ON EULER MACLAURIN SUMMATION FORMULA

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Abstract. This research paper is about the Euler Maclaurin Summation Formula which talks about how to find discrete sums from their integral counterparts and vice versa. It begins with an introduction, then a general proof, and then talks about some of it's applications across mathematics.

1. Introduction

The Euler–Maclaurin summation formula is a powerful analytic tool that bridges discrete and continuous mathematics by relating finite sums to integrals and vice versa. Originally developed by Leonhard Euler and Colin Maclaurin in the 18th century, it has many applications in mathematics.

Given a smooth function, the formula expresses a finite sum $\sum_{n=a}^{b}$ as its integral $\int_a^b f(x)dx$ plus a series of correction terms involving the higher-order derivatives of $f(x)$ and the bernoulli numbers and an error term which is pretty small for suitable curves. The Euler–Maclaurin formula has applications in Stirling's approximation, the analytic continuation of the Riemann zeta function, approximations of Euler's constant γ, and Faulhaber's formula for sums of powers.

The formula can be written as

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)dx + \frac{1}{2}(f(b) + f(a)) + \sum_{i=2}^{n} \frac{b_i}{i!}(f^{(i-1)}(b) - f^{(i-1)}(a)) + R_n$$

where

$$R_n = \frac{(-1)^{n+1}}{(2n)!} \int_a^b B_{2n}(x - \lfloor x \rfloor) f^{(2n)}(x)\, dx$$

R_n represents an error term which is small for suitable curves. The only condition for the formula is that the curve must be k times differential between the boundaries of a and b.

2. Preliminaries

Before we dive into the proof and applications of the formula, it is important to mention some relevant background information.

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2.1. **Taylor Series.** A Taylor series is a different way of representing a function $f(x)$ centred around a point a. This is useful as it allows us to approximate complicated functions with just a few terms. The Taylor series often is an infinite sum involving the functions derivatives. The Taylor series for a function $f(x)$ centred at a point a can be written as

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

Or more compactly, it can be written as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Note that if we take $a = 0$ then we get the maclaurin series. The Taylor series is a powerful tool that can help us approximate complicated functions. Some of the functions and their Taylor Series' are listed below

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The Taylor series for $f(x)$ is required in proving the Euler Maclaurin Summation Formula.

2.2. **Bernoulli Numbers.** The Bernoulli Numbers are a set of numbers discovered in the 17th century by Jacob bernoulli, originalli to be used in context of power sums; however, they pop up in various mathematical topics such as number theory, complex number theory, the euler maclaurin summation formula and more.

The bernoulli numbers are part of a sequence $b_0, b_1, b_2, \ldots, b_n$ and are usually denoted by b_n. They can be defined using the generating function

$$G_{b_n}(t) = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots$$

where

$$G_{b_n}(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}$$

However, since this gives us an infinite polynomial, the generating function is more often written as

$$G_{b_n}(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = \frac{t}{e^t - 1} \quad , \quad |t| < 2\pi$$

Jacob Bernoulli first discovered this during his research on power sums or what we in today's world call the Faulhauber Formula. The sequence of bernoulli numbers is as follows

$$b_0 = 1, b_1 = -\frac{1}{2}, b_2 = \frac{1}{6}, b_3 = 0, b_4 = -\frac{1}{30}, b_5 = 0, \ldots$$

Note that the odd bernoulli numbers other than b_1 are 0.

Additionally, for easier computation, bernoulli numbers can also be found by the recursive formula

$$\sum_{k=1}^{n} \binom{n+1}{k} b_k = 0$$

This formula allows us to calculate higher bernoulli numbers without much diffculty.

Bernoulli numbers show up in analytic number theory; however, they show up in the Euler Maclaurin Summation Formula as the coefficients of the correction terms for the difference bewteen the discreete sum and its integral. Other places where bernoulli numbers might show up is Faulhaubers Formula, Riemann Zeta Function and elsewhere.

2.3. **Bernoulli Polynomials.** Similar to the bernoulli numbers, the bernoulli polynomials are a set of polynomials $B_0(x), B_1(x), B_2(x), B_3(x), B_4(x), \ldots, B_n(x)$, that play a crucial part in the error term in the Euler Maclaurin Summation Formula. Bernoulli Polynomials can be defined using the following generative function.

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n, \quad |t| < 2\pi$$

Note that if you plug in $x = 0$ then you get the nth bernoulli number for your nth bernoulli polynomial. $B_n(x) = B_n(0) = b_n$ The first few bernoulli polynomials are listed below.

$$B_0(x) = 1$$
$$B_1(x) = x - \frac{1}{2}$$
$$B_2(x) = x^2 - x + \frac{1}{6}$$
$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$$

Note that in the Euler Maclaurin Summation Formula we often talk about the Periodic Bernoulli Polynomials denoted by $P_n(x)$ where $P_n(x) = B_n(x - \lfloor x \rfloor)$ where $\lfloor x \rfloor$ is the integer part of x.

