RAMANUJAN GRAPHS

AYUSH BANSAL

1. Introduction

Ramanujan graphs are d regular graphs with its non-trivial eigenvalues all satisfying the bound $|\lambda_i| \leq 2\sqrt{d-1}$. This bound turns out to be tight by the results of Alon-Boppana and Friedman which together imply that Ramanujan Graphs are the best expanders. We first talk about common results in spectral graph theory such as Cheegers inequality and Then we will give two constructions the first by Lubotzky, Phillips and Sarnak which uses heavy duty number theory and then by Marcus, Spielman and Srivastava which uses a probabilistic method. Finally we will wrap up by talking about the Ihara Zeta function and show that Ihara Zeta function of a graph is equivalent to the Riemann Hypothesis if and only if the our graph is Ramanujan.

2. Basic Notation

A graph is a set of vertices V and edges E connecting any two different vertices together. In this paper we will represent graphs in so called adjacency matrices which when analyzed can tell us a great deal about the properties of our graph. An *adjacency matrix* is a way to represent a graph with n vertices in a $n \times n$ square matrix where we put a 1 in A_{ij} and A_{ji} if an edge exists between vertices i and j and a 0 otherwise. In a graph where their are no loops the main diagonal consists of only 0s as no edge exists between vertex i and i.

Example. The adjacency matrix for Figure 1 is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

In particular analyzing the eigenvalues of this adjacency matrix turns out to be fruitful. In this paper we study graphs which are k-regular meaning each vertex has degree of k. Immediately one can notice that any k-regular graph's adjacency matrix has a trivial eigenvalue

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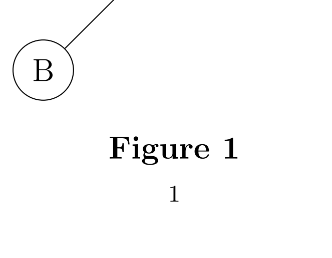


Figure 1

k with its corresponding eigenvector being

$$V = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

This is because when we perform the matrix multiplication $A \cdot V = A_1$ the i'th entry of A_1 is the sum of all the entries in the i'th row of A. By definition, the sum of the entries in i'th row gives the degree of i which is k in our case. In particular:

$$A \cdot V = \begin{bmatrix} \underbrace{1 + 1 + \cdots + 1}_{k} \\ \underbrace{1 + 1 + \cdots + 1}_{k} \\ \vdots \\ \underbrace{1 + 1 + \cdots + 1}_{k} \end{bmatrix} = \begin{bmatrix} k \\ k \\ \vdots \\ k \end{bmatrix} = k \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

The *spectrum* of a graph the ordered set of eigenvalues of its adjacency matrix: $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n$. Analyzing eigenvalues of graphs can help us identify good expander graphs which are graphs which are graphs that are both sparse meaning they contain a minimal number of edges while being highly connected. We will describe a way to measure how connected a graph is in the following section.

Remark 2.1. In a connected and undirected graph the minimum number of edges is $n - 1$ which corresponds to a tree while the maximum number of edges is $\frac{(n)(n-1)}{2}$ where every edge between two vertices is present.

3. Results in Spectral Graph Theory

We begin by discussing the main results from spectral graph theory but first we define the Cheeger constant which helps us measure how connected a graph is.

Definition 3.1. The *Cheeger constant* is defined as

$$h(G) = \min_{0 < |S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|}$$

where S is a subset of the graph G, $|S|$ represents the number of vertices in S, and $|\partial S|$ is the number of edges between S and $G \backslash S$.

Example. The Cheeger Constant for the graph in Figure 2 is 1. The minimum value of $h(G) = 1$ is achieved with the set of green vertices $S = \{D, E\}$ as shown in the figure. In this case we have $|S| = 2$ vertices and $|\partial S| = 2$ edges. This yields $h(G) = \frac{|\partial S|}{|S|} = \frac{2}{2} = 1$. While we have shown the result for the optimal subset we must remember to check *all* possible subsets S in order to calculate $h(G)$.

With the Cheeger constant we now have a way to quantify how connected a graph is. Evidently a good expander graph has a large Cheeger constant since we want our graph to be highly connected. In fact we want the absolute best Cheeger constant which motivates the following question: "In what situations is the Cheeger constant maximized?" It turns

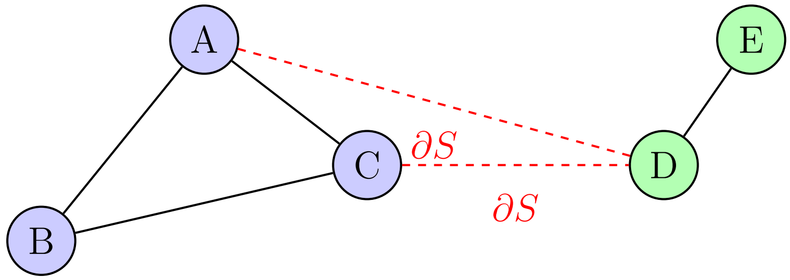


Figure 2

out that calculating the Cheeger constant is NP-hard so the best we can do is bound it. Alon-Milman proved the following theorem in [AM85]:

Theorem 3.2 (Alon-Milman, 1985)**.** *In a d-regular graph G with its second largest eigenvalue being λ_2 we have*

$$\frac{d-\lambda_2}{2} \leq h(G) \leq \sqrt{2d(d-\lambda_2)}$$

Remark 3.3. The motivation for using λ_2 instead of λ_1 stems from λ_2 being the first non-trivial eigenvalue of a graph.

From this inequality it is evident that $h(G)$ is maximized precisely when λ_2 is minimized. Thus we now turn our attention to finding the smallest value of λ_2. Again this problem is NP-hard so the best we can do is find bounds on λ_2. First the lower bound was established by Alon and Boppana [Alo86] in 1991 and then the upper bound was established by Friedman [Fri03] in 2003 almost 2 decades later.

Definition 3.4. The diameter δ of a graph G is the length of the longest shortest path between any two vertices in a graph.

Theorem 3.5 (Alon-Boppana, 1991)**.** *In any d-regular graph with diameter δ let its eigenvalues be $\lambda_1 \geq \lambda_2 \geq \ldots, \geq \lambda_n$ Then*

$$\lambda_2 \geq 2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{\lfloor \delta/2 \rfloor}$$

In this paper we will prove a weaker version of Alon-Boppana in Section 4.

