A TOUR THROUGH ELLIPTIC INTEGRALS

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Abstract. This paper aims to teach the basics of the theory of elliptic integrals. The motivation, resolution techniques and some applications will be talked about. It will require knowledge in calculus. It is also going to include a fair amount of rigorosity, since it is aimed mostly at pure mathematicians.

1. Brief History

[Roy11a] and [Roy11b] give a thorough exposition to the history of the elliptic integrals. To put it shortly, Jakob and Johann Bernoulli worked on the lemniscatic integral, namely

$$\int \frac{1}{\sqrt{1-x^4}},$$

which can be used to compute the arc length of a lemniscate. Then, Faganano went on to work out the curve geometrically, which sparked Euler's interested, who then proved the addition formulae. Then, Legendre spent a lot of time writing his treatise [Leg26], on which he reduces a general expression for elliptic integrals to his three canonical forms. Posteriorly, with Jacobi [Jac29] and Weierstrass, there was a revolution in the field, on which it had been started to think on elliptic functions, that are inverses of elliptic integrals and also have lots of nice properties, such as double periodicity, which will be discussed here.

2. Elliptic Integrals

Now, we shall get to the elliptic integrals themselves.

Definition 2.1 (Elliptic Integral)**.** An elliptic integral is a function in the form

$$f(x) = \int_c^x R(t, \sqrt{p(t)})dt$$

with R being a rational function in $\mathbb{R}$, (that is, it can be expressed as the quotient of two real polynomial functions, constant or not) and $p(t)$ being a real polynomial of degree three or four.

Legendre and Jacobi described some common forms of them, and they shall be shown here.

2.1. **Incomplete Elliptic Integrals.** The Incomplete Elliptic Integrals of the first, second and third kind are, respectively:

$$F(k,x) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{2.1}$$

$$E(k,x) = \int_0^x \sqrt{\frac{1-k^2t^2}{1-t^2}}dt \tag{2.2}$$

$$\Pi(n,k,x) = \int_0^x \frac{dt}{(1-nx^2)\sqrt{(1-x^2)(1-k^2x^2)}} \tag{2.3}$$

Legendre also derived trigonometric forms of them, as [BF71] states:

$$F(\phi \mid k) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2 sin^2\theta}} \tag{2.4}$$

$$E(\phi \mid k) = \int_0^\phi d\theta\sqrt{1-k^2\sin^2\theta} \tag{2.5}$$

$$\Pi(\phi \mid \alpha, k) = \int_0^\phi \frac{d\theta}{(1-\alpha^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \tag{2.6}$$

Proposition 2.2. *The algebraic and trigonometric forms of the elliptic integrals are equivalent.*

Proof. We'll use the change of variable formula from calculus:

$$\int f(u)du = \int f(g(x))g'(x)dx \text{ for } u = g(x).$$

By taking $t = \sin\theta$, we have

$$\int \frac{1}{\sqrt{1-t^2}(1-k^2t^2)}dt = \int \frac{1}{\sqrt{(1-\sin^2\theta)(1-k^2\sin^2\theta)}} \cdot \cos\theta d\theta$$
$$= \int \frac{1}{\cos\theta\sqrt{1-k^2\sin^2\theta}}\cos\theta d\theta = \int \frac{1}{\sqrt{1-k^2\sin^2\theta}}d\theta,$$

$$\int \sqrt{\frac{1-k^2t^2}{1-t^2}}dt = \int \sqrt{\frac{1-k^2\sin^2\theta}{1-sin^2\theta}} \cdot \cos\theta d\theta = \int \sqrt{1-k^2\sin^2\theta}d\theta,$$

and

$$\int \frac{1}{(1-nt^2)\sqrt{(1-t^2)(1-k^2t^2)}}dt =$$

$$\int \frac{1}{(1-n\sin^2\theta)\sqrt{(1-\sin^2\theta)(1-k^2\sin^2\theta)}}\cos\theta d\theta$$

$$= \int \frac{1}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}d\theta.$$

Then, one only has to adjust the limits of integration to show the equality of the definite forms. ■

The former list may also be referred to as Jacobi's algebraic form, while the latter is called, well, Legendre's trigonometric forms. An important result is

Theorem 2.3 (Legendre)**.** *Any elliptic integral can be described as a linear combination of the canonical forms. In other words,*

$$\int_c^x R\left(t, \sqrt{p(t)}\right) dt = c_1F(k,x) + c_2E(k,x) + c_3\mathit{\Pi}(n,k,x) + c_4f(x),$$

with c_1, c_2 *and* c_3 *being real constants, and* $f(x)$ *being a combination of elementary functions.*

