Banach-Tarski Paradox

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Euler Circle

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Theorem (Banach–Tarski 1924)

A solid unit ball $B^3 \subset \mathbb{R}^3$ *can be partitioned into finitely many pairwise-disjoint sets* $B^3 = P_1 \sqcup \cdots \sqcup P_k$ *such that rigid motions* $g_1, \ldots, g_k \in \mathrm{Isom}(\mathbb{R}^3)$ *rearrange them into*

$$g_1 P_1 \;\sqcup\; \ldots \;\sqcup\; g_k P_k \;=\; B^3 \;\sqcup\; B^3,$$

producing two unit balls congruent to the original.

Why is this shocking?

- It appears to "duplicate" volume, clashing with conservation-of-mass intuition.
- The construction crucially invokes the *Axiom of Choice*, making the result both powerful and controversial.

Axiom of Choice (AC)

Definition (Formal statement)

For every family of *non-empty* sets $\{X_i\}_{i \in I}$ there exists a *choice function* $f : I \to \bigcup_{i \in I} X_i$ with $f(i) \in X_i$ for each i.

In plain language:

- Given any collection of bins—no matter how many, and even if you cannot describe what's inside—you are allowed to pick *one* item from each bin *simultaneously*.
- The axiom asserts such a global "picking rule" exists, even when listing or describing it explicitly is impossible.

Why AC is indispensable for Banach–Tarski

- The proof needs a set containing *exactly one* point from each orbit of a *free-group* action on the ball. Finding those representatives requires a choice function on an uncountable family of orbits $\to$ AC.

Free Groups — Quick Overview

Definition

A **free group** on generators $\mathcal{G}$ (denoted $F(\mathcal{G})$) is the group with no relations except those forced by the group axioms.

Concrete realisation

$F(\mathcal{G})$ is *isomorphic* to the set of all *reduced words* built from the symbols $\mathcal{G} \cup \mathcal{G}^{-1}$, with multiplication "write the words back-to-back, then cancel adjacent inverse pairs."

Example

For two generators a, b,

$$F_2 = \langle a, b \rangle \;\cong\; \{\text{reduced words in } a^{\pm 1},\, b^{\pm 1}\}.$$

Typical elements: $b^{-1}a$, $a^2 b^{-3} a^{-1}$, and the identity (empty word) e.

Why F_2 is Paradoxical

The key combinatorial fact (already noted by Hausdorff) is that F_2 can be split into five disjoint pieces which duplicate under left multiplication:

$$F_2 = \{e\} \cup S(a) \cup S(a^{-1}) \cup S(b) \cup S(b^{-1}),$$

where $S(a)$ means "words whose first letter is a" and so on. A quick calculation shows

$$F_2 = S(a) \cup aS(a^{-1}) \quad \text{and} \quad F_2 = S(b) \cup bS(b^{-1}).$$

Thus F_2 is *paradoxical*: it is equidecomposable with two copies of itself—an algebraic analogue of the geometric duplication we seek.

Rotations that Generate F_2

We embed the free group F_2 into the rotation group $\mathrm{SO}(3)$. Recall

$$\mathrm{SO}(3) = \left\{ R \in \mathbb{R}^{3\times 3} \;\middle|\; R^{\mathsf{T}} R = I,\ \det R = 1 \right\},$$

the set of all orientation-preserving rotations of $\mathbb{R}^3$.

Choose

$$\theta = \arccos \frac{1}{3} \;\approx\; 1.23096 \text{ rad.}$$

Define the rotations

$$A:\ \text{rotate by } \theta \text{ about the } x\text{-axis}, \qquad B:\ \text{rotate by } \theta \text{ about the } z\text{-axis}.$$

Theorem (Free-Rotation Lemma)

The subgroup $\langle A, B\rangle \subset \mathrm{SO}(3)$ is isomorphic to the free group F_2; that is, no non-trivial reduced word in $A^{\pm 1}, B^{\pm 1}$ equals the identity.

Explicit Rotation Matrices

Choose $\theta = \arccos \dfrac{1}{3}$ so that $\cos\theta = \frac{1}{3}$ and $\sin\theta = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}$.

$$A = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -2\sqrt{2} \\ 0 & 2\sqrt{2} & 1 \end{pmatrix}, \qquad B = \frac{1}{3}\begin{pmatrix} 1 & -2\sqrt{2} & 0 \\ 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Their inverses (rotations by $-\theta$) are obtained by $\sin\theta \mapsto -\sin\theta$:

$$A^{-1} = A^{\mathsf{T}}, \qquad B^{-1} = B^{\mathsf{T}}.$$

Note every entry is an integer multiple of $\frac{1}{3}$; this 3-adic structure drives the proof that $\langle A, B\rangle$ is free.

