr{M}_{\mu^*}$ is closed under complementation. Now, suppose that B_1 and B_2 are both μ^*-measurable subsets of X. We will prove $B_1 \cup B_2$ is also μ^*-measurable. To do this, let A be an arbitrary subset of X. Then we have

$$\mu^*(A \cap (B_1 \cup B_2)) = \mu^*(A \cap (B_1 \cup B_2) \cap B_1) + \mu^*(A \cap (B_1 \cup B_2) \cap B_1^c),$$

which equals

$$\mu^*(A \cap B_1) + \mu^*(A \cap B_1^c \cap B_2).$$

Then we can use this with the fact $(B_1 \cup B_2)^c = B_1^c \cap B_2^c$ and the measurability of B_1 and B_2. Then

$$\begin{aligned}
&\mu^*(A \cap (B_1 \cup B_2)) + \mu^*(A \cap (B_1 \cup B_2)^c) \\
&= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c \cap B_2) + \mu^*(A \cap B_1^c \cap B_2^c) \\
&= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c) \\
&= \mu^*(A).
\end{aligned}$$

Since A was an arbitrary subset of X, $B_1 \cup B_2$ must be measurable. Thus $\mathscr{M}_{\mu^*}$ is an algebra. Next, to show it is a σ-algebra, suppose (B_i) is an infinite sequence of disjoint μ^*-measurable sets. We will show by induction that

$$\mu^*(A) = \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*(A \cap (\cap_{i=1}^{n} B_i^c))$$

holds for each subset A of X and each positive integer n. In the case where $n = 1$, the equation is just a restatement of the measurability of B_1. For the induction step, note that the μ^*-measurability of B_{n+1} and disjointedness of (B_i) imply that

$$\begin{aligned}
&\mu^*(A \cap (\cap_{i=1}^{n} B_i^c)) \\
&= \mu^*(A \cap (\cap_{i=1}^{n} B_i^c) \cap B_{n+1}) + \mu^*(A \cap (\cap_{i=1}^{n} B_i^c) \cap B_{n+1}^c) \\
&= \mu^*(A \cap B_{n+1}) + \mu^*(A \cap (\cap_{i=1}^{n+1} B_i^c)).
\end{aligned}$$

With this we complete the inductive step. Notice that we do not increase the right-hand side if we replace our initial $\mu^*(A \cap (\cap_{i=1}^{n} B_i^c))$ with $\mu^*(A \cap (\cap_{i=1}^{\infty} B_i^c))$, so by letting the n in the sum in the resulting inequality approach infinity, we find that

$$\mu^*(A) \geq \sum_{i=1}^{\infty} \mu^*(A \cap B_i) + \mu^*(A \cap (\cup_{i=1}^{\infty} B_i)^c).$$

Then, as μ^* is countably subadditive,

$$\begin{aligned}
\mu^*(A) &\geq \sum_{i=1}^{\infty} \mu^*(A \cap B_i) + \mu^*(A \cap (\cup_{i=1}^{\infty} B_i)^c), \\
&\geq \mu^*(A \cap (\cup_{i=1}^{\infty} B_i)) + \mu^*(A \cap (\cup_{i=1}^{\infty} B_i)^c) \\
&\geq \mu^*(A).
\end{aligned}$$

It follows that each preceding inequality must be an equality, and hence $\cup_{i=1}^{\infty} B_i$ is μ^*-measurable. Then $\mathscr{M}_{\mu^*}$ is closed under the formation of unions of disjoint sequences of sets. Since the union of an arbitrary sequence of sets (B_i) in $\mathscr{M}_{\mu^*}$ is the union of a sequence of disjoint sets

$$B_1, B_1^c \cap B_2, B_1^c \cap B_2^c \cap B_3, \ldots$$

, $\mathscr{M}_{\mu^*}$ is closed under formation of countable unions. Thus $\mathscr{M}_{\mu^*}$ is a σ-algebra.

Next, we show the restriction of μ^* to $\mathscr{M}_{\mu^*}$ is a measure. To do this we show that it is countably subadditive. Let (B_i) be a sequence of disjoint sets in $\mathscr{M}_{\mu^*}$. Then

$$\begin{aligned}\mu^*(\cup_{i=1}^{\infty} B_i) &\geq \sum_{i=1}^{\infty} \mu^*((\cup_{i=1}^{\infty} B_i) \cap B_i) + \mu^*((\cup_{i=1}^{\infty} B_i) \cap (\cup_{i=1}^{\infty} B_i)^c) \\ &\geq \sum_{i=1}^{\infty} \mu^*(B_i) + 0.\end{aligned}$$

Since the reverse inequality is given by countable subadditivity, then the restriction of μ^* to $\mathscr{M}_{\mu^*}$ is countably additive and thus a measure. □

Definition 3.16. If $(X, \mathscr{A})$ is a measurable space such that for each $x \in X$ the set $\{x\}$ is in $\mathscr{A}$, a finite or σ-finite measure μ such that $\forall x \in X,\ \mu(\{x\}) = 0$ is called *continuous*.

A more complex definition for continuous and discrete measures is necessary if $\mathscr{A}$ does not contain every $\{x\}$ or if μ is not finite or σ-finite, but that goes beyond the scope of this paper.