Proof. Labahn and Mutrie [LMoWDoCS97] describe algorithms for explicitly finding the combination. ■

2.2. **Complete Elliptic Integrals.** Some specific values are notable in the solution of problems that involve elliptic integrals, therefore, we call those specific cases "complete":

$$K(k) = F\left(k,1\right) = F\left(\frac{\pi}{2} \mid k\right) \tag{2.7}$$

$$E(k) = E\left(k,1\right) = E\left(\frac{\pi}{2} \mid k\right) \tag{2.8}$$

$$\mathit{\Pi}(n,k) = \mathit{\Pi}\left(n,k,1\right) = \mathit{\Pi}\left(\frac{\pi}{2}\middle|\alpha,k\right); \tag{2.9}$$

Now, we'll see how those forms can tackle some problems from the previous section.

Example (Arc length of an ellipse).

Definition 2.4 (Ellipse)**.** An ellipse is defined in $\mathbb{R}^2$ (cartesisan plane) as the locus (i.e. the set of all points that satisfy a condition) of all points (x, y) such that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are called the semi-major and semi-minor axes, respectively.

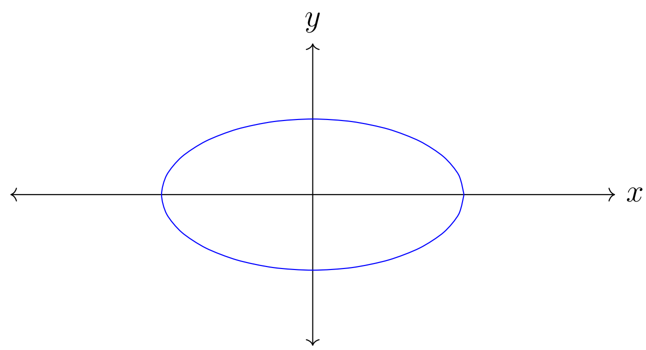

Figure 1. An ellipse.

Definition 2.5 (Parametric Curve)**.** A parametric curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a function that maps a real number to a point in the plane. One can write:

$$\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

for $t \in \mathbb{R}$, $x, y : \mathbb{R} \to \mathbb{R}$.

Mixing both definitions, we have

$$\gamma(t) = \begin{pmatrix} A\cos t \\ B\sin t \end{pmatrix}$$

for $A, B \in \mathbb{R}$. This will be the parametric equation for an ellipse, with A being the semi-major axis and B being the semi-minor axis. One can substitute it into 2.4 and see that

$$\frac{A^2\cos^2 t}{A^2} + \frac{B^2\sin^2 t}{B^2} = 1$$
$$cos^2 t + sin^2 t = 1$$

which is a well-known fact. Thus, both ways to write it are equivalent.

As you should know from basic calculus, if one has a parametric curve $\gamma(t) = (x(t), y(t))$, the arc length $\mathcal{L}$ of the curve at an interval $I \subset R$, $\sup I = a$ and $\inf I = b$,

$$\mathcal{L} = \int_a^b \sqrt{x'^2(t) + y'^2(t)} dt.$$

Taking the theorem and the previously derived parametric equation, one may attempt to compute the arc length of an ellipse γ:

$$\mathcal{L}[\gamma] = \int_0^{2\pi} dt \sqrt{A^2 \sin^2 t + B^2 \cos^2 t} \tag{2.10}$$

which is, at the very least, difficult to compute directly. In fact, this integral does not have a closed-form expression in terms of elementary functions (trigonometric functions, exponentials, logarithms, etc). Of course, if one only wants the arc length of the ellipse, the easier way to do so would be to approximate it numerically; however, we will see that integrals like that are very interesting for studying certain behaviors of functions that have nothing to do directly with ellipses.

We will take 2.10 and generalize it for any arc inside the ellipse:

$$\mathcal{L}(\phi) = \int_0^{\phi} dt \sqrt{A^2 \sin^2 t + B^2 \cos^2 t}.$$

Let's do some work on the integrand:

$$\begin{aligned} &\sqrt{A^2 \sin^2 t + B^2 \cos^2 t} \\ &= B\sqrt{\cos^2 t + \frac{A^2}{B^2} \sin^2 t} \\ &= B\sqrt{1 - \sin^2 t + \frac{A^2}{B^2} \sin^2 t} \\ &= B\sqrt{1 - \frac{B^2 - A^2}{B^2} \sin^2 t}, \end{aligned}$$

and for $k = \dfrac{\sqrt{B^2 - A^2}}{B}$, we attain

$$\sqrt{A^2 \sin^2 t + B^2 \cos^2 t} = B\sqrt{1 - k^2 \sin^2 t},$$

which takes the exact same form of the integrand of the second kind elliptic integral (eq. 2.5) up to a constant. Therefore, it becomes clear that

$$\mathcal{L}(\phi) = B \cdot E\left(\phi; \frac{\sqrt{B^2 - A^2}}{B}\right)$$

is the expression for the length of an elliptic arc in terms of, well, elliptic integrals.

