A Tour Through Elliptic Integrals
and their importance to pure mathematics

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Euler Circle

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Elliptic Integrals

Definition (Elliptic Integral)

An elliptic integral is a function of the form

$$f(x) = \int_c^x R(t, \sqrt{p(t)})dt$$

with R being a rational function in $\mathbb{R}$, (that is, it can be expressed as the quotient of two real polynomial functions, constant or not) and $p(t)$ being a real polynomial of degree three or four.

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Third Kind

$$\Pi(n,k,x) = \int_0^x \frac{dt}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}} \tag{1.3}$$

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$$\Pi(\phi \mid \alpha, k) = \int_0^{\phi} \frac{d\theta}{(1 - \alpha^2 \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}} \tag{1.3}$$

Incomplete Elliptic Integrals

Theorem (Legendre)

Any elliptic integral can be described as a linear combination of the canonical forms. In other words,

$$\int_c^x R\left(t, \sqrt{p(t)}\right) dt = c_1 F(k, x) + c_2 E(k, x) + c_3 \Pi(n, k, x) + c_4 f(x),$$

with c_1, c_2 and c_3 being real constants, and $f(x)$ being a combination of elementary functions.

Proof.

Labahn and Mutrie [LMoWDoCS97] describe algorithms for explicitly finding the combination. □

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$$\Pi(n, k) = \Pi(n, k, 1) = \Pi\left(\frac{\pi}{2} | \alpha, k\right); \tag{1.6}$$

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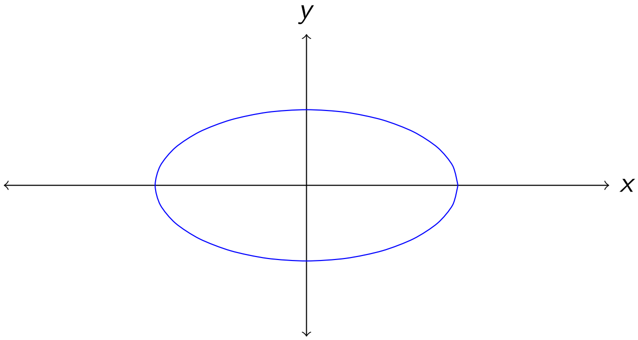

Figure: An ellipse.

The Arc Length of an Ellipse

Definition (Parametric Ellipse)

A parametric curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a function that maps a real number to a point in the plane. One can denote an ellipse using that tool:

$$\gamma(t) = \begin{pmatrix} A \cos t \\ B \sin t \end{pmatrix}$$

for $t \in \mathbb{R}$ and $A, B \in \mathbb{R}$.

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Theorem (Arc length of a parametric curve)

As you should know from basic calculus, if one has a parametric curve $\gamma(t) = (x(t), y(t))$, the arc length $\mathcal{L}$ of the curve at an interval $I \subset R$, $\sup I = a$ and $\inf I = b$,

$$\mathcal{L} = \int_a^b \sqrt{x'^2(t) + y'^2(t)}dt.$$

The Arc Length of an Ellipse

Taking the theorem and the previously defined parametric equation, one may attempt to compute the arc length of an ellipse γ:

$$\mathcal{L}[\gamma] = \int_0^{2\pi} dt \sqrt{A^2 \sin^2 t + B^2 \cos^2 t}. \tag{1.7}$$

which is, at the very least, difficult to compute directly. In fact, this integral does not have a closed-form expression in terms of elementary functions (trigonometric functions, exponentials, logarithms, etc). But we have elliptic integrals, so we can reduce this expression in terms of them!

The Arc Length of an Ellipse

Let's do some work on the integrand:

$$
\begin{aligned}
&\sqrt{A^2 \sin^2 t + B^2 \cos^2 t} \\
&= B\sqrt{\cos^2 t + \frac{A^2}{B^2}\sin^2 t} \\
&= B\sqrt{1 - \sin^2 t + \frac{A^2}{B^2}\sin^2 t} \\
&= B\sqrt{1 - \frac{B^2 - A^2}{B^2}\sin^2 t},
\end{aligned}
$$

The Arc Length of an Ellipse

and for $k = \frac{\sqrt{B^2 - A^2}}{B}$, we attain

$$\sqrt{A^2 \sin^2 t + B^2 \cos^2 t} = B\sqrt{1 - k^2 \sin^2 t},$$

which takes the exact same form of the integrand of the second kind elliptic integral (eq. 1.2) up to a constant. Therefore, it becomes clear that

Arc length of an ellipse

$$\mathcal{L}[\gamma] = B \cdot E\left(2\pi; \frac{\sqrt{B^2 - A^2}}{B}\right)$$

is the expression for the length of an elliptic arc in terms of, well, elliptic integrals.