Definition 3.17. If $(X, \mathscr{A})$ is a measurable space such that for each $x \in X$ the set $\{x\}$ is in $\mathscr{A}$, a finite or σ-finite measure μ such that there exists a countable subset D of X where $\mu(D^c) = 0$ is called *discrete*.

We will be dealing not only with sets, but also functions. As such, we require a definition for measurability of functions.

Proposition 3.18. *Let $(X, \mathscr{A})$ be a measurable space, and let A be a subset of X that belongs to $\mathscr{A}$. For a function $f : A \to [-\infty, \infty]$, the following conditions are equivalent.*

1. *For each real number t the set $\{x \in A : f(x) \leq t\}$ belongs to $\mathscr{A}$.*
2. *For each real number t the set $\{x \in A : f(x) < t\}$ belongs to $\mathscr{A}$.*
3. *For each real number t the set $\{x \in A : f(x) \geq t\}$ belongs to $\mathscr{A}$.*
4. *For each real number t the set $\{x \in A : f(x) > t\}$ belongs to $\mathscr{A}$.*

Proof. The identity

$$\{x \in A : f(x) < t\} = \bigcup_n \{x \in A : f(x) \leq t - \frac{1}{n}\}$$

implies that every set in condition 2 is a union of a sequence of sets satisfying condition 1, so condition 1 implies condition 2. The sets of condition 3 can be expressed in terms of those in condition 2 with the identity

$$\{x \in A : f(x) \geq t\} = A - \{x \in A : f(x) < t\}.$$

Similarly, condition 3 implies condition 4 by means of

$$\{x \in A : f(x) > t\} = \bigcup_n \{x \in A : f(x) \geq t + \frac{1}{n}\}.$$

and condition 4 implies condition 1 with

$$\{x \in A : f(x) \leq t\} = A - \{x \in A : f(x) > t\}.$$

□

Definition 3.19 ($\mathscr{A}$-measurable function)**.** A function $f : A \to [-\infty, \infty]$ is $\mathscr{A}$-measurable if and only if it satisfies one, and thus all, of the above conditions.

Definition 3.20. Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces. A function $f : X \to Y$ is *measurable with respect to* $\mathscr{A}$ *and* $\mathscr{B}$ if for each B in $\mathscr{B}$ the set $f^{-1}(B)$ belongs to $\mathscr{A}$. A *Borel measure* is a function measurable with respect to the Borel σ-algebra on two spaces X and Y.

Proposition 3.21. *Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces, and let $\mathscr{B}_0$ be a collection of subsets of Y such that $\mathscr{B}_0$ generates $\mathscr{B}$. Then a function $f : X \to Y$ is measurable with respect to $\mathscr{A}$ and $\mathscr{B}$ if and only if $f^{-1}(B) \in \mathscr{A}$ for all B in $\mathscr{B}_0$.*

Proof. Going by the definition, every function f that is measurable with respect to $\mathscr{A}$ and $\mathscr{B}$ has the property that $f^{-1}(B) \in \mathscr{A}$ for all B in $\mathscr{B}_0$. Conversely, assume that $f^{-1}(B) \in \mathscr{A}$ for each B in $\mathscr{B}_0$. Let $\mathscr{C}$ be the collection of all subsets B of Y such that $f^{-1}(B) \in \mathscr{A}$. The identities $f^{-1}(Y) = X, f^{-1}(B^c) = (f^{-1}(B))^c$, and $f^{-1}(\cup_n B_n) = \cup_n f^{-1}(B_n)$ imply that $\mathscr{C}$ is a σ-algebra on Y. Since $\mathscr{C}$ includes $\mathscr{B}_0$, it must include the σ-algebra generated by $\mathscr{B}_0$, which is $\mathscr{B}$. Then f is measurable with respect to $\mathscr{A}$ and $\mathscr{B}$. □

4 Integrals in Measure Spaces

To deal with the Riesz Representation Theorem, we need a definition of the integral that is applicable to measure spaces. We will construct it in three parts.
We begin with simple functions. Let $(X, \mathscr{A})$ be a measurable space. We will write $\mathscr{S}$ to represent the collection of all simple real-valued $\mathscr{A}$-measurable functions on X and $\mathscr{S}_+$ to be the collection of nonnegative functions in $\mathscr{S}$.

Definition 4.1. Let $(X, \mathscr{A})$ be a measurable space. For a subset $B \subseteq X$, the *characteristic function* $\mathcal{X}_B$ is the function such that $\mathcal{X}_B(x) = 1$ if $x \in B$ and $\mathcal{X}_B(x) = 0$ if $x \notin B$.

Definition 4.2 (Simple function)**.** Let $(X, \mathscr{A})$ be a measurable space. A simple function $f : X \to \mathbb{C}$ is a function that is a finite linear combination of complex coefficients connected to the characteristic functions of sets in X. This would commonly be expressed in the form

$$f = \sum_{k=1}^{n} a_k \mathcal{X}_{A_k}.$$

As our simple functions are nonnegative and real-valued, they would be $f : X \to [0, +\infty)$. The coefficients would all be nonnegative real numbers.