Example (Arc length of a lemniscate). We'll take a look at the lemniscate. More about it can be found on [Roy11a].

Definition 2.6 (Lemniscate)**.** A lemniscate is a type of curve that resembles an "eight" (8) or an infinity symbol (∞). In polar coordinates, it can be described as

$$r^2 = a^2 \cos 2\theta.$$

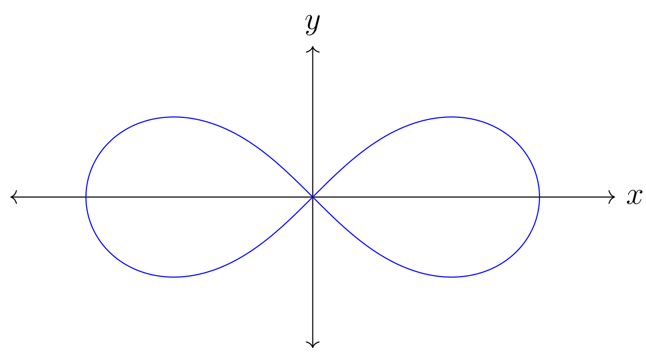

Figure 2. A lemniscate.

The arc length of the polar curve in 2.6 is given by

$$\mathcal{L} = \int_\alpha^\beta d\theta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}.$$

We can compute it:

$$r = \sqrt{a^2 \cos 2\theta} = a\sqrt{\cos 2\theta}$$

$$\left(\frac{dr}{d\theta}\right)^2 = \left(a\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}\right)^2 = a^2 \frac{\sin^2 2\theta}{\cos 2\theta}$$

$$\mathcal{L} = \int_0^{2\pi} d\theta \sqrt{a^2 \frac{\sin^2 2\theta}{\cos 2\theta} + a^2 \cos 2\theta}$$

$$\mathcal{L} = a\int_0^{2\pi} d\theta \sqrt{\frac{1}{\cos 2\theta}}$$

$$\mathcal{L} = a\int_0^{2\pi} d\theta \sqrt{\frac{1}{1 - 2\sin^2\theta}}, \tag{2.11}$$

which is, again, difficult.

Let's now take equation 2.11. First, we note that the right lobe of the lemniscate is located on the interval $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, and since both lobes are symmetric, the total arclength may be calculated by multiplying the length of the right lobe by two. Thus the formula can be put together in a more convenient manner, it being

$$\mathcal{L} = 2a \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \frac{1}{\sqrt{1 - 2\sin^2\theta}},$$

but we also have that

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \frac{1}{\sqrt{1 - 2\sin^2\theta}} = 2\int_{0}^{\frac{\pi}{4}} d\theta \frac{1}{\sqrt{1 - 2\sin^2\theta}}$$

due to the symmetry in the top and bottom parts of the lobe, so

$$\mathcal{L} = 4a \int_{0}^{\frac{\pi}{4}} d\theta \frac{1}{\sqrt{1 - 2\sin^2\theta}}.$$

Now, we'll use the substitution

$$2\sin^2\theta = \frac{1}{2}\sin^2\psi,$$

which can be trivially checked to be valid for all $\theta \in \left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$. Then, note that ψ is valid when (though not only when) it's in the range $\left[0, \frac{\pi}{2}\right]$. Therefore,

$$\mathcal{L} = 4a \int_{0}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - \frac{1}{2}\sin^2\psi}},$$

which clearly fits equation 2.4, by having $k = \frac{1}{\sqrt{2}}$, so

$$\mathcal{L} = 4a \cdot F\left(\frac{\pi}{2}; \frac{1}{\sqrt{2}}\right),$$

and by taking the form $F(\frac{\pi}{2}; k) = K(k)$, we have

$$\mathcal{L} = 4a \cdot K\left(\frac{1}{\sqrt{2}}\right). \tag{2.12}$$

2.3. **Series expansions.** Let's take a known result, which won't be proven here to shorten the text:

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} x^n (-1)^n \binom{-1/2}{n} = \sum_{n=0}^{\infty} x^n (-1)^{2n} \frac{(2n-1)!!}{(2k)!!}$$
$$= \sum_{n=0}^{\infty} x^n \frac{(2n-1)!!}{(2n)!!}$$