Theorem 3.6 (Friedman, 2003)**.** *For a d-regular graph and some $\epsilon > 0$ the probability that*

$$\lambda_i \leq 2\sqrt{d-1} + \epsilon$$

tends to 1 as the number of vertices goes to infinity.

The same $2\sqrt{d-1}$ appears in both the lower and upper bounds thus these bounds are tight. You can't bound λ_2 any better. A special class of graphs which achieve this bound are Ramanujan Graphs which we now explicitly define:

Definition 3.7. A *Ramanujan Graph* is a d-regular graph where all non-trivial eigenvalues satisfy

$$|\lambda_i| \leq 2\sqrt{d-1}$$

Corollary 3.8. *The probability of a random d-regular graph being Ramanujan is* 69% *as n goes to ∞. We refer the reader to [HMY25] for a detailed explanation.*

In fact this bound appears naturally in several other types of graphs as well. One notable example is trees for which we have the following lemma:

Lemma 3.9. *If T is a tree with maximal degree k then all non-trivial eigenvalues of the adjacency matrix satisfy $|\lambda_i| \leq 2\sqrt{d-1}$.*

Proof. Let A be the adjacency matrix of the tree T. Let r be the root of the tree T and for a vertex v_i let $d(r, v_i)$ be its distance from r. Finally let D be a invertible diagonal matrix such that $d_{ii} = \delta^{d(r,v_i)}$ for a parameter $\delta > 0$.

We now construct a matrix $B = DAD^{-1}$ by definition B has the same eigenvalues as A and we know the entries of B explicitly. They are

$$b_{ij} = \begin{cases} \delta^{d(r,v_i)-d(r,v_j)}, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

Now let λ be an eigenvalue of B with a non-zero eigenvector x. The eigenvalue equation is then:

$$\lambda x_i = \sum_{v_i - v_j} \delta^{d(r,v_i)-d(r,v_j)} x_j$$

We choose an index i such that $|x_i|$ is maximal. From this we have $x_i \neq 0$ and $|x_j| \leq |x_i|$. Taking absolute values of the eigenvalue equation gives:

$$|\lambda||x_i| = \sum_{v_i - v_j} \delta^{d(r,v_i)-d(r,v_j)} |x_j| \implies |\lambda||x_i| \leq \sum_{v_i - v_j} \delta^{d(r,v_i)-d(r,v_j)} |x_i|$$

Now if we divide both sides by $|x_i|$ we have

$$|\lambda| \leq \sum_{v_i - v_j} \delta^{d(r,v_i)-d(r,v_j)}$$

This expression gives us a bound on λ which we can now analyze based on the location of v_i in the graph. In a tree, any neighbor v_j of v_i has distance $d(r, v_j) = d(r, v_i) \pm 1$. We have the following cases:

Case 1: v_i is the root
All $\deg(v_i)$ neighbors are a distance 1 away so we have $|\lambda| \leq \deg(v_i) \cdot \delta^{-1} \leq \frac{k}{\delta}$

Case 2: v_i is a leaf
Its single neighbor is one step closer to the root than v_i so we have $|\lambda| \leq \delta^1 = \delta$

Case 3: Any other v_i
v_i has one neighbor closer to the root and all other $\deg(v_i) - 1$ neighbors further away from the root. This means $|\lambda| \leq \delta^1 + (\deg(v_i) - 1)\delta^{-1} \leq \lambda + \frac{k-1}{\delta}$

To minimize the maximum of these bounds, we want δ to minimize $\delta + \frac{k-1}{\delta}$. After some basic calculus we find that this occurs namely when $\delta = \sqrt{k-1}$. With this value for δ all our 3 cases satisfy $|\lambda| \leq 2\sqrt{k-1}$ which is our bound. ■

4. Alon-Boppana

We now discuss the proof of the Alon-Boppana Theorem.

Definition 4.1. An infinite d-regular tree is a graph with infinite vertices of degree d, containing no cycles.

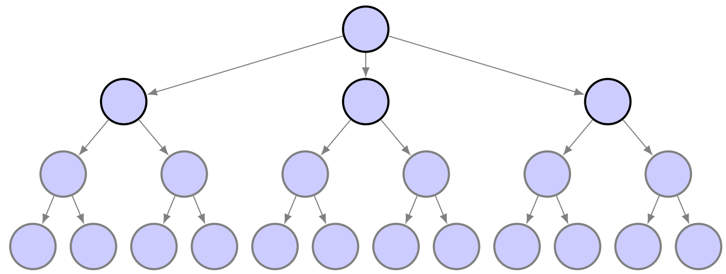


Figure 3. Portion of an Infinite 3-regular Tree

Example. A infinite 3-regular tree is shown in Figure 3 up to 4 generations.

Lemma 4.2. *For a graph with adjacency matrix A the number of paths of length k from vertex i to vertex j is the entry a_{ij} of the matrix A^k*

Proof. We proceed via induction. The base case $k = 1$ is trivial as a path of length 1 between vertex i and vertex j can only exist if their is an edge between them. This is exactly what $A^1 = A$ encodes so we are done.

Assume that all entries of A^k satisfy the lemma. We will show that the matrix A^{k+1} must also satisfy the lemma. The number of paths of length $k+1$ from vertex i to vertex j is equal to the sum of the number of paths of length k from vertex i to a neighbor of vertex j. This is precisely what the matrix multiplication $A^k \cdot A$ does to the new entry a_{ij} of A^{k+1} so our lemma is proven. ∎

Lemma 4.3. *For a graph with adjacency matrix A and eigenvalues λ_i we have:*

$$trace(A^k) = \sum_{i=1}^{n} \lambda_i^k$$

Proof. We will prove this lemma in two parts. First we show that $trace(G) = \sum_{i=1}^{n} \lambda_i$ in any matrix G. Then we show that if the eigenvalues of a graph G are $\lambda_1, \lambda_2, \ldots, \lambda_n$ then the eigenvalues for the matrix G^t are $\lambda_1^t, \lambda_2^t, \ldots, \lambda_n^t$. These two will readily imply our lemma.