Definition 4.3 (Integral for a nonnegative simple function)**.** Let μ be a measure on $(X, \mathscr{A})$. If $f \in \mathscr{S}_+$ and is given by $f = \sum_{i=1}^{m} a_i \mathcal{X}_{A_i}$ where $a_1, \ldots, a_m$ are nonnegative real numbers and $A_1, \ldots, A_m$ are disjoint subsets of X that belong to $\mathscr{A}$, then $\int f \, d\mu$, the integral of f with respect to μ, is defined as $\sum_{i=1}^{m} a_i \mu(A_i)$.

This sum is either a nonnegative real number or $+\infty$. We need to check that $\int f \, d\mu$ depends only on f and not on $a_1, \ldots, a_m$ and $A_1, \ldots A_m$ that f is given by. To do this, we can choose another unrelated pair of sequences $b_1, \ldots, b_n$ and $B_1, \ldots, B_n$ such that f is also given by $\sum_{j=1}^{n} b_j \mathcal{X}_{B_j}$, where $b_1, \ldots, b_n$ are nonnegative real numbers and $B_1, \ldots, B_n$ are disjoint subsets of X that belong to $\mathscr{A}$. Eliminating the sets A_i for which $a_i = 0$ and the sets B_j for which $b_j = 0$, we can assume $\cup_{i=1}^{m} A_i = \cup_{j=1}^{m} B_j$. Then, as $a_i = b_j$ if $A_i \cap B_j \neq 0$ and μ is additive,

$$\sum_{i=1}^{m} a_i \mu(A_i) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \mu(A_i \cap B_j) = \sum_{j=1}^{n} \sum_{i=1}^{m} b_j \mu(A_i \cap B_j) = \sum_{j=1}^{n} b_j \mu(B_j).$$

Thus, $\int f\ d\mu$ does not depend on the representation of f used in its definition. Next, we will check a few properties of this integral we'll use in the next part of the construction.

Proposition 4.4. *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f and g belong to $\mathscr{S}_+$, and let α be a nonnegative real number. Then*

1. $\int \alpha f\ d\mu = \alpha \int f\ d\mu$,
2. $\int (f+g)\ d\mu = \int f\ d\mu + \int f\ d\mu$,
3. *if $f(x) \leq g(x)$ holds at each x in X, then $\int f\ d\mu \leq \int g\ d\mu$.*

Proof. Suppose that $f = \sum_{i=1}^{m} a_i \mathcal{X}_{A_i}$, defined similarly to above, and $g = \sum_{j=1}^{n} b_j \mathcal{X}_{B_j}$, where both $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ are nonnegative real numbers and $A_1, \ldots, A_m$, $B_1, \ldots, B_n$ are disjoint $\mathscr{A}$-measurable sets. We can assume again that $\cup_{i=1}^{m} A_i = \cup_{j=1}^{n}$. Then we have for properties one and two

$$\int \alpha f\ d\mu = \sum_{i=1}^{m} \alpha a_i \mu(A_i) = \alpha \sum_{i=1}^{m} a_i \mu(A_i) = \alpha \int f\ d\mu,$$

$$\begin{aligned}\int (f+g)\ d\mu &= \sum_{i=1}^{m} \sum_{j=1}^{n} (a_i + b_j) \mu(A_i \cap B_j) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \mu(A_i \cap B_j) + \sum_{i=1}^{m} \sum_{j=1}^{n} b_j \mu(A_i \cap B_j) \\ &= \sum_{i=1}^{m} a_i \mu(A_i) + \sum_{j=1}^{n} b_j \mu(B_j) = \int f\ d\mu + \int g\ d\mu.\end{aligned}$$

For the third property, if $\forall x \in X$, $f(x) \leq g(x)$, then $g - f$ is in $\mathscr{S}_+$, and so

$$\int g\ d\mu = \int (f + (g-f))\ d\mu = \int f\ d\mu + \int (g-f)\ d\mu \geq \int f\ d\mu.$$

□

With these properties, we can check a weaker version of the monotone convergence theorem, which will be used as a tool in our definition.

Proposition 4.5. *Let $(X, \mathscr{A}, \mu)$ be a measure space, let f belong to $\mathscr{S}_+$, and let (f_n) be a nondecreasing sequence of functions in $\mathscr{S}_+$ such that $f(x) = \lim_n f_n(x)$ for all $x \in X$. Then $\int f\ d\mu = \lim_n \int f_n\ d\mu$.*

Proof. From third property of 4.4, we can see that

$$\int f_1\ d\mu \leq \int f_2\ d\mu \leq \cdots \leq \int f\ d\mu.$$

Thus $\lim_n \int f_n\ d\mu$ exists, and is less than or equal to $\int f\ d\mu$. Next, we'll show that it is also greater than or equal to, and thus equal to $\int f\ d\mu$. Let $0 < \varepsilon < 1$. We will construct a nondecreasing sequence of functions (g_n) such that $g_n \leq f_n$ for each n and such that $\lim_n \int g_n\ d\mu = (1-\varepsilon)\int f\ d\mu$. Then $(1-\varepsilon)\int f\ d\mu \leq \lim_n \int f_n\ d\mu$ implying $\int f\ d\mu \leq \lim_n \int f_n\ d\mu$, as ε is arbitrary (and can be arbitrarily close to 0).
Suppose that $a_1, \ldots, a_k$ are the nonzero values of f and that $A_1, ..., A_k$ are the sets on which these values occur, so that f is given by $\sum_{i=1}^k a_i \mathcal{X}_{A_i}$. For each n and i let $A(n,i) = \{x \in A_i : f_n(x) \geq (1-\varepsilon)a_i\}$. Then each $A(n,i)$ belongs to $\mathscr{A}$ and for each i the sequence $\{A(n,i)\}_{n=1}^\infty$ is nondecreasing and $A_i = \cup_n A(n,i)$. If we let $g_n = \sum_{i=1}^k (1-\varepsilon)a_i \mathcal{X}_{A(n,i)}$, then $g_n \in \mathscr{S}_+$ and $g_n \leq f_n$, and