We can put it into 2.4 to get

$$\int_0^\phi \frac{d\theta}{\sqrt{1-k^2 sin^2\theta}} = \int_0^\phi d\theta \sum_{n=0}^{\infty} k^{2n} \sin^{2n}\theta \frac{(2n-1)!!}{(2n)!!},$$

thus

$$F(\phi;k) = \sum_{n=0}^{\infty} \int_0^\phi k^{2n} \sin^{2n}\theta \frac{(2n-1)!!}{(2n)!!} d\theta \tag{2.13}$$

is the elliptic integral of the first kind in its trigonometric form expressed as a power series. This type of representation is specially useful for the sake of computation, since one can pick an n-th order approximation by simply picking the appropriate amount of terms, which are much easier integrals. The series expands to

$$F(\phi;k) = \int_0^\phi \left(1 + \frac{1}{2}k^2\sin^2\theta + \frac{3}{8}k^4\sin^4\theta + \frac{5}{16}k^6\sin^6\theta + \dots\right) d\theta \tag{2.14}$$

in its first three terms, for instance. For the second kind, we'll use

$$(1-x)^{1/2} = 1 - \sum_{n=0}^{\infty} \frac{(2n+1)!!}{(2n+2)!!} x^{n+1}.$$

Take $x = k^2\sin^2\theta$, and we have

$$E(\phi;k) = \int_0^\phi d\theta \sum_{n=0}^{\infty} \frac{(2n+1)!!}{(2n+2)!!} k^{2(n+1)} \sin^{2(n+1)}\theta,$$

which can be expanded to its first few terms:

$$E(\phi;k) = \int_0^\phi \left(1 - \frac{1}{2}k^2\sin^2\theta - \frac{1}{8}k^4\sin^4\theta - \frac{1}{16}k^6\sin^6\theta - \dots\right) d\theta. \tag{2.15}$$

For the third kind, we don't really have a nice general expression.

2.4. **Extending the domain.** In the real numbers, the elliptic integrals are really only well defined for $0 \le k < 1$. However, mathematicians do not like to be limited by domains, so we can extend our definition.

Definition 2.7 (Residue)**.** For an analytic complex functions whose Laurent series representation is given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n,$$

its residue at z_0 is

$$\operatorname*{Res}_{z=z_0} f(z) = a_{-1},$$

that is, the coefficient of the -1 term.

Theorem 2.8 (Cauchy's Residue Theorem)**.** *For an analytic complex function, if a countour γ encloses a set of poles $A = \{a_1, a_2, \ldots, a_n\}$,*

$$\int_\gamma f(z)dz = 2\pi i \sum_{a \in A} \operatorname*{Res}_{z=a} f(z),$$

that is, the integral through the countour is equal to the sum of the residues enclosed by it times $2\pi i$.

With those two powerful tools, we can extend our integrals to the complex domain. But first, we need to make some treatment on our square root.

The function $\sqrt{z}$ is multi-valued, that is, there are always two values $\pm\omega$ that satisfy; such behavior is undesirable, since it prevents the formation of poles, which are fundamental for the usage of the residue theorem. To solve that, we will use a technique called *Riemann surfaces.* Let's take a variable w, then define

$$S = \{(z, w) \in \mathbb{C}^2 \mid w^2 = z\}.$$

With that in mind, we can make a mapping

$$S \to \mathbb{C},$$

$$(z, w) \mapsto w$$

that is bijective. By taking $w = w(z) = \sqrt{z}$, we have

$$\sqrt{z} \mapsto w,$$

which is single valued.

This will be the definition used for square root from now, unless specified otherwise.

Proposition 2.9. *$y = \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}$ is analytic with branch points $t = \pm 1$ and $t = \pm\frac{1}{k}$.*

Proof. Analyticity can be easily checked with Cauchy-Riemann equations. Then, we check when the denominator vanishes, which is clearly when

$$t = \pm 1 \text{ or } t = \pm\frac{1}{k}.$$

■

With those poles, we can apply the residue theorem:

$$F(k,x) = \int_\gamma \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}dt$$
$$= 2\pi i \left(\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=-1} f(z) + \operatorname*{Res}_{z=\frac{1}{k}} f(z) + \operatorname*{Res}_{z=-\frac{1}{k}} f(z) \right),$$

with

$$\gamma(t) = x \cdot e^{2\pi i t} \text{ for } t \in [0,1),$$

that is, the complex circle centered at the origin that passes through x.

