Banach-Tarski Paradox

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Abstract

This expository paper will delve into basic ZFC set theory, elementary measure theory, and group actions, in order to present a clear proof to how a solid three-dimensional ball can be "duplicated" via the Banach–Tarski Paradox. In particular, we trace how the Axiom of Choice enables paradoxical decompositions, leading to results that defy classical intuition about volume and congruence. Our goal is to demystify what makes Banach–Tarski possible, showcase its far-reaching implications, and provide readers with a gateway to one of mathematics' most baffling and beautiful results.

1 Introduction

The Banach–Tarski Paradox, first announced by Stefan Banach and Alfred Tarski in 1924 [BT24], proclaims that a solid unit ball in $\mathbb{R}^3$ can be partitioned into finitely many disjoint pieces and, by rigid motions alone, reassembled into two solid balls congruent to the original. This seems to contradict any notion of volume preservation, yet that intuition relies on the pieces being Lebesgue-measurable. The paradox instead employs non-measurable subsets, for which the classical idea of volume simply does not apply. The theorem's origins reach back nearly two decades. Giuseppe Vitali's 1905 construction of a non-measurable subset of $\mathbb{R}$ [Vit05] provided the first spark.

In 1914 Felix Hausdorff extended the idea to the sphere S^2, partitioning it into four congruent parts plus a countable remainder[Hau14]. Banach's 1923 paper on paradoxical decompositions, followed by the joint Banach–Tarski publication a year later, transferred these insights to the solid ball. All such results hinge on Zermelo's Axiom of Choice (AC), which asserts that the Cartesian product of a family of non-empty sets is non-empty. AC ensures the existence of the non-measurable sets used in every paradoxical decomposition and is often viewed as fundamentally intertwined with the Banach–Tarski phenomenon.

The argument presented here will be similar to the one proposed by Hendrickson. We will construct a paradoxical decomposition of the free group on two generators, showing that the group can be duplicated using only its own operation. We then embed a copy of this free group into the rotation group

SO(3); this allows us to duplicate almost every point of the unit ball via rotations alone. Finally, we adapt the construction to cover the remaining points, producing a full paradoxical decomposition of the solid ball.

However, before we prove the Banach-Tarski Paradox, we must lay the foundation of our proof by first proving the Axiom of Choice.

2 Axiom of Choice

Let's take a look at a widely popular logic problem many have seen before.

2.1 Finite Ten-Hat Variant

Ten prisoners each receive a hat, either red or blue. Everyone sees the other nine hats but not his own, and all must shout their guesses *simultaneously.* No information can pass once the hats are placed. The challenge, known as the "ten-hat variant without hearing," asks for a plan that always frees the largest possible number of prisoners.

Proof. The famous solution to this problem is as follows.

1. We first divide them into two arbitrary teams of five. Call them *Team Even* and *Team Odd.*
2. Let *Team Even* pretend that the total number of red hats is *even*; *Team Odd* pretends it is *odd.*
3. Then each prisoner counts the red hats they *see.*

$$\text{If you are on Team Even:} \quad \begin{cases} \text{say } \textsf{red} & \text{if the count is odd,} \\ \text{say } \textsf{blue} & \text{if the count is even;} \end{cases}$$

$$\text{If you are on Team Odd:} \quad \begin{cases} \text{say } \textsf{blue} & \text{if the count is odd,} \\ \text{say } \textsf{red} & \text{if the count is even.} \end{cases}$$

Because exactly one parity assumption matches reality. Every member of the correct team has (total parity) = (seen parity) $\oplus$ (own hat), so he deduces his colour with certainty; all five survive. Naturally, the other team will guess wrong completely and thus will lead to a total number of 5 correct guesses. By using expected value assuming a random probability of guessing the jailer's hat, we also get a maximal value of 5 correct guesses and hence know our solution is optimal.

■

Let's now add a slight twist to the problem. What happens if we have a countably infinite number of prisoners, what's the maximum number of correct guesses in total?

One would think that the answer is simply infinite, as it's likely impossible to guess correctly a "consistently" in order to achieve an infinite number of correct guesses.