$$\lim_n \int g_n\ d\mu = \lim_n \sum_{i=1}^k (1-\varepsilon)a_i\mu(A(n,i)) = \sum_{i=1}^k (1-\varepsilon)a_i\mu(A_i) = (1-\varepsilon)\int f\ d\mu.$$

□

Now we can use this to define the integral of an arbitrary $[0, \infty]$-valued $\mathscr{A}$-measurable function on X. For such a function f, let

$$\int f\ d\mu = \sup\left\{\int g\ d\mu : g \in \mathscr{S}_+ \sup and g \leq f\right\}.$$

Similar properties hold for this new integral. We will check them.

Proposition 4.6. *Let $(X, \mathscr{A}, \mu)$ be a measure space, let f be a $[0, +\infty]$-valued $\mathscr{A}$-measurable function on X, and let (f_n) be a nondecreasing sequence of functions in $\mathscr{S}_+$ such that $f(x) = \lim_n f_n(x)$ for all $x \in X$. Then $\int f\ d\mu = \lim_n \int f_n\ d\mu$.*

Proof. Since (f_n) is nondecreasing,

$$\int f_1 \, d\mu \leq \int f_2 \, d\mu \leq \cdots \leq \int f \, d\mu.$$

Thus $\lim_n \int f_n \, d\mu$ exists and $\lim_n \int f_n \, d\mu \leq \int f \, d\mu$. We now check the reverse. As $\int f \, d\mu$ is the supremum of those elements of $[0, +\infty]$ of the form $\int g \, d\mu$, where g ranges over the set of functions belonging to $\mathscr{S}_+$ such that $g \leq f$. Thus to prove that $\int f \, d\mu \leq \lim_n \int f_n \, d\mu$, we can check that if g is a function in $\mathscr{S}_+$ where $g \leq f$, then $\int g \, d\mu \leq \lim_n \int f_n \, d\mu$. Then suppose that g is such a function. Then $(\min(g, f_n))$ is a nondecreasing sequence of functions in $\mathscr{S}_+$ such that $g = \lim_n(\min(g, f_n))$, so $\int g \, d\mu = \lim_n \int(\min(g, f_n) \, d\mu$. Since $\int(\min(g, f_n)) \, d\mu \leq \int f_n \, d\mu$, it follows that $\int g \, d\mu \leq \lim_n \int f_n \, d\mu$. □

We also will show the other properties that we proved in 4.4 still hold.

Proposition 4.7. *Let $(X, \mathscr{A}, \mu)$ be a measure space, let f and g be $[0, +\infty]$-valued $\mathscr{A}$-measurable functions on X, and let α be a nonnegative real number. Then the following are true.*

1. $\int \alpha f \, d\mu = \alpha \int f \, d\mu$.
2. $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$.
3. *If $f(x) \leq g(x)$ for all $x \in X$, $\int f \, d\mu \leq \int g \, d\mu$.*

Proof. Let us take nondecreasing sequences (f_n) and (g_n) of functions in $\mathscr{S}_+$ such that $f = \lim_n f_n$ and $g = \lim_n g_n$. Then (αf_n) is a nondecreasing sequence of functions in $\mathscr{S}_+$ such that $\alpha f = \lim_n \alpha f_n$. Thus by using the properties of the integral on functions in $\mathscr{S}_+$ we get that

$$\int \alpha f \, d\mu = \lim_n \int \alpha f_n \, d\mu = \lim_n \alpha \int f \, d\mu = \alpha \int f \, d\mu.$$

Next, $(f_n + g_n)$ is a nondecreasing sequence of functions in $\mathscr{S}_+$ such that $f + g = \lim_n(f_n + g_n)$. Thus by using the properties of the integral on functions in $\mathscr{S}_+$ we get that

$$\begin{aligned} \int (f + g) \, d\mu &= \lim_n \int (f_n + g_n) \, d\mu \\ &= \lim_n (\int f_n \, d\mu + \int g_n \, d\mu) = \int f \, d\mu + \int g \, d\mu \end{aligned}$$

For property 3, consider that $f \leq g$, then the set of functions $h_f \subseteq \mathscr{S}_+$ such that $h \leq f,\ \forall h \in h_f$ is a subset of the set of functions $h_g \subseteq \mathscr{S}_+$ such that $h \leq g,\ \forall h \in h_g$, so it must be the case that

$$\int f\ d\mu \leq \int g\ d\mu.$$

□

Definition 4.8 (Integral)**.** Let f be an arbitrary $[-\infty, \infty]$-valued measurable function. Then f^+ is defined as $\max(+f(x), 0)$, and f^- is defined as $\max(-f(x), 0)$. These functions separate the positive and negative parts of $f(x)$ and are measurable. If $\int f^+\ d\mu$ and $\int f^-\ d\mu$ are both finite, then f is *integrable*. Then we can define the *integral* of f as

$$\int f\ d\mu = \int f^+\ d\mu - \int f^-\ d\mu.$$

In the case that $X = \mathbb{R}^d$ and μ is Lebesgue outer measure, this integral is referred to as the *Lebesgue integral*.