3. Elliptic Functions

An interesting approach for elliptic integrals is *inverting* them. We'll check out how Jacobi did so and how we can solve problems with this technique, as taught by [BF71], too.

Definition 3.1 (Jacobi's Elliptic Functions)**.** For

$$u = F(\phi \mid k),$$

am is defined as

$$\phi = F^{-1}(u,k) = \operatorname{am}(u,k)$$

and sn as

$$\operatorname{sn}(u,k) = \sin(\operatorname{am}(u,k)) = \sin\phi. \tag{3.1}$$

This function is called Jacobi's Elliptic Sine.
Analagously,

$$\operatorname{cn}(u,k) = \cos(\operatorname{am}(u,k)) = \cos\phi \tag{3.2}$$

is Jacobi's Elliptic Cosine. Finally, the delta amplitude (a new function with no trigonometric analogous) is

$$\operatorname{dn}(u,k) = \sqrt{1 - k^2 \operatorname{sn}^2(u,k)} = \sqrt{1 - k^2 \sin\phi}. \tag{3.3}$$

It is easy to check that $\operatorname{sn}(u,0) = \sin u$, $\operatorname{cn}(u,0) = \cos u$. An interesting property is that both functions also satisfy an identity analogous to the fundamental trigonometric identity: $\operatorname{sn}^2(u,k) + \operatorname{cn}^2(u,k) = \sin^2\phi + \cos^2\phi = 1$.

Definition 3.2. The complementary modulus k' of k is

$$k' = \sqrt{1-k^2},$$

and

$$K' = K(k').$$

3.1. **Double-periodicity.**

Definition 3.3 (Doubly periodic function)**.** A complex function $f : \mathbb{C} \to \mathbb{C}$ is doubly periodic if there exists two complex numbers ω_1, ω_2 such that $f(x+\omega_1) = f(x+\omega_2) = 0$ and $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$, that is, both complex numbers are not linearly dependent.

Definition 3.4 (Period lattice)**.** A doubly periodic function $f : \mathbb{C} \to \mathbb{C}$ with periods ω_1 and ω_2 describes a period lattice at each point z of its domain, namely

$$\Lambda(z) = z + \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{z + n\omega_1 + m\omega_2 \mid (m,n) \in \mathbb{Z}^2\}.$$

We'll also notate $\Lambda = \Lambda(0)$.

Definition 3.5 (Tiling)**.** A tiling of $\mathbb{C}$ is the union of an arbitrary number of connected regions $\mathcal{R}_n \subset \mathbb{C}$ with $n \in \mathbb{N}$, such that

$$\bigcup_{n=0}^{\infty} \mathcal{R}_n = \mathbb{C}$$

and

$$\mathcal{R}_n \cap \mathcal{R}_m \subseteq \partial\mathcal{R}_n \text{ for } n \neq m,$$

that is, all regions are non-overlapping except at their boundaries.

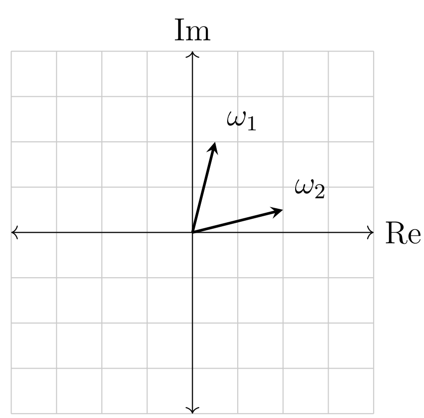

Figure 3. The two periods of a doubly-periodic function.