Lemma 1 – Behaviour on $(1,0,0)^{\mathsf{T}}$

Lemma

Let w be a reduced word of length n in $A^{\pm 1}, B^{\pm 1}$. Then

$$w \cdot (1,0,0)^{\mathsf{T}} = \frac{1}{3^{n}} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix}, \quad a, b, c \in \mathbb{Z},\ 3 \nmid b.$$

Idea. Each factor of $A^{\pm 1}$ or $B^{\pm 1}$ introduces one extra denominator 3. The y-coordinate always carries the factor $\sqrt{2}$; modulo 3 its integer coefficient is never divisible by 3.

Proof of Lemma 1 (Induction Sketch)

Base case $n = 0$. With $w = e$ we have $(1, 0, 0)^{\mathsf{T}}$; the formula gives $a = 1, b = c = 0$.

Inductive step. Assume statement true for length n. Append one generator, say A:

$$A \cdot \frac{1}{3^{\,n}} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix} \;=\; \frac{1}{3^{\,n+1}} \begin{pmatrix} 3a \\ (a + 2c)\sqrt{2} \\ (2b + a) \end{pmatrix}.$$

Because $3 \nmid b$, the new y-coefficient $a + 2c$ is still *not* divisible by 3. Similar calculations hold for A^{-1}, B, B^{-1}.

Thus the property propagates from length n to $n + 1$, completing the induction.

Free-Rotation Lemma – Conclusion

Theorem

The subgroup $G = \langle A, B \rangle \subset \mathrm{SO}(3)$ *is free on two generators; i.e. no non-trivial reduced word in* $A^{\pm 1}, B^{\pm 1}$ *equals the identity.*

Proof.

Suppose a reduced word $w \neq e$ satisfied $w = I$. Apply both sides to $(1, 0, 0)^{\mathsf{T}}$:

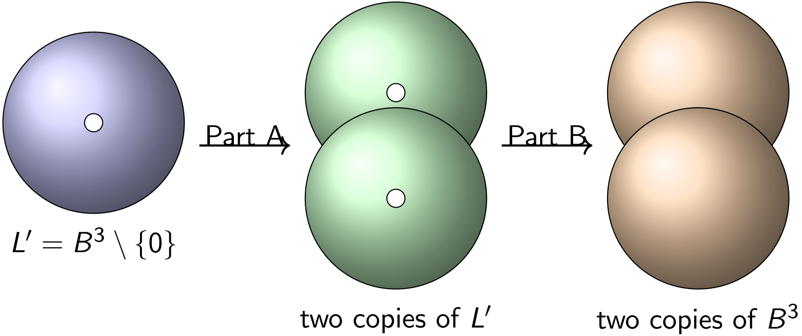
$$(1, 0, 0)^{\mathsf{T}} = w(1, 0, 0)^{\mathsf{T}} = \frac{1}{3^{\,n}} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix} \quad (3 \nmid b).$$

Equality forces $3^{\,n} = 1 \Rightarrow n = 0$, contradicting $w \neq e$. Hence no non-trivial reduced word is the identity and $G \cong F_2$. ∎

Overview of Proof

Transitive Proof (Robinson 2015):

1. Part A: Show that the punctured ball $L' = B^3 \setminus \{0\}$ (centre removed) can be duplicated with finitely many pieces.
2. Part B: Prove that L' is itself *equidecomposable* with the full ball B^3.
   $\Longrightarrow$ the original ball duplicates too.



Part A: Step 1 – Orbits and a Choice Set

Let $L = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ be the unit ball and $L' = L \setminus \{(0, 0, 0)\}$ its punctured version.

Definition (Orbit)

For the rotation group $G = \langle A, B \rangle$ acting on L', two points $p, q \in L'$ are in the same *orbit* if $q = \rho(p)$ for some $\rho \in G$.

By the **Axiom of Choice** pick one point from *every* orbit. Call the resulting set of representatives M.

- Every point of L' is some rotation of a unique element of M.
- Symbolically: $L' = \bigcup_{\rho \in G} \rho M$.

Part A: Step 2 – Fixed Axes and the Set D

Points lying on any rotation axis could be reached by more than one element of G, breaking uniqueness.