5 The Riesz Representation Theorem

At this point, we have almost enough background to get into the theorem. We still require some understanding of locally compact Hausdorff spaces before we begin.

5.1 Locally Compact Hausdorff spaces

Definition 5.1. Let X be a topological space. We call X *Hausdorff* if it has the property that for any two points $a, b \in X$, then there exist disjoint open sets $A, B \subseteq X$ such that $a \in A$ and $b \in B$.

Definition 5.2. Let A be a subset of a topological space X. If B is any set such that $\mathcal{O} = U_b : b \in B$ is a collection of open sets in X with the property that $A \subseteq \bigcup_{b \in B} U_b$, then we call $\mathcal{O}$ an *open cover* of A. If there is a finite subset $C \subseteq B$ such that $A \subseteq \bigcup_{b \in C} U_b$, then $\{U_b : b \in C\}$ is called a *finite subcover* of $\mathcal{O}$.

Definition 5.3. Let X be a topological space, and let $K \subseteq X$ be an arbitrary subset. We call K *compact* if every open cover of K in has a finite subcover. We call the topological space X a *compact space* if every open cover of X has a finite subcover.

Note that a compact set in one topological space is not necessarily a compact set in another.

Definition 5.4. Let X be a topological space. If every point $x \in X$ has a neighborhood with a closure that is a compact subset of X, then X is called *locally compact*.

It follows from these definitions that a locally compact Hausdorff space is a topological space that is locally compact and Hausdorff.

Example. Examples of locally compact Hausdorff spaces include the spaces under the discrete topology (spaces where all subsets are open), Euclidean spaces $\mathbb{R}^d$, and compact Hausdorff spaces.

We will prove the following results, which will assist us in proving the Riesz Representation Theorem.

Proposition 5.5. *Let X be a Hausdorff space and let K and L be compact disjoint subsets of X. Then there are disjoint open subsets U and V of X such that $K \subseteq U$ and $L \subseteq V$.*

Proof. Assume that K and L are nonempty, as we would otherwise have $\varnothing$ as one open set and X as the other. Let us begin with the case where K contains exactly one point, say X. For each y in L there is a pair U_y, V_y of disjoint open sets such that $x \in U_y$ and $y \in V_y$, as X is Hausdorff. Since L is compact, there is a finite set $y_1, \ldots, y_n$ such that $V_{y_1}, \ldots, V_{y_n}$ is an open cover for L. The sets U and V defined by $U = \cap_{i=1}^n U_{y_i}$ and $V = \cup_{i=1}^n V_{y_i}$ are then the required open sets. Now suppose K has more than one element. As we have shown for each x in K there are disjoint open sets U_x and V_x such that $x \in U_x$ and $L \subseteq V_x$, there is a finite set $x_1, \ldots, x_m$ such that $U_{x_1}, \ldots, U_{x_m}$ is an open cover for K. Then we can define $U = \cup_{i=1}^m U_{x_i}$ and $V = \cap_{i=1}^m V_{x_i}$. □

Lemma 5.5.1. *Let X be a locally compact Hausdorff space, let K be a compact subset of X, and let U be an open subset of X that includes K. Then there is an open subset V of X that has a compact closure and satisfies $K \subseteq V \subseteq \bar{V} \subseteq U$.*

Proof. We'll first show the case for a single point. Since X is locally compact, for some point $x \in X$, there is an open neighborhood W whose closure is compact. By replacing W with $W \cap U$, we can ensure that W is included in U. Now we must ensure that $\bar{W}$ is included in U. We'll use 5.5 to choose disjoint open sets V_1, V_2 that separate the compact sets x and $\bar{W} - W$. The closure of $V_1 \cap W$ is then compact and included in W and by extension U. Then, since each point in K has an open neighborhood whose closure is compact and included in U, there is some finite collection of these that covers K. Let V be the union of these sets in the open cover, then V is the desired set. □

Theorem 5.6 (Urysohn's Lemma)**.** *Let X be a normal topological space, and let E and F be disjoint closed subsets of X. Then there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ holds at each x in E and $f(x) = 1$ holds at each x in F.*

Proposition 5.7. *Let X be a locally compact Hausdorff space, let K be a compact subset of X, and let U be an open subset of X that includes K. Then there is a function f that belongs to $\mathscr{K}(X)$ has the property that $\mathcal{X}_K \leq f \leq \mathcal{X}_U$ and $\operatorname{supp}(f) \subseteq U$.*

Proof. Use 5.5.1 to choose an open set V with compact closure such that $K \subseteq V \subseteq \bar{V} \subseteq U$. According to 5.6, there is a continuous function $g : \bar{V} \to [0,1]$ such that $g(x) = 1$ for all $x \in K$ and $g(x) = 0$ for all $x \in (\bar{V} - V)$. Now define the function $f : X \to [0,1]$ by letting $f = g$ on $\bar{V}$ and vanish outside $\bar{V}$. The continuity of f follows as it is continuous on $\bar{V}$ and constant, implying continuous, on $X - V$. The support of f is included in $\bar{V}$ and so is included in U. □

5.2 Radon Measure

Definition 5.8. Let $f : X \to \mathbb{R}$ be a real-valued function where X is an arbitrary set. The support, denoted $\operatorname{supp}(f)$, is the set of points $x \in X$ such that $f \neq 0$.