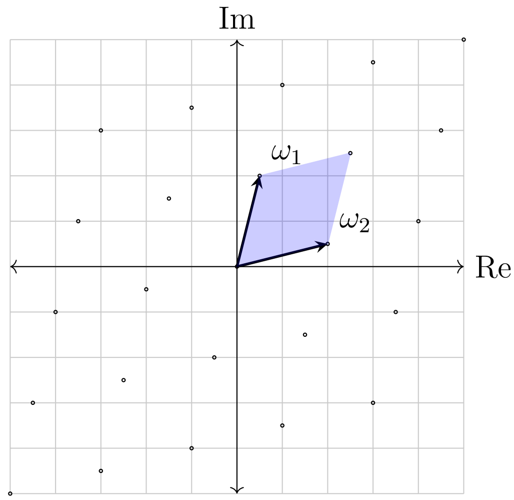

Figure 4. Period lattice at $z = 0$ of a doubly-periodic function

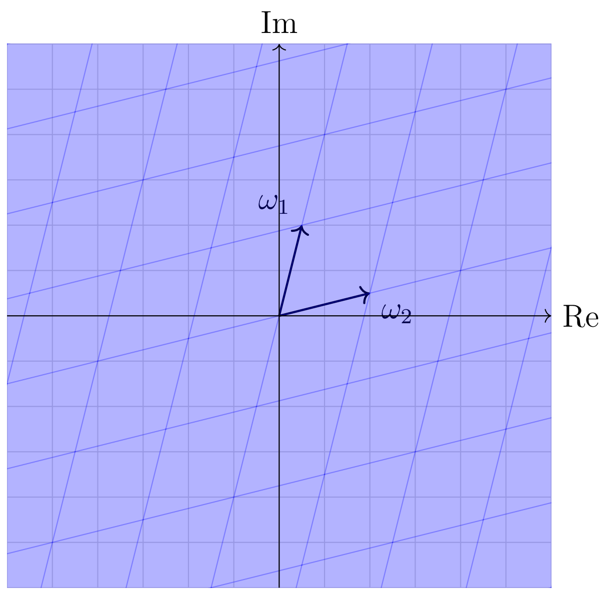

Figure 5. Tiling the complex plane with period parallelograms

From now on, we'll supress the parameter k, except when a different one is used. First, let's take a look at the addition formulae described by Jacobi in [Jac29][p. 35-38]:

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn}(u)\operatorname{cn}(v)\operatorname{dn}(v) + \operatorname{sn}(v)\operatorname{cn}(u)\operatorname{dn}(u)}{1 - k^2 \operatorname{sn}^2(u)\operatorname{sn}^2(v)} \tag{3.4}$$

$$\operatorname{cn}(u+v) = \frac{\operatorname{cn}(u)\operatorname{cn}(v) - \operatorname{sn}(u)\operatorname{sn}(v)\operatorname{dn}(u)\operatorname{dn}(v)}{1 - k^2\operatorname{sn}^2(u)\operatorname{sn}^2(v)} \tag{3.5}$$

$$\operatorname{dn}(u+v) = \frac{\operatorname{dn}(u)\operatorname{dn}(v) - k^2\operatorname{sn}(u)\operatorname{sn}(v)\operatorname{cn}(u)\operatorname{cn}(v)}{1 - k^2\operatorname{sn}^2(u)\operatorname{sn}^2(v)} \tag{3.6}$$

With that in mind, we can prove some results.

Proposition 3.6. $\operatorname{sn}(iu, k) = i\tan\operatorname{am}(u, k')$.

Proof. Let's use the substitution

$$\sin\theta = i\tan\psi \implies \theta = \arcsin(i\tan\psi)$$

$$\int \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int \frac{d\psi}{\sqrt{1+k^2\tan^2\psi}} \cdot i\sec\psi = \int \frac{d\psi}{\cos\psi\sqrt{\frac{\cos^2\psi + k^2\sin^2\psi}{\cos^2\psi}}}$$
$$= \int \frac{d\psi}{\sqrt{1+k^2\sin^2\psi}}.$$

Then, the proposition follows with not much hard work. ∎

Proposition 3.7. sn *has two periods:* $4K$ *and* $2iK'$.

Proof. First, we have

$$\operatorname{sn}(K) = \sin\frac{\pi}{2} = 1$$

by definition. Then, with 3.4,

$$\operatorname{sn}(2K) = \operatorname{sn}(K+K) = \frac{2\operatorname{sn}(K)\operatorname{cn}(K)\operatorname{dn}(K)}{1 - 2k^2\operatorname{sn}^2(K)}$$
$$= 0,$$

$$\operatorname{sn}(4K) = \operatorname{sn}(2K+2K) = \frac{\operatorname{sn}(2K)\operatorname{cn}(2K)\operatorname{dn}(2K) + \operatorname{sn}(2K)\operatorname{cn}(2K)\operatorname{dn}(2K)}{1 - \operatorname{sn}^2(2K)\operatorname{sn}^2(2K)} = 0,$$

$$\operatorname{sn}(u+4K) = \frac{\operatorname{sn}(u)\operatorname{cn}(4K)\operatorname{dn}(4K) + \operatorname{sn}(4K)\operatorname{cn}(u)\operatorname{dn}(u)}{1 - k^2\operatorname{sn}^2(4K)\operatorname{sn}^2(u)} = \operatorname{sn}(u)\operatorname{cn}(4K)\operatorname{dn}(4K)$$
$$= \operatorname{sn}(u)\sqrt{1 - k^2\operatorname{sn}^2(4K)}^2 = \operatorname{sn}(u).$$