The Arc Length of an Ellipse

However, due to the symmetry of an ellipse, one can see that its total arc length is equal to four times the length of its arc from 0 to $\frac{\pi}{2}$. With that in mind, we restructure:

Arc length of an ellipse

$$\mathcal{L}[\gamma] = 4 \cdot B \cdot E\left(\frac{\sqrt{B^2 - A^2}}{B}\right)$$

is the expression for the length of an elliptic arc in terms of, well, elliptic integrals, with $E(k)$ being the complete elliptic integral of the second kind.

The Arc Length of a Lemniscate

Definition (Lemniscate)

A lemniscate is a type of curve that resembles an "eight" (8) or an infinity symbol (∞). In polar coordinates, it can be described as

$$r^2 = a^2 \cos 2\theta.$$

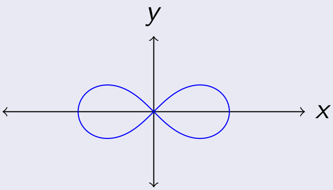

Figure: A lemniscate.

The Arc Length of a Lemniscate

The arc length of the polar curve in 3 is given by

$$\mathcal{L} = \int_{\alpha}^{\beta} d\theta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}.$$

We can compute it:

$$r = \sqrt{a^2 \cos 2\theta} = a\sqrt{\cos 2\theta}$$

$$\left(\frac{dr}{d\theta}\right)^2 = \left(a\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}\right)^2 = a^2\frac{\sin^2 2\theta}{\cos 2\theta}$$

$$\mathcal{L} = \int_0^{2\pi} d\theta \sqrt{a^2\frac{\sin^2 2\theta}{\cos 2\theta} + a^2 \cos 2\theta}$$

The Arc Length of a Lemniscate

$$\mathcal{L} = a\int_0^{2\pi} d\theta \sqrt{\frac{1}{\cos 2\theta}}$$

$$\mathcal{L} = a\int_0^{2\pi} d\theta \sqrt{\frac{1}{1-2\sin^2\theta}}, \tag{1.7}$$

which is, again, difficult.
First, we note that the right lobe of the lemniscate is located on the interval $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, and since both lobes are symmetric, the total arclength may be calculated by multiplying the length of the right lobe by two. Thus the formula can be put together in a more convenient manner, it being

$$\mathcal{L} = 2a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \frac{1}{\sqrt{1-2\sin^2\theta}},$$

The Arc Length of a Lemniscate

but we also have that

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \frac{1}{\sqrt{1-2\sin^2\theta}} = 2\int_0^{\frac{\pi}{4}} d\theta \frac{1}{\sqrt{1-2\sin^2\theta}}$$

due to the symmetry in the top and bottom parts of the lobe, so

$$\mathcal{L} = 4a\int_0^{\frac{\pi}{4}} d\theta \frac{1}{\sqrt{1-2\sin^2\theta}}.$$

Now, we'll use the substitution

$$2\sin^2\theta = \frac{1}{2}\sin^2\psi,$$

The Arc Length of a Lemniscate

which can be trivially checked to be valid for all $\theta \in \left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$. Then, note that ψ is valid when (though not only when) it's in the range $\left[0, \frac{\pi}{2}\right]$. Therefore,

$$\mathcal{L} = 4a \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - \frac{1}{2}\sin^2\psi}},$$

which clearly fits equation 1.1, by having $k = \frac{1}{\sqrt{2}}$, so

$$\mathcal{L} = 4a \cdot F\left(\frac{\pi}{2}; \frac{1}{\sqrt{2}}\right),$$

and since $F(\frac{\pi}{2} \mid k) = K(k)$, we have

Arc length of a lemniscate

$$\mathcal{L} = 4a \cdot K\left(\frac{1}{\sqrt{2}}\right). \tag{1.7}$$

Series Expansion (First Kind)

Lemma

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} x^n (-1)^n \binom{-1/2}{n} = \sum_{n=0}^{\infty} x^n (-1)^{2n} \frac{(2n-1)!!}{(2k)!!}$$

$$= \sum_{n=0}^{\infty} x^n \frac{(2n-1)!!}{(2n)!!}$$

for real x.