However, with the Axiom of Choice, we can actually come to the baffling conclusion that the prisoners can guess in such a way that there will be a finite number of incorrect guesses.

2.2 Axiom of Choice Concept

Imagine an endless supply of boxes, each containing at least one marble. The Axiom of Choice (AC) states that somehow one can pick exactly one marble from *every* box, even when the collection of boxes is so large or so unstructured that no explicit rule for choosing is available. In daily life we rarely face such "rule–less" situations: if the set of boxes is countable we may index them $B_1, B_2, \ldots$ and define "always take the leftmost marble." Problems arise only when the family of sets is so vast (say, uncountably many boxes, each with no distinguished elements) that no uniform picking rule exists inside ordinary Zermelo–Fraenkel (ZF) set theory; AC can be used to assert a choice function anyway.

Theorem 2.1 (**Axiom of Choice**). *Let $\{X_i\}_{i\in I}$ be a family of non-empty sets. Then there exists a* choice function

$$f\colon I \longrightarrow \bigcup_{i\in I} X_i$$

such that $f(i) \in X_i$ for every $i \in I$.

Because this is an axiom, we are unable to prove it within ZF set theory. However, we can show it is equivalent to other well-known theorems or lemmas. To do this, we shall introduce some new ideas and finally state our claim.

2.2.1 Background

Definition 2.2 (Indexed family). *A collection $\{X_i\}_{i\in I}$ obtained by mapping each index i in some set I to a non-empty set X_i.*

Definition 2.3 (Cartesian product). $\prod_{i\in I} X_i := \{f : I \to \bigcup_i X_i \mid f(i) \in X_i\}$. *A choice function (see AC) is simply an element of this product.*

Definition 2.4 (Partial order (poset)). *A relation $\le$ that is reflexive ($x \le x$), antisymmetric ($x \le y$ and $y \le x$ force $x = y$), and transitive ($x \le y \le z$ implies $x \le z$). A set equipped with such a relation is called a* poset.

Definition 2.5 (Chain). *A subset of a* poset *in which any two elements are comparable: for all x, y in the* chain, *$x \le y$ or $y \le x$.*

Definition 2.6 (Upper bound & maximal element)**.** *An* upper bound u *for a set* C *satisfies* $c \leq u$ *for all* $c \in C$*. A* maximal element m *has no strict larger neighbour (*$m \leq x$ *implies* $x = m$*).*

Definition 2.7 (Ordinal)**.** *An abstract label for the position of an element in a* well-order*, extending the counting numbers into the transfinite.*

2.2.2 AC Equivalence

From the background, we find define the following Lemma and Principle which are widely accepted by mathematicians.

Theorem 2.8 (Zorn's Lemma[Zor35])**.** *Let* $(P, \leq)$ *be a non-empty partially ordered set such that* every *chain in* P *has an upper bound in* P*. Then* P *contains a maximal element; that is, some* $m \in P$ *satisfies* $m \leq x \implies x = m$ *for all* $x \in P$*.*

Theorem 2.9 (Well-Ordering Theorem)**.** *For any set* S *there exists a binary relation* $\preccurlyeq$ *on* S *such that* $(S, \preccurlyeq)$ *is a well-ordered set; that is,* $\preccurlyeq$ *is a total order and every non-empty subset of* S *has a least element under* $\preccurlyeq$*.*

Proposition 2.10. *The Axiom of Choice, Well-Ordering Princinple, and Zorn's Lemma are logically equivalent statements.*

Proof. We shall break the proof into different cases to show equivalence.

(i) *Axiom of Choice Implies Well–Ordering Theorem:* Fix an arbitrary set S. For every non–empty subset $T \subseteq S$ use AC to choose a *designated* element $c(T) \in T$. Define a relation $\preceq$ by recursion:

   (a) At stage 0 set $a_0 := c(S)$.

   (b) Given the set of *already chosen* elements $A_\alpha = \{a_\beta : \beta < \alpha\}$ at ordinal stage α, if $S \backslash A_\alpha \neq \varnothing$ let $a_\alpha := c(S \backslash A_\alpha)$; otherwise terminate.