In the case that X is a locally compact Hausdorff space, we will denote by $\mathscr{K}(X)$ the set of continuous functions $f : X \to \mathbb{R}$ for which $\operatorname{supp}(f)$ is compact. Then $\mathscr{K}(X)$ is a vector space over $\mathbb{R}$.

Proposition 5.9. *Let X and Y be Hausdorff topological spaces, and let $f : X \to Y$ be continuous. Then f is Borel measurable (measurable with respect to $\mathscr{B}(X)$ and $\mathscr{B}(Y)$).*

Proof. The continuity of f implies that if U is open in Y, then $f^{-1}(U)$ is an open and thus Borel subset of X. As the collection of Borel sets is generated by the open sets, f is Borel measurable by 3.21. □

Definition 5.10. A *Radon measure* on a measurable space $(X, \mathscr{A})$ is a Borel measure that is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets.

Let U be an open subset of a locally compact Hausdorff space X. We will write $f \prec U$ to indicate that $0 \leq f \leq \mathcal{X}_U$ and $\text{supp}(f) \subseteq U$. Before we can prove the theorem, we will require a few more lemmas.

Lemma 5.10.1. *Let X be a Hausdorff space, let K be a compact subset of X, and let U_1 and U_2 be open subsets of X such that $K \subseteq U_1 \cup U_2$. Then there are compact sets K_1 and K_2 such that $K = K_1 \cup K_2, K_1 \subseteq U_1$, and $K_2 \subseteq U_2$.*

Proof. Let $L_1 = K - U_1$ and $L_2 = K - U_2$. Then L_1 and L_2 are disjoint and compact, and so according to 5.5 they can be separated by disjoint open sets, which we'll call V_1 and V_2. If we define K_1 and K_2 by $K_1 = K - V_1$ and $K_2 = K - V_2$, then K_1 and K_2 are compact, are included in U_1 and U_2 respectively, and have K as their union. □

Lemma 5.10.2. *Let X be a locally compact Hausdorff space, let f belong to $\mathscr{K}(X)$, and let $U_1, \ldots, U_n$ be open subsets of X such that* $\text{supp}(f) \subseteq \bigcup_{i=1}^n U_i$*. Then there are functions $f_1, \ldots, f_n$ in $\mathscr{K}(X)$ such that $f = f_1 + f_2 + \cdots + f_n$ and such that for each i the support of f_i is included in U_i. Furthermore, if f is nonnegative, then $f_1, \ldots, f_n$ can be chosen so that all are nonnegative.*

Proof. If $n = 1$ we can let f_1 be f. So we begin by supposing $n = 2$. By using 5.10.1, we construct compact sets K_1 and K_2 such that $K_1 \subseteq U_1$, $K_2 \subseteq U_2$, and $\text{supp}(f) = K_1 \cup K_2$, then use 5.7 to construct functions h_1 and h_2 in $\mathscr{K}(X)$ such that $\mathcal{X}_{K_i} \leq h_i \leq \mathcal{X}_{U_i}$ and $\text{supp}(h_i) \subseteq U_i$ for $i = 1, 2$. Define functions g_1 and g_2 by $g_1 = h_1$ and $g_2 = h_2 - \min(h_1, h_2)$. Then g_1 and g_2 are non-negative, have supports included in U_1 and U_2. and satisfy $g_1(x) + g_2(x) = \max(h_1, h_2)(x) = 1$ at each x in the support of f. Then we can complete the case where $n = 2$ by letting $f_1 = g_1$ and $f_2 = g_2$.
The general case can be dealt with by induction. We can use what we have proven to write f as the sum of two functions, having supports included in $\cup_{i=1}^{n-1} U_i$ and U_n, then use the induction hypothesis to decompose the first of these functions to the sum of $n - 1$ valid functions. □

Lemma 5.10.3. *Let X be a locally compact Hausdorff space, and let μ be a Radon measure on X. If U is an open subset of X, then*

$$\begin{aligned}\mu(U) &= \sup\left\{\int f\, d\mu : f \in \mathscr{K}(X) \text{ and } 0 \leq f \leq \mathcal{X}_U\right\}\\ &= \sup\left\{\int f\, d\mu : f \in \mathscr{K}(X) \text{ and } f \prec U\right\}\end{aligned}$$

Proof. We can see that $\mu(U)$ is greater than or equal to the first supremum, which is greater than or equal to the second. Therefore it is enough to show that

$$\mu(U) \leq \sup\left\{\int f\, d\mu : f \in \mathscr{K}(X) \text{ and } f \prec U\right\}.$$

Let α be a number that satisfies $\alpha < \mu(U)$, and as μ is Radon, we can choose a compact subset K of U such that $\alpha < \mu(K)$. We can get a function f in $\mathscr{K}(X)$ satisfying $\mathcal{X}_K \leq f$ and $f \prec U$ from 5.7. Then $\alpha < \int f\, d\mu$, and so

$$\alpha < \sup\left\{\int f\, d\mu : f \in \mathscr{K}(X) \text{ and } f \prec U\right\}.$$

Since α was arbitrary and less than $\mu(U)$, we're done. □

We can now prove the theorem for which this paper is named, the Riesz Representation Theorem, originally proven by Riesz in [Rie09]. An interesting paper comparing and constructing several different proofs for the theorem that have appeared can be found in [Nar11].