Definition

Let $D \subset L'$ be the set of points fixed by *some* non-trivial element of G.

D is small:

$$D \ = \ \text{countably many lines} \quad \Longrightarrow \quad \text{Lebesgue measure}(D) = 0.$$

We first partition $L' \setminus D$ (almost the whole ball) and tackle D later.

Part A: Step 3 – A Four-Piece Partition of $L' \setminus D$

Define

$$X = \bigcup_{i=1}^{\infty} A^{-i}M \quad \big(\text{all points reached by repeated } A^{-1}\big).$$

$$\begin{aligned} P_1 &= S(A)M \cup M \cup X, \\ P_2 &= S(A^{-1})M \setminus X, \\ P_3 &= S(B)M, \\ P_4 &= S(B^{-1})M, \end{aligned}$$

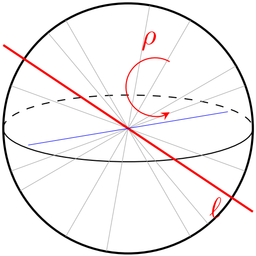
where $S(A)$ means "rotations whose reduced word starts with A", etc.

Using the identities $AP_2 = P_2 \cup P_3 \cup P_4$ and $BP_4 = P_1 \cup P_2 \cup P_4$, we obtain two disjoint unions, each equal to $L' \setminus D$:

Part B: Step 1 - Fixed Points Equidecomposability

Theorem

Let D be the countable union of rotation axes in $G = \langle A, B \rangle$. Then $L' \setminus D$ and L' are equidecomposable.



- Gray lines = the countable set D of rotation axes.
- Pick a red line ℓ that avoids all gray axes.
- Rotate by ρ (irrational angle) about ℓ: copies $\rho^n(D)$ never intersect each other.
- The disjoint union $E = D \cup \rho(D) \cup \rho^2(D) \cup \cdots$ lets us swap E with $\rho(E)$,
- Both $L' \setminus E$ and $\rho(E) \cup (L' \setminus E)$ equal L'. Hence $L' = (L' \setminus E) \sqcup E$ is equidecomposable with $(L' \setminus E) \sqcup \rho(E) = L' \setminus D$. ■

Part B: Step 2 - Circle Without a Point

Lemma

A circle S^1 is equidecomposable with $S^1 \setminus \{p\}$ (remove one point).

Proof.

WLOG use the unit circle and remove $p = (1, 0)$. Let A be the set $\{(\cos n, \sin n) : n \in \mathbb{N}\}$. The angle 1 rad is an irrational multiple of π, so the points of A are distinct and countably infinite. Rotate S^1 by -1 radian to get A'. This sends p back onto the circle, but every point of A moves to a new location still in $S^1 \setminus \{p\}$. Hence

$$S^1 = A \cup B, \quad S^1 \setminus \{p\} = A' \cup B,$$

where $B = S^1 \setminus A$. The two partitions use the same finite set of pieces, proving equidecomposability. ∎

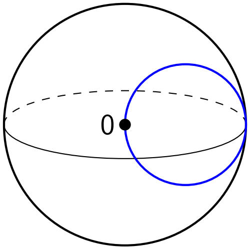
Application to the ball

Part B: Step 3 - Center Equidecomposability

Theorem

A unit ball with its centre removed is equidecomposable with the full ball.

Proof. Draw any circle through the centre that lies completely inside the ball. The circle is equidecomposable with itself minus that point ("circle–minus–a–point lemma"). Replace the missing centre by the circle's pieces; the rest of the ball stays unchanged, so the two balls are equidecomposable. ■



Putting It All Together

- **Duplication.** Four AC–constructed pieces duplicate the punctured ball $L' \setminus D$.
- **Axes are negligible.** The countable union D of rotation axes can be "spiralled" away, so $L' \setminus D \sim L'$.
- **Centre absorbed.** A circle through the centre is equidecomposable with the same circle minus that point, hence $L' \sim B^3$.

$$\boxed{B^3} \sim \boxed{L'} \sim \boxed{(L' \setminus D) \sqcup (L' \setminus D)} \implies B^3 \text{ duplicates.}$$

Does this work in 1 D or 2 D?
The construction hinges on a *free group on two generators* sitting inside $\mathrm{SO}(3)$; $\mathrm{SO}(2)$ and $\mathrm{SO}(1)$ are abelian, so no such free subgroup—and therefore no Banach–Tarski paradox—exists in the plane or on the line.

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References & Resources

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