   Because exactly one element is removed at each stage, the process stops after $|S|$ stages or fewer. Declare $a_\beta \preceq a_\gamma$ precisely when $\beta \leq \gamma$. By construction, every non–empty subset of S contains the least–indexed element a_β, hence $(S, \preceq)$ is a well–ordered copy of S.

(ii) *Well–Ordering Theorem Implies Zorn's Lemma:* Assume every set can be well–ordered. Let $(P, \leq)$ be any poset where each chain has an upper bound. Choose a well–order $\triangleleft$ of P. Build, by transfinite recursion on $\triangleleft$-rank, a new chain $C \subseteq P$:

   > Include an element $p \in P$ whenever it is $\triangleleft$-least among those $\geq$ all earlier members of C.

Because every chain in P has an upper bound, the union $u := \bigvee C$ exists and dominates C. Suppose u is *not* maximal; then some $q > u$ exists. But $q \notin C$ (otherwise $q \le u$), so by the construction rule q could have been added to C, contradicting its definition. Hence u is maximal, proving Zorn's Lemma.

(iii) *Zorn's Lemma Implies Axiom of Choice:* Given a family $\{X_i\}_{i \in I}$ of non–empty sets, consider the poset $(\mathcal{P}, \subseteq)$ of *partial* choice functions, i.e.

$$\mathcal{P} = \big\{\, f\colon J \to \bigcup X_i \mid J \subseteq I,\ f(j) \in X_j\ (\forall j \in J) \big\}.$$

Any chain of partial functions has an upper bound obtained by taking the union of their graphs, so Zorn's conditions hold. A maximal element $f_{\max}$ therefore exists. If its domain were a proper subset of I, choose $i_0 \notin \operatorname{dom} f_{\max}$ and *extend* by setting $\tilde{f} := f_{\max} \cup \{(i_0, x_{i_0})\}$ with some $x_{i_0} \in X_{i_0}$; this contradicts maximality. Thus $\operatorname{dom} f_{\max} = I$ and $f_{\max}$ is a full choice function. AC follows. ■

We now have the proper tools required to derive a baffling answer to our modified hat problem.

2.3 Countably Infinite Hat Variant

Imagine prisoners indexed by all integers $\ldots, -2, -1, 0, 1, 2, \ldots$; prisoner k sees every hat with index $> k$. They speak *simultaneously.* Can a plan guarantee only finitely many errors?

Proof. A color sequence is a point in $\{0,1\}^{\mathbb{Z}}$ (0 = blue, 1 = red). Declare

$$x \sim y \quad \Longleftrightarrow \quad x \text{ and } y \text{ differ at only finitely many indices.}$$

Each $\sim$-class is huge and has no natural "first" element.
For example:

$$\begin{aligned} r &= \ldots 0\,0\,0\,0\,0\,0\,0\,0 \ldots \\ r' &= \ldots 0\,0\,0\,0\,\mathbf{1}\,0\,0\,0 \ldots && \text{(flip at index 0)} \\ r'' &= \ldots 0\,0\,\mathbf{1}\,0\,0\,0\,0\,0 \ldots && \text{(flips at indices } -2, 3) \end{aligned}$$

The three rows belong to the *same* class because only finitely many entries differ. A *different* class might start with an alternating pattern:

$$\begin{aligned} a &= \ldots 0\,1\,0\,1\,0\,1\,0\,1 \ldots \\ a' &= \ldots 0\,1\,0\,1\,0\,1\,0\,\mathbf{0}\,1 \ldots && \text{(finite tweak at one spot)} \end{aligned}$$

Again, $a \sim a'$ since they differ finitely often, but $a \nsim r$ because the disagreement set is infinite.

Solving the puzzle would be easy if we could pick one *representative sequence* from every $\sim$-class: prisoner k would

1. Observe the tail of hats with index $> k$,
2. Find the unique class containing that tail,
3. Speak the bit in position k of the chosen representative.

Because each representative differs from the true sequence at only finitely many indices, at most finitely many prisoners guess wrong. Selecting those representatives requires a global choice function—precisely what AC guarantees.