3. Proof

Thetrapezoidal rule states that any sum of $f(x)$ can be approximately represented by it's integral plus the average of it's endpoints. It can be written as

$$\sum_{x=a}^{b} f(x) \approx \int_a^b f(x)dx + \frac{1}{2}((f(b) + f(a))$$

Let f be a smooth function. Therefore, it's sum S can be written as $S = \sum_{k=a}^{b} f(k)$. However, we can also write this sum as a sum of integrals of $f(k)$ evaluated along the interval k and $k+1$. Or

$$\sum_{k=a}^{b} f(k) = \sum_{k=a}^{b} \int_k^{k+1} f(k)dx$$

Moreover, We can add and subtract the function $f(x)$ and then integrate the real part of the function, or

$$S = \sum_{k=a}^{b} f(k) = \sum_{k=a}^{b} \left(\int_k^{k+1} f(x)\, dx - \int_k^{k+1} (f(x) - f(k))\ dx \right)$$

This can be simplified to give us

$$\sum_{k=a}^{b} f(k) = \int_a^{b+1} f(x)\, dx - \sum_{k=a}^{b} \int_k^{k+1} (f(x) - f(k))\ dx$$

Next, we can expand $f(x)$ using it's taylor series which gives us

$$f(x) = f(k) + f'(k)(x-k) + \frac{f''(k)}{2!}(x-k)^2 + \cdots$$

Adding $f(k)$ on both sides gives us

$$f(x) - f(k) = f'(k)(x-k) + \frac{f''(k)}{2!}(x-k)^2 + \cdots$$

Therefore, if we integrate this expanded function over the interval $k, k+1$ we get

$$\int_k^{k+1} (f(x) - f(k))\, dx = f'(k) \int_k^{k+1} (x-k)\, dx + \frac{f''(k)}{2!} \int_k^{k+1} (x-k)^2\, dx + \cdots$$

As the derivatives evaluated at k can be pulled out of the integral as they are just a constant. Next if we calculate each integral of $(x-k)^n$ for the nth term, we get $\frac{1}{n+1}$. Therefore, we can rewrite this integral as

$$\int_k^{k+1} (f(x) - f(k)) dx = \frac{1}{2} f'(x) + \frac{1}{6} f''(x) + \frac{1}{24} f^{(3)}(x) + \cdots$$

Next, if we sum this from a to b we get

$$\sum_{k=a}^{b} \int_k^{k+1} (f(x) - f(k)) dx = \frac{1}{2} \sum_a^b f'(x) + \frac{1}{6} \sum_a^b f''(x) + \frac{1}{24} \sum_a^b f^{(3)}(x) + \cdots$$

Thus, we can rearrange the formula which gives us

$$\sum_{k=a}^{b} f(k) = \int_a^{b+1} f(x) dx + \frac{1}{2} \sum_a^b f'(x) + \frac{1}{6} \sum_a^b f''(x) + \frac{1}{24} \sum_a^b f^{(3)}(x) + \cdots$$

Next, if we rearrange the integral to be from a to b and write the correction terms as a sum of a series of a combination of bernoulli numbers and $i!$, we get the original formula (without the error term) which is

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x) dx + \frac{1}{2}(f(b) + f(a)) + \sum_{i=2}^{n} \frac{b_i}{i!} (f^{(i-1)}(b) - f^{(i-1)}(a))$$

However this only holds true as n approaches ∞ and thus to avoid a lot of calculations we take a finite n which means we need to add an extra error term which will be small for large values of n. Thus we get

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x) dx + \frac{1}{2}(f(b) + f(a)) + \sum_{i=2}^{n} \frac{b_i}{i!} (f^{(i-1)}(b) - f^{(i-1)}(a)) + R_n$$

Which is the Euler Maclaurin Summation Formula.

4. Applications

The Euler Maclaurin Summation Formula has applications all throughout mathematics ranging from applied mathematics to analytic number theory. In this paper I will be focusing on its application in **Stirling's Formula**, **Faulhaber's Formula**, **Euler's Constant**, **Riemann Zeta Function**.

4.1. **Stirling's Formula.** The factorial function denoted by $n!$ is a function that is the product of all the natural numbers less than and equal to n. it is defined as $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$. While this is easy to calculate manually for relatively small numbers, for larger values of n it gets very cumbersome. To overcome this we can use Stirling's Formula.