Theorem 5.11. *Let X be a locally compact Hausdorff space, and let I be a positive linear functional on $\mathscr{K}(X)$. Then there is a unique Radon measure μ on X such that for all f in $\mathscr{K}(X)$,*

$$I(f) = \int f\, d\mu.$$

Proof. We start with showing the uniqueness of μ. Suppose that μ and ν are regular Borel measures on X such that $\int f\, d\mu = \int f\, d\nu = I(f)$ for each f in $\mathscr{K}(X)$. By 5.10.3, $\mu(U) = \nu(U)$ for each open subset U of X and then from outer regularity $\mu(A) = \nu(A)$ for each Borel subset A of X. Then $\mu = \nu$, so μ is unique.

We now begin the construction of a Radon measure representing the functional I. We will follow off of some ideas from 5.10.3 and the definition of outer regularity. We define a function μ^* on the open subsets of X as

$$\mu^*(U) = \sup\{I(f) : f \in \mathscr{K} \text{ and } f \prec U\},$$

and then extend it to all subsets of X with

$$\mu^*(A) = \inf\{\mu^*(U) : U \text{ is open and } A \subseteq U\}.$$

These are consistent in the sense that an open set is assigned the same value by both. Now we'll show that μ^* is an outer measure on X and every Borel subset of X is μ^*-measurable.
The relation $\mu^*(\varnothing) = 0$ and the monotonicity of μ^* are clear. We will check the countable subadditivity of μ. First suppose (U_n) is a sequence of open subsets of X. We will verify that

$$\mu^*(\bigcup_n U_n) \leq \sum_n \mu^*(U_n).$$

Let f be a function in $\mathscr{K}(X)$ and $f \prec \cup_n U_n$. Then $\text{supp}(f)$ is a compact subset of $\cup_n U_n$, so there is a positive integer N such that $\text{supp}(f) \subseteq \cup_{n=1}^N U_n$. By 5.10.2, f is the sum of functions $f_1, \dots, f_N$ that belong to $\mathscr{K}(X)$ and satisfy $f_n \prec U_n$ for $n = 1, \dots N$. Then

$$I(f) = \sum_{n=1}^N I(f_n) \leq \sum_{n=1}^N \mu^*(U_n) \leq \sum_{n=1}^\infty \mu^*(U_n).$$

With the above definition of μ^* on the open subsets of X, the inequality is verified.
Now, suppose (A_n) is an arbitrary sequence of subsets of X. The inequality $\mu^*(\cup_n A_n) \leq \sum_n \mu^*(A_n)$ is clear if $\sum_n \mu^*(A_n) = +\infty$. Suppose then that $\sum_n \mu^*(A_n) < +\infty$, let ε be a positive real number, and for each n use the above definition for $\mu^*(A)$ for subsets of X to choose an open set U_n that includes A_n and satisfies $\mu^*(U_n) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}$. Then from our inequality for open sets,

$$\mu^*(\cup_n A_n) \leq \mu^*(\cup_n U_n) \leq \sum_{n=1}^\infty \mu^*(U_n) \leq \sum_{n=1}^\infty \mu^*(A_n) + \varepsilon.$$

Since ε is arbitrary, the relation $\mu^*(\cup_n A_n) \leq \sum_n \mu^*(A_n)$ follows, and so μ^* is countably subadditive and thus an outer measure.
Since the family of μ^*-measurable sets forms a σ-algebra, we can show that every Borel subset of X is μ^*-measurable by checking that all open subsets of X are μ^*-measurable. So let U be an open subset of X. Then we can prove U is μ^*-measurable by showing that

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \cap U^c)$$

holds for each subset A of X where $\mu^*(A) < +\infty$. Suppose A is such a set, ε is a positive real number, and use $\mu^*(A) = \inf \{\mu^*(U) : U \text{ is open and } A \subseteq U\}$ to choose an open set V that includes A and satisfies $\mu^*(V) < \mu^*(A) + \varepsilon$. If we show that

$$\mu^*(V) > \mu^*(V \cap U) + \mu^*(V \cap U^c) - 2\varepsilon,$$

then

$$\mu^*(A) + \varepsilon > \mu^*(A \cap U) + \mu^*(A \cap U^c) - 2\varepsilon,$$

and since ε is arbitrary, that $\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \cap U^c)$ holds and U is μ^*-measurable. Thus, to show that it is the case, choose a function f_1 in $\mathscr{K}(X)$ that satisfies $f_1 \prec V \cap U$ and $I(f_1) > \mu^*(V \cap U) - \varepsilon$, let $K = \operatorname{supp}(f_1)$, and choose f_2 in $\mathscr{K}(X)$ such that $f_2 \prec V \cap K^c$, and $I(f_2) > \mu^*(V \cap K^c) - \varepsilon$. Since $f_1 + f_2 \prec V$ and $V \cap U^c \subseteq V \cap K^c$,

$$\mu^*(V) \geq I(f_1 + f_2) > \mu^*(V \cap U) + \mu^*(V \cap U^c) - 2\varepsilon.$$