For the imaginary period, we first have that

$$F(\pi \mid k) = \int_0^\pi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^\pi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} + \int_{\frac{\pi}{2}}^\pi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},$$

but it is easy to check that, due to the symmetry of the integrand, both integrals have the same value, which implies

$$F(\pi \mid k) = 2K(k).$$

Then, by 3.6,

$$\operatorname{sn}(2iK') = i\tan\operatorname{am}(2K', k') = i\tan\pi = 0$$
$$\operatorname{sn}(u+2iK') = \frac{\operatorname{sn}(u)\operatorname{cn}(2iK')\operatorname{dn}(2iK') + \operatorname{sn}(2iK')\operatorname{cn}(u)\operatorname{dn}(u)}{1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(2iK')} = \operatorname{sn}(u).$$

■

Proposition 3.8. cn *has periods* $4K$ *and* $2iK'$*, too.*

Proof.

$$\operatorname{cn}(4K) = \sqrt{1-\operatorname{sn}^2(4K)} = 1.$$

Then, we can simply apply 3.5:

$$\operatorname{cn}(u+4K) = \frac{\operatorname{cn}(u)\operatorname{cn}(4K) - \operatorname{sn}(u)\operatorname{sn}(4K)\operatorname{cn}(u)\operatorname{cn}(4K)}{1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(4K)} = \operatorname{cn}(u)\operatorname{cn}(4K) = \operatorname{cn}(u).$$

Analogously,

$$\operatorname{cn}(2iK') = \sqrt{1-\operatorname{sn}^2(2iK')} = 1,$$
$$\operatorname{cn}(u+2iK') = \frac{\operatorname{cn}(u)\operatorname{cn}(2iK') - \operatorname{sn}(u)\operatorname{sn}(2iK')\operatorname{cn}(u)\operatorname{cn}(2iK')}{1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(2iK')} = \operatorname{cn}(u)\operatorname{cn}(2iK') = \operatorname{cn}(u).$$

■

Proposition 3.9. dn *has periods* $2K$ *and* $2iK'$*.*

Proof.

$$\operatorname{dn}(2K) = \sqrt{1-k^2\operatorname{sn}^2(2K)} = 1,$$
$$\operatorname{dn}(2iK') = \sqrt{1-k^2\operatorname{sn}^2(2iK')} = 1,$$
$$\operatorname{dn}(u+2K) = \frac{\operatorname{dn}(u)\operatorname{dn}(2K) - k^2\operatorname{sn}(u)\operatorname{sn}(2K)\operatorname{cn}(u)\operatorname{cn}(2K)}{1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(2K)} = \operatorname{dn}(u)\operatorname{dn}(2K) = \operatorname{dn}(u),$$
$$\operatorname{dn}(u+2iK') = \frac{\operatorname{dn}(u)\operatorname{dn}(2iK') - k^2\operatorname{sn}(u)\operatorname{sn}(2iK')\operatorname{cn}(u)\operatorname{cn}(2iK')}{1-k^2\operatorname{sn}^2(u)\operatorname{sn}^2(2iK')} = \operatorname{dn}(u)\operatorname{dn}(2iK') = \operatorname{dn}(u).$$

■

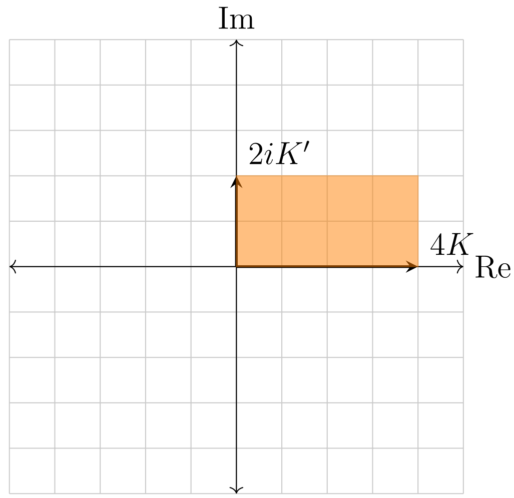

Figure 6. Illustration of the periods of Jacobi's sn and cn.

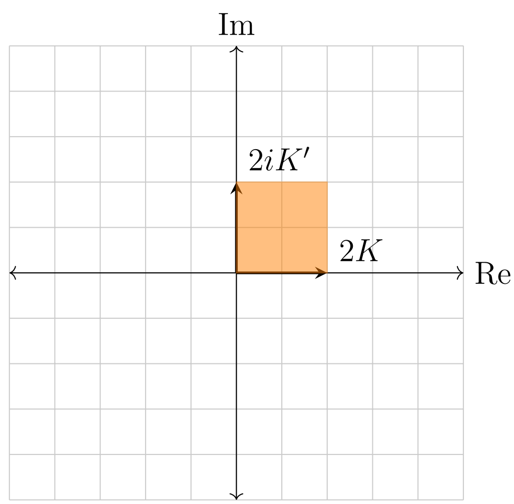

Figure 7. Illustration of the periods of Jacobi's dn.