Stirling's Formula is a formula that can be used to approximate the factorials of very high numbers. the formula is such that as $n \to \infty$

$$n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$$

However to prove this we could use the Euler Maclaurin Summation Formula. taking the natural log of $n!$ and rearranging the terms in the log function gives us

$$\ln(n!) = \ln n + \ln(n-1) + \ln(n-2) + \cdots + \ln 2 + \ln 1 = \sum_{r=1}^{n} ln(r)$$

Next, if we apply the euler maclaurin formula to this sum we get

$$\sum_{k=2}^{n} \ln k = \int_1^n \ln x \, dx + \frac{1}{2}(\ln 1 + \ln n) + \sum_{i=2}^{m} \frac{b_i}{(i)!} \left(f^{(i-1)}(n) - f^{(i-1)}(1)\right) + R_m$$

where m is a variable which we assign a suitable value to as per to calculate the accuracy of our result. Calculating the integral and simplifying gives us

$$\sum_{k=2}^{n} \ln k = n \ln n - n + 1 + \frac{1}{2}(\ln n) + \sum_{i=2}^{m} \frac{b_i}{i!} \left(f^{(i-1)}(n) - f^{(i-1)}(1)\right) + R_m$$

Next, if we find the higher order derivatives of $f(x) = \ln x$ we find that

$$f^{(n)}(x) = \frac{(n-1)!}{x^n}(-1)^{n-1}$$

Therefore, computing $\sum_{i=2}^{m} \frac{b_i}{(i)!} \left(f^{(i-1)}(n) - f^{(i-1)}(1)\right)$, as $n \to \infty$ we can discard the $-f^{(i-1)}(1)$ terms as they become negligible, which means we are left with $\sum_{k=1}^{m} \frac{B_i}{(i)!} \left(f^{(i-1)}(n)\right)$ Next, substituting and simplifying we get

$$\sum_{i=2}^{m} \frac{b_i(i-2)!}{i!n^{i-1}}(-1)^{i-2} = \frac{1}{6n} - \frac{1}{30n^3} + \cdots$$

Substituting and exponentiating gives us

$$n! \sim n^n e^{-n} \sqrt{n} e^{\frac{1}{6n} - \frac{1}{30n} + \cdots}$$

where $+1$ is absorbed by the higher order coefficients as $n \to \infty$. Next, $e^{\frac{1}{6n} - \frac{1}{30n^3} + \cdots}$ tends to a constant C which works out to be $\sqrt{2\pi}$.

Finally, Substituting everything together and rearranging we arrive at

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

which is Stirling's Formula.

4.2. **Faulhabers Formula.** Power sums are defined as the sum of the natural numbers from one to n raised to some power p. They can be written as $\sum_{k=1}^{n} k^p$. These power sums are found in different fields of mathematics ranging from calculus to statistics. To calculate these power sums we can use Faulhabers Formula.

Faulhaber's Formula is a formula that can be used to work out the value of these power sums, $\sum_{k=1}^{n} k^p$. The Euler Maclaurin Summation Formula can be used to derive the Faulhabers formula which can be written as

$$\sum_{k=1}^{n} k^p = \frac{1}{p-1} \sum_{i=1}^{p} \binom{p+1}{i} b_i n^{p+1-i}$$

where b_i is the ith bernoulli number and $\binom{p+1}{i}$ is the binomial coefficient of $p+1$ over i. The formula always returns a polynomial in terms of n.

To begin with, we let $f(x) = x^p$. Applying the Euler Maclaurin Summation Formula to this sum gives us

$$\sum_{k=1}^{n} k^p = \int_1^n x^p dx + \frac{1}{2}(f(n) + f(1)) + \sum_{i=2}^{k} \frac{b_i}{i!}(f^{i-1}(n) - f^{i-1}(1)) + R_m$$

Solving the integral and computing the higher order derivatives we get

$$\sum_{k=1}^{n} k^p = \frac{n^{p+1} - 1}{p+1} - \frac{1}{2}(n^p + 1) + \sum_{i=2}^{k} \frac{b_i}{i!} \left(\frac{p!}{(p-i+1)!} n^{p-i+1} - \frac{p!}{(p-i+1)!} \right) + R_m$$

Further simplification and grouping of all the terms we get

$$\sum_{k=1}^{n} k^p = A_0 n + A_1 n + A_2 n^2 + \cdots + A_p n^p + A_{p+1} n^{p+1}$$

where A_t are the coefficients of the terms in the form of some combination of Bernoulli numbers and the binomial coefficient. This means that we can factor out the Bernoulli numbers and the binomial coefficients and be left with

$$\sum_{k=1}^{n} k^p = \frac{1}{p-1} \sum_{i=1}^{p} \binom{p+1}{i} b_i n^{p+1-i}$$

4.3. **Euler's constant.** Eulers constant, denoted by γ is a constant that pops up in many mathematical fields such as analytic number theory and more. It is defined as the limit of the difference between the Harmonic Series and the Natural log, or

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right)$$

The constant approaches to be a value of $\gamma = 0.5772\ldots$ and it is still unknown whether Euler's constant is rational or irrational. A method to find Euler's Constant can be given by the Euler Maclaurin Summation Formula.