We prove the first part by finding the characteristic polynomial of the adjacency matrix A in two different ways. Since the characteristic polynomial is unique we will be able to compare coefficients of the polynomials we find and then conclude. In particular we look at the coefficient of X^{n-1}. We know that

$$P(X) = det(\begin{bmatrix} A - IX \end{bmatrix}) = det\left(\begin{bmatrix} a_{11} - X & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - X & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - X \end{bmatrix}\right)$$

The coefficient of X^{n-1} when evaluated turns out to be is $-\sum_{i=1}^{n} a_{ii} = -trace(A)$. Moreover the characteristic polynomial is also defined as

$$P(X) = (X - \lambda_1)(X - \lambda_2) \ldots (X - \lambda_n)$$

whose coefficient of X^{n-1} is $-\sum_{i=1}^{n} \lambda_i$. Thus by comparing coefficients of X^{n-1} we conclude

$$-\text{trace}(A) = -\sum_{i=1}^{n} \lambda_i \implies \text{trace}(A) = \sum_{i=1}^{n} \lambda_i$$

.

Now we prove the second part. Let the eigenvector for λ_i be v. We proceed with induction and show that if A^k has eigenvalue λ_i^k then A^{k+1} has eigenvalue λ_i^{k+1}. The base case $k = 1$ is trivial. For the inductive step we have that $A^k \cdot v = \lambda_i^k v$. Multiplying both sides of this equation by A we have that

$$AA^k v = A\lambda_i^k v \implies A^{k+1}v = \lambda_i^k(Av) = A^{k+1}v = \lambda_i^{k+1}v$$

Thus λ_i^{k+1} is an eigenvalue for the matrix A^{k+1}. Repeating this for all eigenvectors concludes the second part. ■

Now we are ready to prove a weaker version of Alon-Bopanna.

Theorem 4.4. *Let G be a d-regular graph and the eigenvalues of the adjacency matrix be $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let the eigenvalue with the greatest magnitude be σ. We have that for any integer k*

$$\sigma \geq 2\sqrt{d-1} \cdot \left(1 - O\left(\frac{\log k}{k}\right)\right)$$

Proof. By lemma 4.2 we know that the number of closed paths (i.e. paths from vertex i back to vertex i) is $\sum_{i=1}^n a_{ii}$ which is also just the trace of a matrix. The path length must be even in order for us to have a closed path so we will consider path lengths of $2k$ rather than k. The number of closed paths of length $2k$ starting from a single vertex v_i in a graph is lower bounded by the number of closed paths of length $2k$ in an infinite d-regular graph. This is because in G you may have cycles which add additional paths which are not counted in the infinite graph. However every path in the infinite graph is counted in the regular graph since the infinite graph is a covering.

In fact, the number of closed paths of length $2k$ in a infinite d-regular graph is at least $C_k \cdot (d-1)^k$ where C_k is the k^{th} Catalan number. We can see this since a path of length $2k$ consists of going forward k times and going backwards k times. Each time you go forward you have at least $d-1$ choices and each time you go backward you are forced to go back the way you came. Thus in our path we have $(d-1)^k$ choices for where to go. It is well known the number of arrangements of when to go forward or backwards is C_k as you can never go back more times than you go forward; it makes no sense to go forward, back, back as you would be outside the graph. Thus we have

$$trace(A^{2k}) = \sum_{i=1}^n \# \text{ of closed paths from } v_i \geq n \cdot C_k(d-1)^k = n \cdot \frac{1}{k+1}\binom{2k}{k}(d-1)^k \tag{4.1}$$

By lemma 4.3 we know

$$\text{trace}(A^{2k}) = \sum_i^n \lambda_i^{2k}$$

We now bound $\sum_{i=1}^n \lambda_i^{2k}$ as follows:

$$d^{2k} + n\sigma^{2k} \geq d^{2k} + (n-1)\sigma^{2k} \geq \sum_{i=1}^n \lambda_i^{2k} = \text{trace}(A^{2k}) \tag{4.2}$$

Combining equations 4.1 and 4.2 we have that

$$d^{2k} + n\sigma^{2k} \geq trace(A^k) \geq n \cdot \frac{1}{k+1}\binom{2k}{k}(d-1)^k$$

We can approximate the right hand side via Stirling's formula to get

$$\sigma \geq \sqrt[2k]{\frac{1}{k+1}\binom{2k}{k}(d-1)^k - \frac{d^{2k}}{n}} \approx \sqrt[2k]{2^{2k}(d-1)^k \cdot \Omega(k^{-1.5})} = 2\sqrt{d-1}\left(1 - O\left(\frac{log(k)}{k}\right)\right)$$

which concludes. ■

5. LPS Construction of Ramanujan Graphs

The first explicit constructions of Ramanujan Graphs was given by Lubotzky, Phillips, and Sarnak in 1988 and we often refer to the construction as the LPS construction. The construction is number theory heavy and utilizes Cayley Graphs which we will define below

Definition 5.1. We call p a *quadratic residue* modulo q if their exists an integer x such that $p \equiv x^2 \pmod q$.

Remark 5.2. We shorten the words quadratic residue to QR and non-quadratic residue to NQR for convenience.

Definition 5.3. The Legendre symbol is denoted $(\frac{p}{q})$ where:

$$\left(\frac{p}{q}\right) = \begin{cases} 1, & \text{if } p \text{ is a } QR \text{ modulo } q \\ -1, & \text{if } p \text{ is } NQR \text{ modulo } q \\ 0, & \text{if } q \mid p \end{cases}$$

Lemma 5.4. *For every odd prime p there are exactly $\frac{p-1}{2}$ QR and NQR modulo p.*

Proof. Evidently assume that $x^2 \equiv y^2 \pmod p$ for two distinct integers x and y. Then

$$(x-y)(x+y) \equiv 0 \pmod p \implies x+y \equiv 0 \pmod p$$

as $x \neq y$. So in fact all values of $1^2, 2^2, \ldots, (\frac{p-1}{2})^2$ are QR and they are each distinct for size reasons. This leaves the rest of the integers $p - 1 - \frac{p-1}{2} = \frac{p-1}{2}$ being NQR. Note we ignore 0 since $p \mid 0$. ■

Lemma 5.5. *If $p \equiv 1 \pmod 4$ is a prime then their exists an integer i such that $i^2 \equiv -1 \pmod p$.*

Proof. By Wilson's theorem since $(p-1)! \equiv -1 \pmod p$ and $p \equiv 1 \pmod 4$ then

$$(p-1)! \equiv \left(\left(\frac{p-1}{2}\right)!\right)^2 \cdot (-1)^{\frac{p-1}{2}} \equiv \left(\left(\frac{p-1}{2}\right)!\right)^2 \equiv -1 \pmod p$$

So a value of i which works is just $\left((\frac{p-1}{2})!\right)^2$ ■

With the number theory preliminaries out of the way we now state what a Cayley Graph is.