Assuming AC, fix a representative for each class once and for all. Each prisoner applies the three-step rule above. Only the finitely many indices where the actual sequence and its representative disagree are guessed incorrectly, so $\boxed{\text{all but finitely many prisoners survive}}$. ■

In summary, elementary parity saves exactly five lives in the finite puzzle, while the infinite puzzle achieves near-perfect survival only under the Axiom of Choice.

3 Free Groups

The algebraic engine of what drives the paradoxical result of decomposing a three-dimensional ball into two copies relies on the paradoxical nature of what is known as free groups.

3.1 The Paradoxical Nature

We shall first introduce what free groups are.

Definition 3.1 (Free Monoid)**.** *Let A be a non-empty set, called the* alphabet*. The* free monoid *A^* is the collection of all finite strings (words) in the letters of A, including the empty word ε, with concatenation as the operation.*

Definition 3.2 (Free Group)**.** *To obtain the* free group *on A, denoted $F(A)$, we adjoin for every letter $a \in A$ a formal inverse a^{-1} and then declare the relations $aa^{-1} = \varepsilon = a^{-1}a$. The resulting quotient of the free monoid on $A \cup A^{-1}$ is a group; if $|A| = n$ we also write F_n.*

The adjective "free" comes from the following universal property: any map ϕ from A into an arbitrary group G extends to a unique homomorphism $F(A) \to G$. Intuitively, $F(A)$ contains *all* words one can spell with A, subject only to the cancellations aa^{-1} and $a^{-1}a$ and nothing else. This minimalism is what grants the group its enormous flexibility.

Definition 3.3 (Reduced word)**.** *A word in the symbols $a^{\pm 1}$ $(a \in A)$ is* reduced *if no letter stands immediately next to its inverse. A single rewriting rule,*

$$xx^{-1} \longrightarrow \varepsilon, \qquad x^{-1}x \longrightarrow \varepsilon,$$

successively applied, removes every such pair and produces a reduced word.

Proposition 3.4 (Normal form)**.** *Every element of $F(A)$ can be represented by a* unique *reduced word. In particular, two reduced words are equal in $F(A)$ iff they are identical as strings.*

Proof. Repeatedly canceling adjacent xx^{-1} or $x^{-1}x$ terminates, because each step shortens the word; the result is reduced and represents the same group element.

Conversely, if two distinct reduced words could represent the same element, their juxtaposition would reduce to ε while containing no canceling neighbours—a contradiction. ■

The smallest non-trivial free group is F_1, which is isomorphic to $\mathbb{Z}$ by sending the generator to 1; each reduced word is a^k or a^{-k}, encoding an integer k. The group F_2 on two generators a, b is already vastly richer: it is non-abelian, infinitely generated by its reduced words, and—most crucially for us—admits a paradoxical decomposition.

Before describing that decomposition we need a few lines of vocabulary about how groups may act on sets.

Definition 3.5 (Group action, free action)**.** *A* group action *of G on a set X is a map $G \times X \to X$, $(g, x) \mapsto gx$ satisfying $1x = x$ and $g(hx) = (gh)x$. The action is* free *if $gx = x$ implies $g = 1$.*

Definition 3.6 (Paradoxical subset)**.** *Let G act on X. A subset $Z \subseteq X$ is* paradoxical *if there are pairwise disjoint sets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq Z$ and group elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that*

$$Z \;=\; \bigsqcup_{i=1}^{n} g_i A_i \;=\; \bigsqcup_{j=1}^{m} h_j B_j.$$

In words, two disjoint selections of pieces—moved around by group elements—each cover the whole of Z. If some *action of G contains a paradoxical set, we call G itself paradoxical.*

Armed with these notions we return to $F_2 = \langle a, b \rangle$. Partition its elements into five blocks:

$$F_2 \;=\; \{\varepsilon\} \sqcup W(a) \sqcup W(a^{-1}) \sqcup W(b) \sqcup W(b^{-1}),$$

where, for instance, $W(a)$ is the set of all reduced words beginning with a. Because every reduced word starts either with a, a^{-1}, b, or b^{-1} (or is empty), the union is indeed F_2.