Firstly, rewrite the harmonic series using the euler maclaurin summation formula. therefore,

$$\sum_{k=1}^{n}\frac{1}{k} = \int_{1}^{n}\frac{1}{x}dx + \frac{1}{2}\left(\frac{1}{n}+1\right) + \sum_{i=1}^{k}\frac{b_i}{i!}\left((-1)^i(i!)n^{-(i+1)} + (-1)^i\right) + R_m$$

calculating the integral and simplifying further gives us

$$\sum_{k=1}^{n}\frac{1}{k} = \ln n + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + \cdots + \gamma$$

Here, Eulers constant, γ pops out as a constant to correct the difference between the integral and the sum and the correction terms as the correction terms would eventually converge to 0 as $n \to \infty$. This means that with a bit of rearranging we can rewrite Euler's Constant as

$$\gamma = \sum_{k=1}^{n}\frac{1}{k} - \ln n - \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \cdots$$

This allows us to rewrite Euler's Constant as its original definition plus some error terms which means we can rewrite it as it's original definition + a series of error terms. This means we can calculate Eulers Constant upto a very high accuracy even with a low value of n. Using this we can estimate Euler's Constant to approximately be $\gamma = 0.5772156649\ldots$.

4.4. **Riemann Zeta function.** The Riemann Zeta function is a function that gives the sum of the reciprocals of the whole numbers raised to some power s. The function is denoted by $\zeta(s)$ and has various applications in analytical number theory. The function in its simplest form can be written as

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

For example, $\zeta(1)$ gives the sum of the reciprocal of all the natural numbers, this value actually diverges. $\zeta(2)$ gives us the sum of the reciprocals of all the squares of the natural numbers. $\zeta(2)$ is also known as the Basel Problem and has stumped mathematicians for centuries until Leonhard Euler later proved the sum to equal exactly $\frac{\pi^2}{6}$ however he did calculate to a high accuracy the value using the Euler Maclaurin Summation Formula. We can rewrite the zeta function $\zeta(s)$ using the Euler Maclaurin Summation Formula.

To start with, let $f(x) = \frac{1}{x^s}$
using the Euler Maclaurin Summation Formula to rewrite the zeta function, we get

$$\sum_{n=1}^{N} \frac{1}{n^s} = \int_1^N \frac{1}{x^s} dx + \frac{1}{2N^s} + \sum_{i=2}^{k} \frac{b_i}{i!} \left({}^n p_i \cdot \frac{1}{N^{k-i}} \right)$$

where ${}^n p_i = \frac{n!}{(n-i)!}$ and is called the descending factorial and can be written as ${}^n p_i = n(n-1)(n-2)\cdots(n-i+1)(n-i)$. Note that here the function evaluated at 1 gets absorbed into the integral as we already accounted for that there. Additionally, the higher order derivative evaluated at 1 is also not needed as we are only approximating the function anyway and thus we do not need to account for it.

Upon further simplification we can write the function as

$$\sum_{n=1}^{N} \frac{1}{n^s} = \frac{N^{1-s} - 1}{1-s} + \frac{1}{2N^s} + \frac{s}{12N^{s-1}} + \frac{s(s-1)(s-2)}{720N} + \cdots$$

if we split the sum to infinity into a sum from 1 to N and a sum from N to infinity we can rearrange the integral and the formula to get

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + \frac{1}{2N^s} + \frac{s}{12N^{s-1}} + \frac{s(s-1)(s-2)}{720N} + \cdots$$

This approximation best works for large values of N as the subsequent correction terms also converge to 0 quicker.

One advantage of this method of using the Euler Maclaurin Summation Formula is that it helps figure out the values for the Zeta function evaluated at negative integers which does not make sense the traditional way at first glance.

The significance of the zeta function in number thory is to figure out the primes as the Zeta function is closely related to it. The Euler Maclaurin Summation Formula could be used to investigate the Riemann Zeta Function more closely and potentially be used to answer the Riemann Hypothesis which states that the zeroes of the function for a complex number s, the real part for all of them equal to $\frac{1}{2}$.

5. Conclusion

The Euler Maclaurin Summation Formula is a very powerful formula, to find the value of a sum using it's integral, and has applications in many fields such as deriving Stirling's Formula, deriving Faulhaber's Formula, finding the value of Euler's Constant or expressing the Riemann Zeta function in a way that it's easier to compute.

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