Definition 5.6. Let G be a group and let S be a set of group G such that $I \notin S$. The *Cayley Graph* is then defined as a set of vertices and edges where every element of the group G becomes an vertex with a directed edge between any two vertices a and b if $ab^{-1} \in S$.

Remark 5.7. We prevent the identity from being in S as to avoid self-loops.

Definition 5.8. The *center* of a group is denoted $Z(G)$ and is the set of elements in G that commute with every other element in G

Now we will study some special groups which the LPS construction uses.

Definition 5.9. The general linear group $GL(n, \mathbb{F}_p)$ is the group of all $n \times n$ invertible matrices for which each entry in the matrix is taken modulo p.

Definition 5.10. The projective general linear $PGL(n, \mathbb{F}_p) = GL(n, \mathbb{F}_p)/Z(G)$ is a group of equivalences classes of $n \times n$ matrices with each entry taken modulo p. Two matrices A and B are in the same equivalence class if one is a non-zero multiple of the other. That is, $A = \lambda B$ for $\lambda \in \mathbb{F}$

Definition 5.11. The special general linear group $SL(n, \mathbb{F}_p)$ is a group which consists of all matrices that are a part of $GL(n, \mathbb{F}_p)$ which have determinant 1.

Definition 5.12. The projective special general linear group $PSL(n, \mathbb{F}_p) = SL(n, \mathbb{F}_p)/Z(G)$ is a subgroup of $PGL(n, \mathbb{F}_p)$ consisting of all equiv lance classes that contain at least one matrix with determinant 1.

Lemma 5.13. *The number of equivalence classes in $PGL(2, \mathbb{F}_p)$ is $p(p^2 - 1)$.*

Proof. The order of the $PGL(2, \mathbb{F}_p)$ is determined by its definition as a quotient group, $PGL(n, \mathbb{F}_p) = GL(n, \mathbb{F}_p)/Z(G)$. The order of $GL(2, \mathbb{F}_p)$ can be found by counting the number of 2×2 invertible matrices over $\mathbb{F}_p$. This is equivalent to choosing two linearly independent vectors from $\mathbb{F}_p^2$ yielding $|GL(2, \mathbb{F}_p)| = (p^2 - 1)(p^2 - 1)$. The center of this group $Z(G)$ consists of all non-zero vector scalar matrices so $|Z| = |\mathbb{F}_p^*| = p - 1$. Thus

$$|PGL(2, \mathbb{F}_p)| = \frac{(p^2 - 1)(p^2 - 1)}{p - 1} = p(p^2 - 1)$$

. ■

Lemma 5.14. *Any matrix with a quadratic residue for its determinant represents an equivalence class in $PSL(2, \mathbb{F}_p)$.*

Proof. Consider $A \in GL(2, \mathbb{F}_p)$. Then $\det(A)$ can either be a QR or a NQR. Suppose that $\det(A)$ is a QR. Now note that since we are dealing with 2×2 matrices $\det(\lambda A) = \lambda^2 \det(\mathrm{A})$ which is again a QR. So $\lambda^{-1} A \in SL(2, \mathbb{F}_p)$. In particular, all scalar multiples of $c^{-1} A$ belong in the same equivalence class. Now if $\det(A)$ is a NQR then we can never have a QR by the same reasoning above. ■

Lemma 5.15. *The number of equivalence classes in $PSL(2, \mathbb{F}_p)$ is $\frac{p(p^2-1)}{2}$.*

Proof. By 5.4 we know that we have exactly half QR and half NQR thus the number of equivalence classes in $PSL(2, \mathbb{F}_p)$ is exactly half of those in $PGL(2, \mathbb{F}_p)$ which is $\frac{p(p^2-1)}{2}$. ■

Now we will state a theorem without proof as it requires a lot of heavy duty machinery.

Theorem 5.16. *Jacobi's Four Square Theorem, 1834 The number of solutions to $a^2 + b^2 + c^2 + d^2 = n$ is 8 times the sum of all diviosrs of n which are not divisible by 4.*

Corollary 5.17. *If $p \equiv 1 \pmod 4$ is an odd prime then their exist $8(p + 1)$ solutions to $a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$. In particular their exist exactly $p + 1$ solutions such that $a_0 > 1$ and a_0 is odd.*

Proof. Each solution must have exactly 3 odd numbers and one even number in order for $p \equiv 1 \pmod 4$. So the number of solutions where a_0 is the odd one is $\frac{8(p+1)}{4} = 2(p+1)$ and we again divide by two for when a_0 is positive or negative. This gives us $p+1$ solutions as we wanted. ∎

Finally now we are ready to state the LPS construction. Proving that this construction actually satisfies the bounds for a Ramanujan Graph is extremely technical and we refer the reader to [LPS88].

Theorem 5.18. *Lubotzky-Phillips-Sarnak Ramanujan Graph Construction Let $p, q \equiv 1 \pmod 4$ be two primes and by 5.5 i an integer satisfying $i^2 \equiv -1 \pmod p$ We know by 5.17 that their exist $q+1$ solutions to $a_0^2 + a_1^2 + a_2^2 + a_3^2 = q$ for which $a_0 > 1$ and a_0 is odd. Associate each solution a_0, a_1, a_2, a_3 the matrix:*

$$a = \begin{bmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{bmatrix}$$

We use this set of matrices as the generating set S for the Cayley graph. It turns out that their are both bipartite and non-bipartite constructions of a $q+1$ regular Ramanujan graph.

Case 1: *q is a NQR modulo p*
Here, the Cayley graph is constructed using the group $PGL(2, \mathbb{F}_p)$ and the generating set S. This produces a bipartite Ramanujan graph with $p(p^2-1)$ vertices.

Case 2: *q is a QR modulo p*
Here, the determinant of any generator $a \in S$ is a square in $\mathbb{F}_p$ ensuring that all elements of the generating set S belong to the group $PSL(2, \mathbb{F}_p)$. The corresponding Cayley graph is therefore built on this smaller group resulting in a non-bipartite Ramanujan graph with $\frac{p(p^2-1)}{2}$ vertices.