These results have quite interesting properties, as it will be shown below.

3.2. **The Complex Torus.**

Definition 3.10 (Quotient Group)**.** The quotient group G/H is the set of all left cosets $gH = \{gh \mid h \in H\}$, with g being iterated over the elements of G.

Definition 3.11 (Torus)**.** The "canonical" torus is the cartesian product $S^1 \times S^1$, with S^1 being the real unit circle. We'll denote by "torus" any topological space that is homeomorphic to the canonical one.

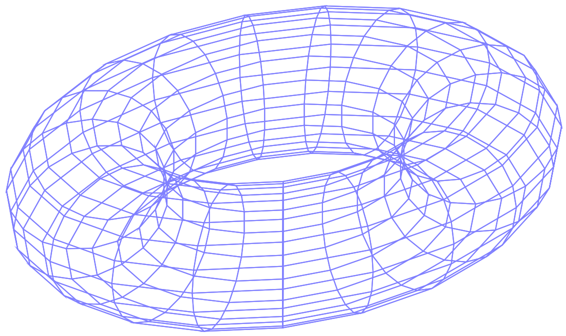

Figure 8. Embedding of a torus in $\mathbb{R}^3$ (it looks like a doughnut)

Definition 3.12 (Complex Torus)**.** The quotient group $\mathbb{C}/\Lambda$, with Λ being a lattice, is called a complex torus.

Theorem 3.13. *$\mathbb{C}/\Lambda$ is a torus.*

Proof. Miranda [Mir95][p. 8-10] gives a comprehensive explanation and formal proof. ■

Considering the period parallelograms described, we can have some geometric intuition behind the construction of the torus, as shown visualy in fig. 9.

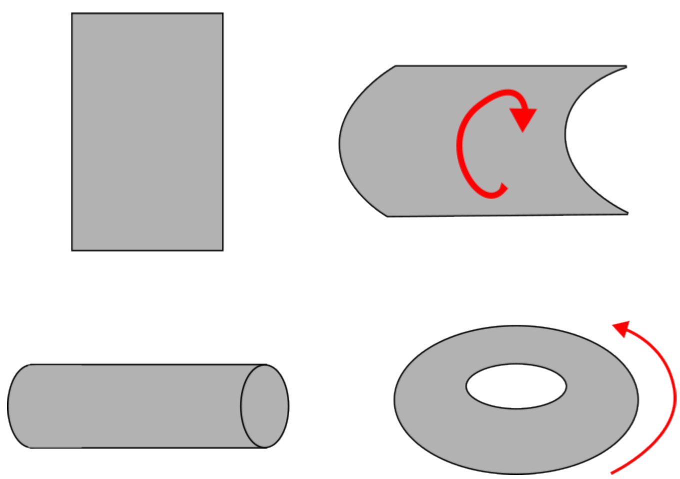

Figure 9. Diagram showing how to construct a torus from a rectangle

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References

[BF71] Paul F. Byrd and Morris D. Friedman, *Handbook of elliptic integrals for engineers and scientists*, 2 ed., Springer Berlin, Heidelberg, 1971.

[Jac29] C.G.J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, Sumtibus fratrum, 1829.

[Leg26] A.M. Legendre, *Traité des fonctions elliptiques et des intégrales eulériennes,: (1826) méthodes pour construire les tables elliptiques. recueil des tables elliptiques. traité des intégrales eulériennes. appendice*, Traité des fonctions elliptiques et des intégrales eulériennes,: avec des tables pour en faciliter le calcul numérique, Imprimerie de Huzard-Courcier, 1826.

[LMoWDoCS97] G. Labahn, M. Mutrie, and University of Waterloo. Department of Computer Science, *Reduction of elliptic integrals to legendre normal form*, Report (University of Waterloo. Department of Computer Science), University of Waterloo, Computer Science Department, 1997.

[Mir95] R. Miranda, *Algebraic curves and riemann surfaces*, Dimacs Series in Discrete Mathematics and Theoretical Comput, American Mathematical Society, 1995.

[Roy11a] Ranjan Roy, *Sources in the development of mathematics*, ch. 36, Cambridge University Press, 2011.

[Roy11b] ———, *Sources in the development of mathematics*, ch. 37, Cambridge University Press, 2011.