Series Expansion (First Kind)

We can put it into 1.1 to get

$$\int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 sin^2\theta}} = \int_0^{\phi} d\theta \sum_{n=0}^{\infty} k^{2n} \sin^{2n}\theta \frac{(2n-1)!!}{(2n)!!},$$

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which implies

Theorem

$$F(\phi \mid k) = \sum_{n=0}^{\infty} \int_0^\phi k^{2n} \sin^{2n}\theta \frac{(2n-1)!!}{(2n)!!} d\theta \tag{1.8}$$

Series Expansion (Second Kind)

For the second kind, we'll use

$$(1-x)^{1/2} = 1 - \sum_{n=0}^{\infty} \frac{(2n+1)!!}{(2n+2)!!} x^{n+1}.$$

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Take $x = k^2 \sin^2 \theta$, and we have

$$E(\phi \mid k) = \int_0^{\phi} d\theta \sum_{n=0}^{\infty} \frac{(2n+1)!!}{(2n+2)!!} k^{2(n+1)} \sin^{2(n+1)} \theta,$$

Series Expansion (Second Kind)

The previous result can be expanded to its first few terms:

Theorem

$$E(\phi \mid k) = \int_0^{\phi} \left(1 - \frac{1}{2}k^2 \sin^2\theta - \frac{1}{8}k^4 \sin^4\theta - \frac{1}{16}k^6 \sin^6\theta - \dots\right) d\theta. \tag{1.9}$$

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Unfortunately, we don't really have a nice general expression for the integrals of the third kind.

Extending the domain

In the real numbers, the elliptic integrals are really only well defined for $0 \leq k < 1$. However, mathematicians do not like to be limited by domains, so we can extend our definition.

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Definition (Residue)

For an analytic complex functions whose Laurent series representation is given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

its residue at z_0 is

$$\operatorname*{Res}_{z=z_0} f(z) = a_{-1},$$

that is, the coefficient of the -1 term.

Residue theorem

Theorem (Cauchy's Residue Theorem)

For an analytic complex function, if a countour γ encloses a set of poles $A = \{a_1, a_2, \ldots, a_n\}$,

$$\int_\gamma f(z)dz = 2\pi i \sum_{a \in A} \operatorname*{Res}_{z=a} f(z),$$

that is, the integral through the contour is equal to the sum of the residues enclosed by it times $2\pi i$.

Cutting the domain

With those two powerful tools, we can extend our integrals to the complex domain. But first, we need to make some treatment on our square root.

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With those two powerful tools, we can extend our integrals to the complex domain. But first, we need to make some treatment on our square root. The function $\sqrt{z}$ is multi-valued, that is, there are always two values $\pm\omega$ that satisfy; such behavior is undesirable, since it prevents the formation of poles, which are fundamental for the usage of the residue theorem. To solve that, we will use a technique called *Riemann surfaces*. Let's take $z = Re^{i\theta}$, with real R and θ, and

Definition (Square root)

Let's take a variable w, then define

$$S = \{(z, w) \in \mathbb{C}^2 \mid w^2 = z\}.$$

With that in mind, we can make a mapping

$$S \to \mathbb{C},$$

$$(z, w) \mapsto w$$

that is bijective. By taking $w = w(z) = \sqrt{z}$, we have

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Lemma

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Proof.

Analyticity can be easily checked with Cauchy-Riemann equations. Then, we check when the denominator vanishes, which is clearly when

$$t = \pm 1 \text{ or } t = \pm \frac{1}{k}.$$

□

With those poles, we can apply the residue theorem:

$$F(k,x) = \int_\gamma \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt$$

$$= 2\pi i \left(\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=-1} f(z) + \operatorname*{Res}_{z=\frac{1}{k}} f(z) + \operatorname*{Res}_{z=-\frac{1}{k}} f(z) \right),$$

with

$$\gamma(t) = x \cdot e^{2\pi i t} \text{ for } t \in [0,1),$$

that is, the complex circle centered at the origin that passes through x.

Jacobi's Elliptic Functions

Definition (Jacobi's Elliptic Functions)

Jacobi described inverses of elliptic integrals in [Jac29]. For

$$u = F(\phi \mid k),$$
$$\phi = F^{-1}(u, k) = \mathrm{am}(u, k),$$

we have

$$\mathrm{sn}(u, k) = \sin(\mathrm{am}(u, k)) = \sin\phi. \tag{2.1}$$

$$\mathrm{cn}(u, k) = \cos(\mathrm{am}(u, k)) = \cos\phi \tag{2.2}$$

$$\mathrm{dn}(u, k) = \sqrt{1 - k^2\,\mathrm{sn}^2(u, k)} = \sqrt{1 - k^2 \sin\phi}. \tag{2.3}$$

Addition Formulae

From now on, we'll supress the parameter k, except when a different one is used. First, let's take a look at the addition formulae described by Jacobi in [Jac29][p. 35-38]:

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$$\operatorname{sn}(u+v) = \frac{\operatorname{sn}(u)\operatorname{cn}(v)\operatorname{dn}(v) + \operatorname{sn}(v)\operatorname{cn}(u)\operatorname{dn}(u)}{1 - k^2\operatorname{sn}^2(u)\operatorname{sn}^2(v)} \tag{2.4}$$

$$\operatorname{cn}(u+v) = \frac{\operatorname{cn}(u)\operatorname{cn}(v) - \operatorname{sn}(u)\operatorname{sn}(v)\operatorname{dn}(u)\operatorname{dn}(v)}{1 - k^2\operatorname{sn}^2(u)\operatorname{sn}^2(v)} \tag{2.5}$$

$$\operatorname{dn}(u+v) = \frac{\operatorname{dn}(u)\operatorname{dn}(v) - k^2\operatorname{sn}(u)\operatorname{sn}(v)\operatorname{cn}(u)\operatorname{cn}(v)}{1 - k^2\operatorname{sn}^2(u)\operatorname{sn}^2(v)} \tag{2.6}$$

Double periodicity

Definition (Doubly periodic function)

A function $f : \mathbb{C} \to \mathbb{C}$ is doubly periodic if there exists two complex ω_1 and ω_2 such that

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and $\frac{\omega_1}{\omega_2} \not\in \mathbb{R}$.

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and $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$.

Theorem

sn *and* cn *are doubly periodic with periods* $4K$ *and* $2iK'$*.* dn *is doubly periodic with periods* $2K$ *and* $2iK'$*.*

Period parallelograms

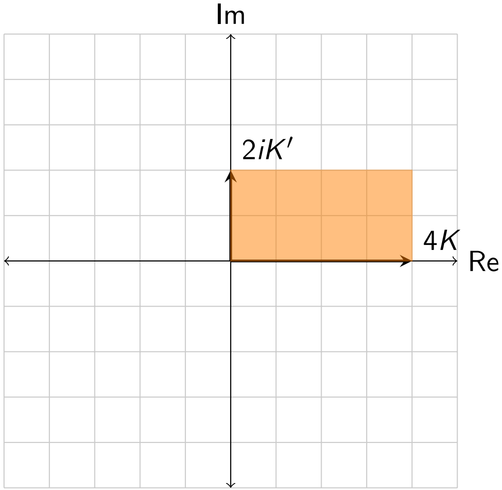

Figure: Illustration of the periods of Jacobi's sn and cn.

Period parallelograms

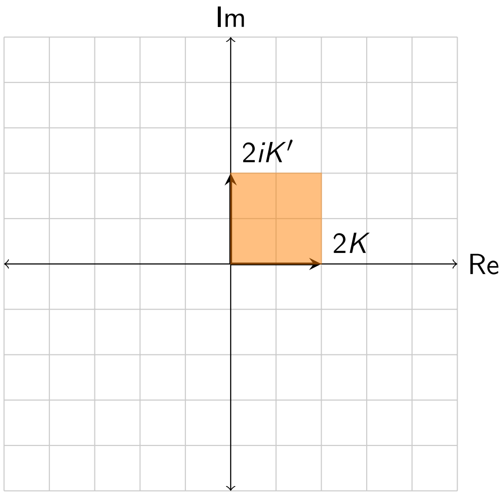

Figure: Illustration of the periods of Jacobi's dn.

Tori

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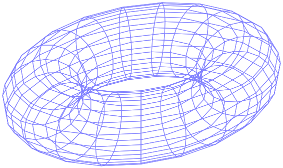

Figure: Embedding of a torus in $\mathbb{R}^3$ (it looks like a doughnut)

The Complex Torus

Theorem

$\mathbb{C}/\Lambda$ *is a torus.*

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Proof.

Miranda [Mir95][p. 8-10] gives a comprehensive explanation and formal proof. □

The Complex Torus

Considering the period parallelograms described, we can have some geometric intuition behind the construction of the torus, as shown visualy in fig. 3.

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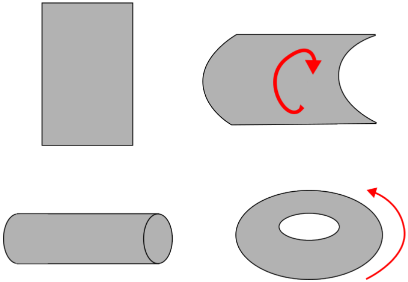

Figure: Diagram showing how to construct a torus from a rectangle

References

C.G.J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, Sumtibus fratrum, 1829.

G. Labahn, M. Mutrie, and University of Waterloo. Department of Computer Science, *Reduction of elliptic integrals to legendre normal form*, Report (University of Waterloo. Department of Computer Science), University of Waterloo, Computer Science Department, 1997.

R. Miranda, *Algebraic curves and riemann surfaces*, Dimacs Series in Discrete Mathematics and Theoretical Comput, American Mathematical Society, 1995.

Thank you!

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