A few years later after LPS and Margulis [Mar88] in 1994 Moregenstern [Mor94] extended their work by generalizing the construction to cover all graphs of degree $p^k + 1$ where p is a prime. Since then no major advancements have been made in finding explicit construction for non bipartite Ramanujan graph. In fact, we still don't have a construction of a degree 7 non-bipartite Ramanujan Graph. However their has been work done on constructing bipartite Ramanujan Graphs which we talk about in the following section:

6. Existence and Construction of Infinite Bipartite Ramanujan Graphs

This construction was so ahead of its time because it was still an open problem weather infinite bipartite Ramanujan graphs existed or not. This proof not only answers this question but details a method to construct them even though the proof is probabilistic in nature. First, let us preface the motivation for using bipartite graphs. When we try bounding λ_2 it is very hard to keep all the negative eigenvalues greater than $-2\sqrt{d-1}$. In essence, when you try bounding λ_2 you will mess up the bound for some negative eigenvalue. Thus, if we only consider bipartite graphs it suffices to only bound λ_2 as we get the lower bound for free since eigenvalues are symmetric about 0. The proof is as follows:

Lemma 6.1. *Eigenvalues are symmetric about* 0 *for a bipartite graph.*

Proof. By the definition of a bipartite graph we can split up the graph into having two sub-square matrices as follows: $\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}$. Now consider an eigenvector $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ of the matrix $\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}$. Immediately we have that

$$\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} A_1 \cdot u_2 \\ A_2 \cdot u_1 \end{bmatrix} = k \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Now we want to show that $-k$ is also an eigenvalue however this is immediately obvious as

$$\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} -u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} A_1 \cdot u_2 \\ -A_2 \cdot u_1 \end{bmatrix} = -k \cdot \begin{bmatrix} -u_1 \\ u_2 \end{bmatrix}$$

■

Definition 6.2. A *2-lift* in a process where you generate a new graph G' with $2n$ vertices from a graph G with n vertices. The process is as follows:

(1) Take the graph G and double it. Each vertex gets duplicated with the corresponding vertex for X in the original graph G being X' in the new graph. It is evident that if any two vertices X and Y are connected in the original graph G then X' and Y' are connected as well.

(2) We now have the option of leaving edges between X and Y and X' and Y' alone or crossing them in the sense that we instead join X with Y' and X' with Y.

Example. We emphasize that **multiple** $2-$lifts exist and this is an example of a 2-lift. See Figure 4 and Figure 5

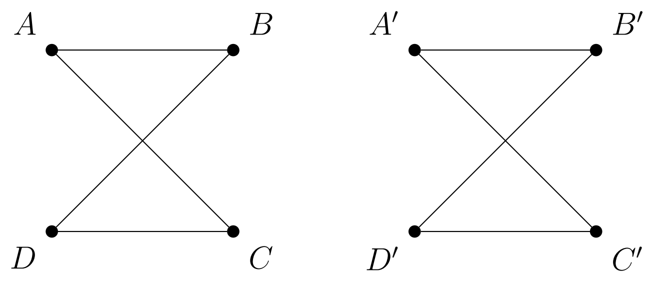


Figure 4. Doubled Graph

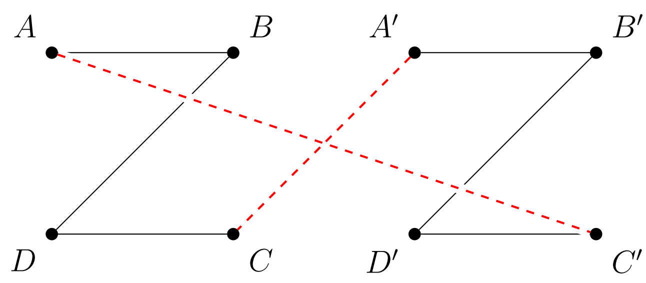


Figure 5. Final Graph

Remark 6.3. Observe that when we perform a 2-lift our graph always remains bipartite since we never introduce any new odd cycles.

Definition 6.4. The adjacency matrix A' of the 2-lift is called the *doubled adjacency matrix*.

If the adjacency matrix of the original graph was A then the doubled adjacency matrix (i.e. Figure 4) would look like

$$\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

Now after you cross some edges your adjacency matrix would change but still be symmetric about the diagonal.

Example. The adjacency matrix for Figure 5 would be

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Representing 2-lifts in adjacency matrices gets quite hard as the size of the matrices grows exponentially which makes it difficult to analyze the matrices eigenvalues. This means we need a new and better way to analyze the eigenvalues of the adjacency matrix of 2-lift. This way must not double the size of the adjacency matrix but still encode the same information about the eigenvalues.

Definition 6.5. The signed adjacency matrix A_s for a graph G is the adjacency matrix with

$$a_{ij} = \begin{cases} -1, & \text{if edges } I - J' \text{ and } I' - J \text{exist} \\ 1, & \text{otherwise} \end{cases}$$

Example. The signed adjacency matrix for 5 is

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Definition 6.6. The eigenvalues of A are called the *old eigenvalues*.

Definition 6.7. The eigenvalues of A_s are called the *new eigenvalues*.

It turns out that their does exist a way to analyze the eigenvalues of the doubled adjacency matrix without the issues of size. This comes from the following theorem:

Theorem 6.8. *The union of the signed adjacency matrix and the adjacency matrix are the eigenvalues of the matrix of the doubled graph.*

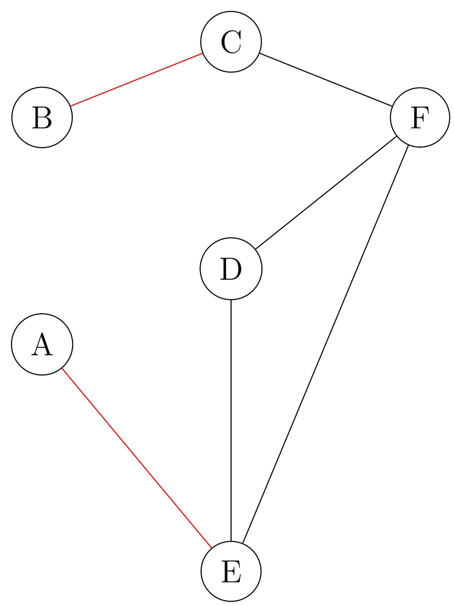
Proof. The doubled adjacency matrix can be written as $\hat{A} = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}$. Observe that $A = A_1 + A_2$ and $A_s = A_1 - A_2$. Suppose that (α, v) and (β, u) are the eigen-pairs of A and A_s, respectively. Then the following are eigen-pairs of $\hat{A}$:

$$\left(\alpha, \begin{bmatrix} v \\ v \end{bmatrix}\right) \quad \text{and} \quad \left(\beta, \begin{bmatrix} u \\ -u \end{bmatrix}\right)$$

Since the eigenvectors of the form $\begin{bmatrix} v \\ v \end{bmatrix}$ and $\begin{bmatrix} u \\ -u \end{bmatrix}$ are mutually orthogonal, and there are $2n$ of them, they span all the eigenvectors of $\hat{A}$. ■

Definition 6.9. A *matching* on a graph consisting is a set of i edges for which no two edges share a vertex.