Next observe that left-multiplication by a strips the initial a^{-1} from any word in $W(a^{-1})$, producing a word that starts with *not* a; more precisely,

$$aW(a^{-1}) \;=\; \{\text{reduced words } \textit{not} \text{ starting with } a\}.$$

Similarly $bW(b^{-1})$ is the set of words *not* starting with b. Consequently

$$F_2 \;=\; W(a) \sqcup aW(a^{-1}) \quad \text{and} \quad F_2 \;=\; W(b) \sqcup bW(b^{-1}).$$

We have manufactured two disjoint unions of subsets whose images under the shifts a and b each reassemble into the *entire* group. Unfortunately, this does cause problems with the identity element, e, which we will have to tackle when mapping this into $\mathbb{R}^3$.

Theorem 3.7 (Paradoxical decomposition of F_2)**.** *The five pieces* $\{\varepsilon\}, W(a), W(a^{-1}), W(b), W(b^{-1})$ *create a paradoxical decomposition of* F_2*.*

Proof. Both equalities displayed above are unions of *pairwise disjoint* sets, because left-multiplication is a bijection. Thus F_2 contains two rearrangements of itself built from fewer than all of its parts, certifying paradoxicality. ■

This paradoxical logic in F_2 will serve as the crux of our proof of the Banach-Tarski Paradox going forward.

3.2 Embedding F_2 in $SO(3)$

Our algebraic five-piece paradox for the free group $F_2 = \langle a, b \rangle$ is still only a concept until we can translate it into physical motions of space. The natural setting for rigid motions that preserve the origin is the *special orthogonal group*

$$SO(3) = \left\{ A \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) \;\middle|\; A^T A = I,\ \det A = 1 \right\}.$$

Elements of $SO(3)$ are (proper) rotations: each fixes a unique axis through the origin and acts as an ordinary planar rotation on the perpendicular plane. To embed F_2 in this group we must choose two rotations A and B whose non-trivial combinations never collapse to the identity.

Set

$$\theta = \arccos\left(\tfrac{1}{3}\right) \quad \Longrightarrow \quad c = \cos\theta = \tfrac{1}{3}, \quad s = \sin\theta = \tfrac{2\sqrt{2}}{3}.$$

With the usual right-handed coordinate axes, we can denote the transformation matrices of A and B as

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -2\sqrt{2} \\ 0 & 2\sqrt{2} & 1 \end{pmatrix} \quad \text{(rotate by } \theta \text{ around the } x\text{-axis)},$$

$$B = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & -2\sqrt{2} & 0 \\ 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{(rotate by } \theta \text{ around the } z\text{-axis)}.$$

One checks directly that $A^T A = B^T B = I$ and $\det A = \det B = 1$, so $A, B \in SO(3)$.

At first sight these matrices look innocent, yet they already encode a vast amount of information. The key observation is that both A and B send the "lattice"

$$\mathcal{S} = \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \;\middle|\; x_1, x_3 \in \mathbb{Z},\ x_2 \in \sqrt{2}\,\mathbb{Z} \right\}$$

to itself: multiplying by $c = \frac{1}{3}$ or $s = \frac{2\sqrt{2}}{3}$ either keeps an integer an integer or turns $\sqrt{2}$ into $\pm\sqrt{2}$ or $2\sqrt{2}$. That simple fact lets us *track* the effect of any word in A, B on the coordinate values of a point.

Proposition 3.8. *Let w be a reduced word of length n in the generators $A^{\pm 1}$, $B^{\pm 1}$ of the free subgroup $G = \langle A, B\rangle \subset SO(3)$. Acting on the vector $(1,0,0)^T$ one obtains*

$$w \cdot (1,0,0)^T \;=\; \frac{1}{3^{\,n}} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix}, \qquad \textit{with } a, b, c \in \mathbb{Z} \textit{ and } 3 \nmid b.$$

Proof. We proceed with simple induction on the addition of each additional rotation. For the base case, we let $n = 1$ and do casework on if $w = A, B, A^{-1}, B^{-1}$. With simple algebraic bashing, we can prove that the base case works.

