Ramanujan Graphs

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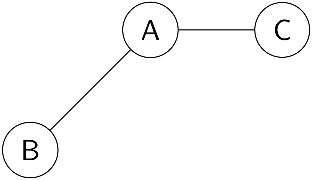
Adjacency Matrix

Definition

An *adjacency matrix* is a way to represent a graph in a square matrix where we put a 1 in A_{ij} and A_{ji} if an edge exists between vertices i and j and a 0 otherwise.



$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Figure: Interested in Eigenvalues

Expander Graphs

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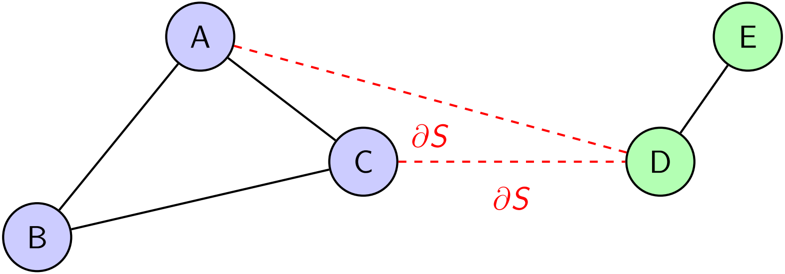
Remark

Their is no way to calculate $h(G)$ is polynomial time. PNP.

Cheeger Constant

Example:

The Cheeger Constant for this graph is 1. In particular when $|S| = 2$ and $|\partial S| = 2$ we have $h(G) = \frac{|\partial S|}{|S|} = \frac{2}{2} = 1$



Cheegers Inequality

Theorem (Alon-Milman, 1985)

Let G be a d-regular graph on n vertices and let the eigenvalues of its adjacency matrix be: $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$ then:

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}$$

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$$A \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \underbrace{1+1+\cdots+1}_{d} \\ \underbrace{1+1+\cdots+1}_{d} \\ \vdots \\ \underbrace{1+1+\cdots+1}_{d} \end{bmatrix} = \begin{bmatrix} d \\ d \\ \vdots \\ d \end{bmatrix} = d \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Bounding Non-Trivial Eigenvalues

Theorem (Alon-Boppana, 1991)

In any d-regular graph with diameter δ let its eigenvalues be $\lambda_1 \geq \lambda_2 \geq \ldots, \geq \lambda_n$ Then

$$\lambda_2 \geq 2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{\lfloor \delta/2 \rfloor}$$

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Theorem (Weaker version of Alon-Boppana, 1991)

Among the non-trivial eigenvalues let the eigenvalue with the greatest magnitude be σ. We have that:

$$\sigma \geq 2\sqrt{d-1} \cdot (1 - O(1))$$

Proof of the Weaker version of Alon-Boppana

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- For any k the entry a_{ij} of A^k is the number of paths from i to j

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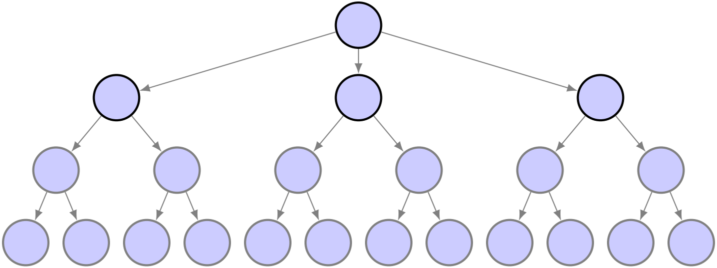

Proof of the Weaker version of Alon-Boppana

Proof Continued:

- $trace(A^{2k}) \geq n \cdot \#$ of closed paths $= n \cdot C_k(d-1)^k = n \cdot \frac{1}{k+1}\binom{2k}{k}(d-1)^k$

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- $d^{2k} + n(\sigma^{2k}) \geq \sum \lambda_i^{2k} = trace(A^{2k})$
- $\sigma \geq 2\sqrt{d-1} \cdot (1 - O(1))$

Bounding Non-Trivial Eigenvalues

Theorem (Friedman, 2003)

For a random d-regular graph and some $\epsilon > 0$ the probability that

$$\sigma \leq 2\sqrt{d-1} + \epsilon$$

tends to 1 as the number of vertices goes to infinity.

We have now established both a lower bound and upper bound on the non-trivial eigenvalues.

Ramanujan Graphs

Definition

Ramanujan Graphs are d-regular graphs for which all non-trivial eigenvalues satisfy $|\lambda_i| \leq 2\sqrt{d-1}$

Now some questions to consider:

1. Can we find explicit constructions of Ramanujan Graphs?
2. Do infinite Ramanujan Graphs exist?

Construction of Ramanujan Graphs

Constructions (Lubotzky-Phillips-Sarnak 1988, Margulis 1988)

Provided an explicit construction for the case where $d - 1 = p$ using Cayley Graphs

Constructions (Morgenstern, 1994)

Generalized the construction to $d - 1 = p^k$

Existence of Infinite Bipartite Ramanujan Graphs

Construction (Marcus Spielman Srivastava, 2015)

- Take a complete $K_{d,d}$ graph
- Perform a 2-lift
- Interlacing families to conclude that their exists at least one 2-lift

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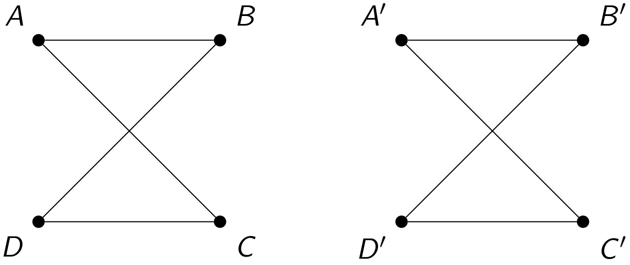

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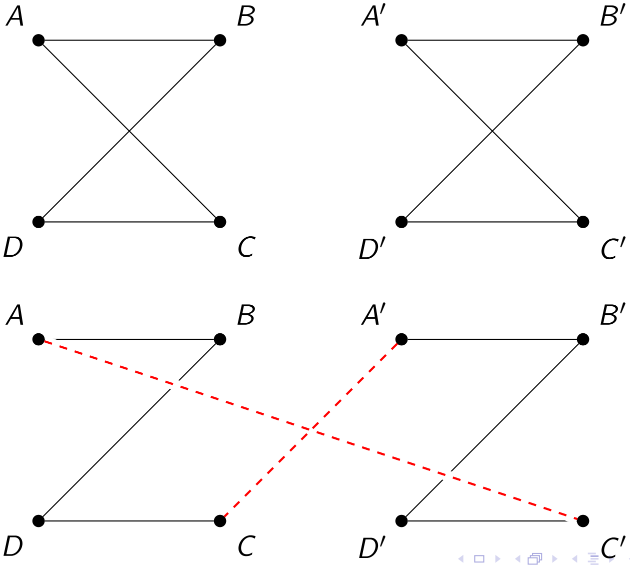

Acknowledgments

Acknowledgments

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