Example. In the following graph $m_2 = 7$ and an example of a possible matching with 2 edges is:



Definition 6.10. The *matching polynomial* for a graph G where represents the number of matchings with i edges is:

$$M(X) = \sum_{i \geq 0} (-1)^i m_i X^{n-2i}$$

where m_i

Example. The matching polynomial for 6 is

$$M_G(x) = (-1)^0 m_0 x^{6-0} + (-1)^1 m_1 x^{6-2} + (-1)^2 m_2 x^{6-4} + (-1)^3 m_3 x^{6-6}$$

which after computing m_i gives

$$M_G(x) = x^6 - 6x^4 + 7x^2 - 1$$

Matchings on graphs are studied widely and are quite nice because of several recurrences which are present. We present one here below:

Lemma 6.11. *Let the matching polynomial on a graph G be μ_g. Suppose two vertices a and b are connected by an edge e. Then we have: $\mu_g(X) = X \cdot \mu_{g-e}(X) + \mu_{g-a-b}$ where $\mu_{g-q}(X)$ represents the matching polynomial of the graph without q.*

Proof. Evidently the number of matchings on a graph can be equal to the number of matchings with two vertices plus the number of matchings without those two vertices which is what we count above. ∎

Lemma 6.12. *The expected characteristic polynomial of the signed matrices after the 2-lift is equal to the matching polynomial. In other words*

$$\mathbb{E}(A_s(G)) = \mu_g$$

Proof. The key idea is to find the characteristic polynomial by evaluating the determinant via the Leibniz formula.

$$E(A_s(G)) = E(\det(XI - A_s(G)) = E\left(\sum_{\pi \in S_n} sgn(\pi) x^{|a:\pi(a)=a|} \prod_{a:\pi(a)\neq a} (S(a,\pi(a)))\right)$$

Above we are summing over every permutation of the adjacency matrix as per Leibniz's formula. We refer to points which are unchanged after the permutation as fixed points (i.e $S(a,\pi(a)) = S(a,a)$). In each permutation the only way we can get a factor of x is if we have a fixed point in our permutation as the main diagonal consists of only x. Thus the degree of x is the number of fixed points. The coefficient of of $x^{|a:\pi(a)=a|}$ is the product of all other terms in our permutation times $sgn(\pi)$ by definition. By linearity of expectation this expression is also equal to

$$\sum_{\pi \in S_n} sgn(\pi) x^{|a:\pi(a)=a|} E\left(\prod_{a:\pi(a)\neq a} (S(a,\pi(a)))\right)$$

One can see that each permutation consists of cycle(s) in the sense that $a \to b \to \ldots \to a$ and so on. Here, the only way to get a non-zero expected value term is that all remaining non-fixed points are part of a cycle of length 2 meaning $S(a,\pi(a)) = S(a,b)$ and $S(b,\pi(b)) = S(b,a)$. Since each entry which is not in the main diagonal is either 1 or -1 which equal probability and $S(a,b) = S(b,a)$ as the matrix is symmetric we have a net contribution from each involution of 1.

Now suppose that their exists a cycle which is not of length 2. Then each term $S(i,\pi(i))$ is independently 1 or -1 in that cycle so the expected value of the cycle's product and thus the product of the permutation is 0.

The number of permutations which consist of just involutions are precisely the matching on a graph as each one corresponds to a particular edge and they are all disjoint, which finishes proving this lemma. ∎

Corollary 6.13. *If G is a tree with adjacency matrix A then $A = \mu_g$*

Proof. The proof follows the exact same manner as above but we have no cycles which are present as the graph is a tree so we have an equality. ∎

Definition 6.14. A path tree $T_a(G)$ is a tree rooted at vertex a such that paths start at a and do not contain any vertex twice.

Godsil proved the following:

Theorem 6.15. *$\mu_G(x)$ divides $\mu_{T_a(G)}(x)$*

This implies that $\mu_G(x)$ also divides the characteristic polynomial of $T_a(G)$ by corollary 6.13. Suppose our graph G has maximal degree d then by lemma 3.9 all eigenvalues of $\mu_G(x)$ are bounded by $2\sqrt{d-1}$. This implies the following theorem:

Theorem 6.16. *All real roots of $\mu_g(x)$ lie in the interval* $(-2\sqrt{d-1}, 2\sqrt{d-1})$

Definition 6.17. Two real-rooted polynomials P and Q are said to be *interlacing polynomials* if they satisfy either of the following two cases:

(1) The roots of P and Q are $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_{n-1}$ satisfy $\alpha_1 \leq \beta_1 \leq \alpha_2 \cdots \leq \beta_{n-1} \leq \alpha_n$
(2) The roots of P and Q are $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ satisfy $\alpha_1 \leq \beta_1 \leq \alpha_2 \cdots \leq \alpha_n \leq \beta_n$

We only care about the second case since all polynomials we deal with have the same number of roots, but nevertheless we state the two cases for completion. A common example of interlacing polynomials is a polynomial and its derivative.

Definition 6.18. Two monic polynomials P and Q are said to have a *common interlacing* if their exists another polynomial G which interlaces both P and Q.

Definition 6.19. An *interlacing family* of polynomials exists if all polynomials can be represented as a tree where each node has exactly two children such that the node is the common interlacing of the two children.