Then our inequality holds, so every Borel subset of X is μ^*-measurable.
For the next part of the proof, suppose A is a subset of X and f belongs to $\mathscr{K}(X)$. We will show that if $\mathcal{X}_A \leq f$, then $\mu^*(A) \leq I(f), while if 0 \leq f \leq \mathcal{X}_A$ and if A is compact, then $I(f) \leq \mu^*(A)$. First, assume $\mathcal{X}_A \leq f$. Let ε be such that $0 < \varepsilon < 1$, and define U_ε as $U_\varepsilon = \{x \in X : f(x) > 1 - \varepsilon\}$. Then U_ε is open, and each g in $\mathscr{K}(X)$ where $g \leq \mathcal{X}_{U_\varepsilon}$ also has the property that $g \leq \frac{1}{1-\varepsilon} f$, so $\mu^*(U_\varepsilon) \leq \frac{1}{1-\varepsilon} I(f)$. As $A \subseteq U_\varepsilon$ and since ε can be arbitrarily close to 0, $\mu^*(A) \leq I(f)$. Next, suppose $0 \leq f \leq \mathcal{X}_A$ and A is compact. Let U be an open set including A. Then $f \prec U$ and so $I(f) \leq \mu^*(U)$. Since U was an arbitrary open set including A, $I(f) \leq \mu^*(A)$.
We will finish the proof by verifying that a restriction of our construction is a measure that satisfies the conditions of the theorem. Let μ be the restriction of μ^* to the Borel subsets of X, let μ_1 be the restriction of μ^* to the σ-algebra

of $\mathscr{M}_{\mu^*}$ of μ^*-measurable sets. We will show μ and μ_1 are Radon measures, and

$$\int f\, d\mu = \int f\, d\mu_1 = I(f)$$

for all f in $\mathscr{K}(X)$. By 3.15, μ_1 is a measure on $\mathscr{M}_{\mu^*}$ and since $\mathscr{B}(X) \subseteq \mathscr{M}_{\mu^*}$ as we have shown earlier, μ is a measure on $\mathscr{B}(X)$. By 5.7, for each compact subset K of X there is a function f belonging to $\mathscr{K}(X)$ where $\mathcal{X}_K \leq f$. This implies that μ and μ_1 are finite on compact sets. The outer regularity of μ and μ_1 follows from our earlier equation $\mu^*(A) = \inf\{\mu^*(U) : U \text{ is open and } A \subseteq U\}$, and the inner regularity follows from $\mu^*(U) = \sup\{I(f) : f \in \mathscr{K} \text{ and } f \prec U\}$. Then μ and μ_1 are Radon. Next, we'll look at the identity $\int f\, d\mu = \int f\, d\mu_1 = I(f)$. Since each function in $\mathscr{K}(X)$ is the difference of two nonnegative functions in $\mathscr{K}(X)$, we can focus on the nonnegative functions in $\mathscr{K}(X)$. Let f be such a nonnegative function, and ε be a positive number. For each positive integer n define a function $f_n : X \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } f(x) \leq (n-1)\varepsilon, \\ f(x) - (n-1)\varepsilon & \text{if } (n-1)\varepsilon < f(x) \leq n\varepsilon, \\ \varepsilon & \text{if } n\varepsilon < f(x). \end{cases}$$

Then each f_n belongs to $\mathscr{K}(X)$, $f = \sum_n f_n$, and there is a positive integer N such that $f_n = 0$ if $n > N$. Let $K_0 = \text{supp}(f)$ and for each positive integer n let $K_n = \{x \in X : f(x) \geq n\varepsilon\}$. Then $\varepsilon\mathcal{X}_{K_n} \leq f_n \leq \varepsilon\mathcal{X}_{K_{n-1}}$ for all positive integers n, so the properties of the integral imply $\varepsilon\mu(K_n) \leq I(f_n) \leq \varepsilon\mu(K_{n-1})$ and $\varepsilon\mu(K_n) \leq \int f_n\, d\mu \leq \varepsilon\mu(K_{n-1})$ for each n. As $f = \sum_{n=1}^N f_n$, it follows that

$$\sum_{n=1}^{N} \varepsilon\mu(K_n) \leq I(f) \leq \sum_{n=1}^{N-1} \varepsilon\mu(K_n)$$

and

$$\sum_{n=1}^{N} \varepsilon\mu(K_n) \leq \int f\, d\mu \leq \sum_{n=1}^{N-1} \varepsilon\mu(K_n).$$

Thus $I(f)$ and $\int f\, d\mu$ both lie in the interval $[\sum_{n=1}^{N} \varepsilon\mu(K_n), \sum_{n=1}^{N-1} \varepsilon\mu(K_n)]$, with length $\varepsilon\mu(K_0) - \varepsilon\mu(K_nN)$. As ε is arbitrary, $I(f)$ and $\int f\, d\mu$ must be equal. It is clear that $\int f\, d\mu = \int f\, d\mu_1$, so our proof for the Riesz Representation Theorem is complete. $\square$

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