Stokes' Theorem

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Euler Circle

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What is the Stokes' Theorem?

- Stokes' Theorem connects a surface integral of the curl of a vector field with a line integral around its boundary.
- It's a higher-dimensional generalization of:
  - Fundamental Theorem of Calculus
  - Green's Theorem in the plane
- Converts difficult surface integrals into easier line integrals and vice versa.

Stokes' Theorem (Formal Statement)

$$\boxed{\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}}$$

- $\vec{F}$: Vector field in $\mathbb{R}^3$
- $\nabla \times \vec{F}$: Curl of the vector field
- S: Oriented smooth surface in 3D
- ∂S: Boundary of surface S
- $d\vec{S}$: Vector surface element (normal vector)
- $d\vec{r}$: Differential vector along the curve

What is Curl?

- Curl measures local rotation or swirling strength of a vector field.
- Mathematically:

$$\nabla \times \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$$

- Each term tells you the rotational tendency about one of the coordinate axes.
- Think of placing a tiny paddle wheel at a point. Curl tells how fast it spins and in which direction.

Line vs Surface Integrals

- A **line integral** measures the flow along a curve:

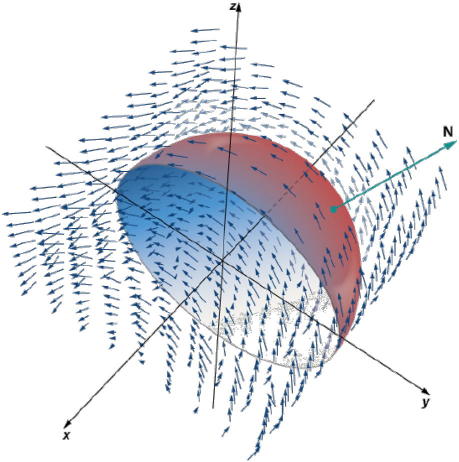
$$\oint_{\partial S} \vec{F} \cdot d\vec{r}$$

- A **surface integral** measures the rotation across the surface:

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

- Stokes' Theorem says: The total rotation inside equals the circulation around the edge.

Stokes' Theorem Visual

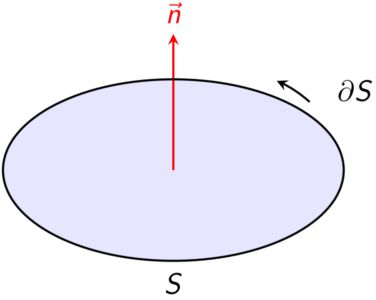


A visual showing how the curl across a surface relates to the circulation along its boundary.

Visual Intuition

- Imagine small paddlewheels placed on a surface.
- Curl tells us how fast each wheel spins.
- The total “spin” across the surface equals the amount of flow around the edge.
- This is what Stokes’ Theorem reveals!

Orientation Diagram



Right-hand rule: Fingers curl along ∂S, thumb points in direction of $\vec{n}$

Fundamental Theorem of Calculus (1D Stokes)

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

- This is the 1D version of Stokes' Theorem.
- It connects a derivative over an interval to the values of the function at the boundary points.
- In this case:
  - $f'(x)$: Derivative of function f
  - $[a, b]$: The interval (our "surface")
  - $f(b) - f(a)$: Difference across the boundary (our "line integral")
- It's the simplest and most familiar case. Everything else is a generalization of this!

Green's Theorem (2D Stokes)

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy}$$

- Green's Theorem is just Stokes' Theorem applied to a flat surface in the *xy*- plane.
- Only the *z*- component of the curl matters.

Unifying 1D, 2D, and 3D

- Multivariable calculus is built on a beautiful pattern:
  1. **1D**: Fundamental Theorem of Calculus

     $$\int_a^b f'(x)\,dx = f(b) - f(a)$$

  2. **2D**: Green's Theorem

     $$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \vec{k}\,dA$$

  3. **3D**: Stokes' Theorem

     $$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

- Each one relates a derivative inside a region to values on the boundary.
- All these tell the same story in different languages.
- 1D talks about values at endpoints
- 2D talks about circulation around a curve enclosing an area

Surface Parametrization

- A surface S can be written as:

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

- The surface element becomes:

$$d\vec{S} = \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) du\, dv$$

- Parametrization is essential for computing surface integrals.

Sketch of the Proof

- Parametrize the surface using $\vec{r}(u, v)$
- Define:

$$P = \vec{F} \cdot \frac{\partial \vec{r}}{\partial u}, \quad Q = \vec{F} \cdot \frac{\partial \vec{r}}{\partial v}$$

- Apply Green's Theorem in the parameter domain
- Then convert back using the cross product:

$$(\nabla \times \vec{F}) \cdot (\vec{r}_u \times \vec{r}_v)$$

Applications of Stokes' Theorem

- Electromagnetism (Faraday's Law): curl of electric field relates to changing magnetic field.
- Fluid dynamics: understand vortex motion or rotation in flows.
- Simplify computation of line or surface integrals.
- Foundation of differential forms and generalizations in higher dimensions.

Summary

- Stokes' Theorem links local rotation to boundary circulation.
- Generalizes several core theorems (FTC, Green's).
- Builds intuition for fields and flow in space.
- Appears in physics, engineering, and geometry.