We present an equivalent statement to Definition 6.18

Lemma 6.20. *The statement two monic polynomials P and Q have a common interlacing is equivalent to showing that the polynomial $R = t \cdot P + (1-t)Q$ is real rooted for all $t \in [0, 1]$*

Proof. We first show that the backward direction. For now, let us assume that both of these polynomials P and Q each have n distinct roots. By our assumption we also know that our polynomial R has n real roots, say $\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)$ as well. In order to show that the existence of R implies a common interlacing it suffices to show that between the roots α_1 and β_1 their exists a root $\gamma_1(t)$. As t varies from 0 to 1 inclusive, $\gamma_1(t)$ starts at α_1 and goes to β_1. Suppose instead that in between of α_1 to β_1 their exists another root say, α_2, then at the precise moment when $\gamma_1(t) = \alpha_2$, we have that

$$0 = R(\gamma_1(t)) = tP(\gamma_1(t)) + (1-t)Q(\gamma_1(t)) = \underbrace{tP(\alpha_2)}_{0} + (1-t)Q(\alpha_2) \implies Q(\alpha_2) = 0$$

Which means that Q has a shared root with P violating our assumption, so in fact no such α_2 can exist and thus our roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ alternate which implies a common interlacing.

Now for the forward direction. Again, we assume that the polynomials P and Q have n distinct roots. Since a common interlacing exists we know that their exists points $\lambda_1, \lambda_2, \ldots, \lambda_n$ which split the roots of P and Q. We will show that their exists a root for R in each of the intervals $(\lambda_i, \lambda_{i+1})$ for $i \in [1, n-1]$. As both polynomials are monic we know that at each of the λ_i's both polynomials P and Q must have the same sign (they cross the axis in the same direction at each of their roots). Thus finally by the intermediate value theorem we know that since $R(\lambda_i) = \underbrace{tP(\lambda_i)}_{+} + \underbrace{(1-t)Q(\lambda_i)}_{+} \implies R(\lambda_i) > 0$. Moreover $R(\lambda_{i+1}) = \underbrace{tP(\lambda_{i+1})}_{-} + \underbrace{(1-t)Q(\lambda_{i+1})}_{-} \implies R(\lambda_{i+1}) < 0$ so their must exist a root for R between λ_i and λ_{i+1} which finishes. ■

Lemma 6.21. *Let $f_1, f_2, \ldots, f_t$ be monic real-rooted polynomials with degree n. Let*

$$F = \sum_{i=1}^{t} f_i$$

If $f_1, f_2, \ldots, f_t$ form an interlacing family then there exists an f_i for which the largest root of f_i is at most the largest root of F

Proof. Let the polynomial which interlaces all of f_i be $P(x)$. Let the second largest root of $P(x)$ be α_{n-1}. Then all f_i must have a root $\geq \alpha_{n-1}$ and in particular we must have that $f_i(\alpha_{n-1}) \leq 0$ since the end behavior of f_i is positive. This implies that $F(\alpha_{n-1}) \leq 0$ as well. However since F also has end behavior which is positive F must have another root β_n. Since

$$F(\beta_n) = 0 = \sum_{i=1}^{n} f_i(\beta_n)$$

their must exist an f_i whose roots lie in between α_{n-1} and β_n inclusive because if their didn't all

$$f_i(\beta_n) < 0 \implies \sum_{i=1}^{n} f_i(\beta_n) < 0 \neq 0$$

■

Now the final piece of the puzzle was what Marcus, Spielman, Srivastava proved in 2018 which we state below we refer the reader to the details in [MSS18].

Theorem 6.22. *The characteristic polynomials of all signed matrices $f_1, f_2, \ldots, f_i$ form an interlacing family*

Now since the characteristic polynomials of all signed matrices form an interlacing family we know that their exists a signing f_r for which the largest root is bounded by the largest root of $\sum_{i=1}^{i} f_i$, which have roots already bounded by $\sqrt{d-1}$. Since we have bounded the largest eigenvalue we have simultaneously bounded the lowest eigenvalue as well thus proving that the new doubled graph is in fact Ramanujan.

7. Ihara Zeta Function

It turns out that their exists a zeta function for a graph which was first motivated by the Selberg Zeta function for Riemann surfaces. Ihara [IHA66] pioneered the Ihara Zeta function which shows an equivalence between Ramanujan graphs and the Riemann hypothesis. This function is takes on a 'new' form of primes in the form of prime cycle equivalence classes which we define below. Let X be a k-regular graph in the rest of the section with $q = k - 1$.

First we talk about the Riemann Hypothesis.

Definition 7.1. A *closed geodesic* γ is a proper walk on a graph which starts and ends on the same vertex. Let $\ell(\gamma)$ represent the number of edges in the closed geodesic.

We let the notation γ^r mean that we traverse the closed geodesic γ r times.

Definition 7.2. A *prime geodesic* is a closed geodesic which cannot be represented by a power of a shorter closed geodesic.

Definition 7.3. A *prime geodesic cycle* is an equivalence class of a prime geodesic.

Remark 7.4. The reason we need an equivalence class is in order to distinguish paths which are the same but start on different vertices. For example consider the paths $(1,2,3,1)$ and $(2,3,1,2)$; they are the same but have different starting points thus the need to define an equivalence class.

Definition 7.5. The *Ihara Zeta Function* is the following:

$$Z_X(s) = \prod_p (1 - q^{-s\cdot\ell(p)})^{-1}$$

Here the product is over all prime geodesic cycles p, $\ell(p)$ represents the length of the cycle p and s is a complex variable.

This definition is motivated by the following theorems.

Theorem 7.6. *Let A be the adjacency matrix of the graph X. For $g = \frac{(q-1)|X|}{2}$ we have*

$$Z_X(s) = (1-u^2)^{-g} \cdot det(I - Au + qu^2 I)^{-1}$$

where $u = q^{-s}$

Theorem 7.7. *$Z_X(s)$ satisfies the Riemann Hypothesis if and only if X is a Ramanujan Graph.*

Proof. Notice that the poles of $Z_X(s)$ in the region $0 < \Re(s) < 1$ are determined by the zeros of the determinant term. Let $\phi(z)$ be the characteristic polynomial of A. The determinant $\det(I - Au + qu^2 I)$ is zero if and only if there exists an eigenvalue z of A such that $1 - zu + qu^2 = 0$.

