Euler Maclaurin Summation Formula

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Overview of the formula

Consider a function $f(x)$ and find it's sum for integers between a and b, or $\sum_{x=a}^{b} f(x)$.
Next consider it's integral between a and b, or $\int_a^b f(x)$

Question

Which is bigger, the integral or the discrete sum?

Overview of the formula

The answer is the integral is always lesser than the discrete sums thus you can write the sum as the integral plus correction terms, or

$$\sum_{x=a}^{b} f(x) = \int_{a}^{b} f(x)dx + c$$

where c are the correction terms

Overview of the formula

Firstly, to make it more accurate, we can find the average of the value of $f(x)$ at it's endpoints. Next the correction terms are in terms of the sum of the bernoulli numbers multiplied by the difference in higher order derivatives evaluated at it's endpoints, the whole divided by $i!$. or

$$\sum_{x=a}^{b} f(x) = \int_a^b f(x)dx + \frac{1}{2}(f(a) + f(b)) + \sum_{i=1}^{n} \frac{b_i}{i!} \left(f^{(i-1)}(b) - f^{(i-1)}(a) \right) + R_n$$

where R_m is an error term.

Stirling's Formula

Consider $f(x) = x!$
taking the natural log on both sides we get

$$\ln x! = \ln x + \ln(x-1) + \cdots + \ln 2 + 0$$

or

$$\ln(x!) = \sum_{i=1}^{x} \ln i$$

Stirling's Formula

Plugging in the formula for the sum where $f(x) = \ln x$ gives us

$$\sum_{k=2}^{n} \ln k = \int_1^n \ln x \, dx + \frac{1}{2}(\ln 1 + \ln n) + \sum_{i=2}^{m} \frac{b_i}{(i)!} \left(f^{(i-1)}(n) - f^{(i-1)}(1) \right) + R_m$$

where the second sum involving the higher order derivatives simplifies to

$$\sum_{i=2}^{m} \frac{b_i (i-2)!}{i! n^{i-1}} (-1)^{i-2} = \frac{1}{6n} - \frac{1}{30n^3} + \cdots$$

Further simplification and then exponentiating gives us

$$n! \sim n^n e^{-n} \sqrt{n} e^{\frac{1}{6n} - \frac{1}{30n} + \cdots}$$

where $e^{\frac{1}{6n} - \frac{1}{30n} + \cdots}$
This value actually converges to exactly $\sqrt{2\pi}$ Therefore, putting it all together gives us

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

Which is Stirling's Formula.

Faulhaber's Formula

Consider the sum of the first n whole numbers squared. Now consider the sum of the first n whole numbers cubed. What about raised to the power 5? What about raised to the power p?

Faulhaber's formula allows us to find the power sum of the first n whole numbers raised to some power p.

Faulhaber's Formula

Consider the sum $\sum_{k=1}^{n} k^p$.
plugging in the formula we get

$$\sum_{k=1}^{n} k^p = \int_1^n x^p dx + \frac{1}{2}(f(n) + f(1)) + \sum_{i=2}^{k} \frac{b_i}{i!}(f^{i-1}(n) - f^{i-1}(1)) + R_m$$

Simplification gives us

$$\sum_{k=1}^{n} k^p = \frac{n^{p+1} - 1}{p+1} - \frac{1}{2}(n^p + 1) + \sum_{i=2}^{k} \frac{b_i}{i!} \left(\frac{p!}{(p-i+1)!} n^{p-i+1} - \frac{p!}{(p-i+1)!} \right) + R_m$$

or

$$\sum_{k=1}^{n} k^p = A_0 n + A_1 n + A_2 n^2 + \cdots + A_p n^p + A_{p+1} n^{p+1}$$

Faulhaber's Formula

Therefore, we can conclude that

$$\sum_{k=1}^{n} k^p = \frac{1}{p-1} \sum_{i=1}^{p} \binom{p+1}{i} b_i n^{p+1-i}$$

Which is Faulhaber's Formula

Euler's Constant

Consider the harmonic series $\sum_{k=1}^{n} \frac{1}{k}$. As $n \to \infty$ this value actually diverges. Now consider $\ln x$. These two graphs look pretty similar.
If you find the difference between the two graphs as $n \to \infty$ we get a constant value which is called Euler's Constant!
We can write Euler's Constant (denoted by γ) as

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln x \right)$$

Euler's Constant

Plugging the harmonic series into the formula gives us

$$\sum_{k=1}^{n} \frac{1}{k} = \int_{1}^{n} \frac{1}{x} dx + \frac{1}{2}\left(\frac{1}{n} + 1\right) + \sum_{i=1}^{k} \frac{b_i}{i!}\left((-1)^i (i!) n^{-(i+1)} + (-1)^i\right) + R_m$$

Further simplification gives us

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + \cdots + \gamma$$

Euler's Constant

With a bit of rearranging we can work out eulers constant to be

$$\gamma = \sum_{k=1}^{n} \frac{1}{k} - \ln n - \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \cdots$$

We do this as with a finite small value of n we can calculate γ to be $\gamma = 0.5772156649\ldots$

Riemann zeta function

The Basel problem is the sum of the reciprocals of the whole numbers squared, or $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Euler proved this converges to precisely $\frac{\pi^2}{6}$.

What about the sum of the reciprocals of the whole unmbers cubed? What about raised to the power 4. What about raised to a fraction or even a complex number?

The Riemann Zeta function ($\zeta(s)$)is the sum to infinity of the reciprocals of the whole numbers raised to some power s. It can be written as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Riemann Zeta Function

Let $f(x) = \frac{1}{x^s}$. plugging the Riemann Zeta function into the Euler Maclaurin Summation Formula we get

$$\sum_{n=1}^{N} \frac{1}{n^s} = \int_1^N \frac{1}{x^s} dx + \frac{1}{2N^s} + \sum_{i=2}^{k} \frac{b_i}{i!} \left({}^n p_i \cdot \frac{1}{N^{k-i}} \right)$$

Upon further simplification we can write the function as

$$\sum_{n=1}^{N} \frac{1}{n^s} = \frac{N^{1-s} - 1}{1 - s} + \frac{1}{2N^s} + \frac{s}{12N^{s-1}} + \frac{s(s-1)(s-2)}{720N} + \cdots$$

if we split the sum to infinity into a sum from 1 to N and a sum from N to infinity we can rearrange the integral and the formula to get

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + \frac{1}{2N^s} + \frac{s}{12N^{s-1}} + \frac{s(s-1)(s-2)}{720N} + \cdots$$

This means we can approximate the Zeta function using large values of N

The End