First we show the backwards direction. That is, if X is a Ramanujan graph then $Z_X(s)$ satisfies the Riemann Hypothesis. Let z_0 be an non-trivial eigenvalue of A. The corresponding values of u_0 that determine the poles of $Z_X(s)$ are the roots of the quadratic equation $qu^2 - z_0 u + 1 = 0$. By the quadratic formula we have

$$u_0 = \frac{z_0 \pm \sqrt{z_0^2 - 4q}}{2q}$$

Since our graph is Ramanujan we know that $|z_0| \leq 2\sqrt{q} = 2\sqrt{k-1}$. This implies that the discriminant $z_0^2 - 4q \leq 0$. Therefore the roots u_0 are a complex conjugate pair or real and equal if $z_0^2 = 4q$. Both cases are resolved in the same manner as below. WLOG say $z_0^2 - 4q < 0$

Here the roots are both complex conjugates of each other and by Vietas we know that the product of the roots is $\frac{1}{q}$. Then we must have

$$u_0 \cdot \bar{u_0} = |u_0|^2 = \frac{1}{q} \implies |u_0| = \frac{1}{\sqrt{q}}$$

Recalling the substitution $u = q^{-s}$, the condition $|u_0| = q^{\frac{-1}{2}}$ which means that $|q^{-s_0}| = q^{-\Re(s_0)} = q^{\frac{-1}{2}}$. This forces $\Re(s_0) = \frac{1}{2}$, thus all non-trivial poles lie on the critical line.

Now we will show the forward direction that if $Z_X(s)$ satisfies the Riemann Hypothesis then X is a Ramanujan graph. The poles of the zeta function $Z_X(s)$ occur when the determinant term is zero, that is $\det(I-Au+qu^2I)=0$. This happens exactly when the matrix $Au-qu^2I$ has an eigenvalue of 1. This condition can now be rewritten as an $1-\lambda u+qu^2=0$. By the Riemann hypothesis we know that for any pole s_0 the real part $\Re(s_0)=\frac{1}{2}$ so this forces the magnitude of $u_0=q^{-s_0}$ to be $|u_0|=q^{\frac{-1}{2}}=\frac{1}{\sqrt{q}}$. Now if we rearrange $1-\lambda u+qu^2=0$ we get $|\lambda_0|=|u_0^{-1}+qu_0|$ which by the triangle inequality implies

$$|\lambda_0|=|u_0^{-1}+qu_0|\leq|u_0^{-1}|+|qu_0|=\sqrt{q}+q(\frac{1}{\sqrt{q}})=2\sqrt{q}$$

Which is what we needed in order for X to be Ramanujan. ■

8. Expander-Mixing Lemma

Lemma 8.1 (Alon's Expander Mixing lemma)**.** *Let G be a d-regular, n-vertex graph. Then for any two subsets $S,T\subset V(G)$, we have*

$$\left|e(S,T)+e(S\cap T)-\frac{d}{n}|S||T|\right|\leq\sigma\sqrt{\left(|S|-\frac{|S|^2}{n}\right)\left(|T|-\frac{|T|^2}{n}\right)}\leq\sigma\sqrt{|S||T|}.$$

If S and T are disjoint, we have

$$\left|e(S,T)-\frac{d}{n}|S||T|\right|\leq\sigma\sqrt{\left(|S|-\frac{|S|^2}{n}\right)\left(|T|-\frac{|T|^2}{n}\right)}\leq\sigma\sqrt{|S||T|}.$$

Proof. Let A be the adjacency matrix of a d-regular graph on n vertices. Let $\{v_1,v_2,\ldots,v_n\}$ be a corresponding orthonormal basis of eigenvectors. For a d-regular graph, the principal eigenvector is $v_1=\frac{1}{\sqrt{n}}\mathbf{1}$, corresponding to the eigenvalue $\lambda_1=d$.

For any two subsets of vertices $S,T\subseteq V$, let $\mathbf{1}_S$ and $\mathbf{1}_T$ be their respective indicator vectors. We can express these vectors in the eigenbasis as:

$$\mathbf{1}_S=\sum_{i=1}^n a_iv_i\quad\text{and}\quad\mathbf{1}_T=\sum_{i=1}^n b_iv_i$$

The coefficients are found via inner products: $a_i=\langle\mathbf{1}_S,v_i\rangle$ and $b_i=\langle\mathbf{1}_T,v_i\rangle$. In particular, the coefficients for v_1 are $a_1=\langle\mathbf{1}_S,\frac{1}{\sqrt{n}}\mathbf{1}\rangle=\frac{|S|}{\sqrt{n}}$ and $b_1=\langle\mathbf{1}_T,\frac{1}{\sqrt{n}}\mathbf{1}\rangle=\frac{|T|}{\sqrt{n}}$. A simple calculation yeilds that we have, $\sum_{i=1}^n a_i^2=\langle\mathbf{1}_S,\mathbf{1}_S\rangle=|S|$ and $\sum_{i=1}^n b_i^2=\langle\mathbf{1}_T,\mathbf{1}_T\rangle=|T|$.

The term $e(S,T)+e(S\cap T)$ represents the number of ordered pairs of vertices (s,t) with $s\in S$ and $t\in T$ that are connected by an edge, which can be written as $\sum_{s\in S,t\in T}A_{st}=\mathbf{1}_S^TA\mathbf{1}_T$. Evaluating this expression in the eigenbasis gives:

$$\mathbf{1}_S^TA\mathbf{1}_T=\left(\sum_{i=1}^n a_iv_i\right)^TA\left(\sum_{j=1}^n b_jv_j\right)=\sum_{i,j=1}^n a_ib_j\lambda_jv_i^Tv_j=\sum_{i=1}^n\lambda_ia_ib_i$$

We can further simplify this by using, $v_i^Tv_j=\delta_{ij}$.

Some calculations with Cauchy-Shwarz show that:

$$\begin{aligned}\left|e(S,T)+e(S\cap T)-\frac{d}{n}|S||T|\right| &= \left|\sum_{i=1}^{n}\lambda_i a_i b_i-\lambda_1 a_1 b_1\right| \\ &= \left|\sum_{i=2}^{n}\lambda_i a_i b_i\right| \\ &\leq \sigma\sum_{i=2}^{n}|a_i b_i| \\ &\leq \sigma\sqrt{\left(\sum_{i=2}^{n}a_i^2\right)\left(\sum_{i=2}^{n}b_i^2\right)} \\ &= \sigma\sqrt{\left(|S|-a_1^2\right)\left(|T|-b_1^2\right)} \\ &= \sigma\sqrt{\left(|S|-\frac{|S|^2}{n}\right)\left(|T|-\frac{|T|^2}{n}\right)}.\end{aligned}$$

■

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