Similarly, for the inductive step, we do casework on whether the additional rotation is A, B, A^{-1}, or B^{-1}. The details of the algebra will be omitted, but can be found in [Wan24]. ■

As an example of Prop 3.8's validity, consider the case when $w = AB^{-1}AB$ and $v_0 = (1,0,0)$.

k	word prefix w_k	$3^{\,k} v_k$
0	ε	$(1,0,0)$
1	A	$(3,0,0)$
2	AB^{-1}	$(1,-2\sqrt{2},0)$
3	$AB^{-1}A$	$(3,-2\sqrt{2},2)$
4	$AB^{-1}AB$	$(1,-2\sqrt{2},2)$

At $k = 4$ the vector is $3^{-4}(1,-2\sqrt{2},2)$; here $b_4 = -2 \not\equiv 0 \pmod 3$.

Proposition 3.9 (Free Rotation Lemma)**.** *The subgroup $\langle A, B\rangle \subset SO(3)$ is isomorphic to the free group F_2; that is, no non-trivial reduced word in $A^{\pm 1}, B^{\pm 1}$ equals the identity.*

Proof. Suppose some reduced, non-empty word w in $A^{\pm 1}, B^{\pm 1}$ were the identity. Acting on v_0 would return v_0, yet by Proposition 3.8

$$w(v_0) = 3^{-n}(a_n, b_n\sqrt{2}, c_n)^T = (1,0,0)^T,$$

so b_n would have to be 0—contradicting $3 \nmid b_n$. Therefore *no non-trivial reduced word collapses.* ■

Now, we have gained an explicit isomorphism:

$$\langle A, B\rangle \;\cong\; F_2 \quad \text{inside } SO(3).$$

4 Measurability

Because the paradox is seemingly able to double the volume of a sphere, it begs the question of what volume is. The aim of this section is to not only define what volume is, but also show a similar consequence of the Axiom of Choice relating to "volume" in $\mathbb{R}$.

4.1 Background

It's important to follow rigorous definitions when proving such a paradox.

Definition 4.1 (Equidecomposable sets). *Let $\mathcal{G}$ be a group of rigid motions of $\mathbb{R}^3$ (we take $\mathcal{G} = \mathrm{Iso}(\mathbb{R}^3)$, all rotations and translations). Two bounded sets $E, F \subset \mathbb{R}^3$ are* equidecomposable *if there exist finitely many pairwise disjoint pieces $E_1, \dots, E_k \subset E$ and motions $g_1, \dots, g_k \in \mathcal{G}$ such that*

$$E = \bigsqcup_{i=1}^{k} E_i, \qquad F = \bigsqcup_{i=1}^{k} g_i E_i.$$

Proposition 4.2 (Transitivity of Equidecomposability). *For bounded sets $E, F, G \subset \mathbb{R}^3$ let $\mathcal{G} = \mathrm{Iso}(\mathbb{R}^3)$. If $E \sim_{\mathcal{G}} F$ and $F \sim_{\mathcal{G}} G$, then $E \sim_{\mathcal{G}} G$.*

Proof. Assume $E = \bigsqcup_{i=1}^{k} E_i$ and, by applying motions $g_1, \dots, g_k \in \mathcal{G}$, the pieces form $F = \bigsqcup_{i=1}^{k} g_i E_i$. Next, write $F = \bigsqcup_{j=1}^{\ell} F_j$ and move those pieces with $h_1, \dots, h_\ell \in \mathcal{G}$ to obtain $G = \bigsqcup_{j=1}^{\ell} h_j F_j$.

Trace each original piece E_i through the two stages: first g_i takes it into F; afterwards some h_j (the one assigned to the F_j containing $g_i E_i$) moves it into G. The composition $h_j \circ g_i$ is still a rigid motion, and only finitely many such compositions arise.

Re-declare the pieces to be $E_1, \dots, E_k$ and use the composed motions $h_j\, g_i$. These finitely many pieces now build G directly, so E and G are equidecomposable. ■

The Banach–Tarski theorem asserts that a solid ball B^3 is equidecomposable with *two disjoint* copies of itself.

To be more rigorous, for the rest of our paper, we will be using Lebesgue measure rather than volume as it is applicable in all subsets of $\mathbb{R}^n$.

Definition 4.3 (Lebesgue measure on $\mathbb{R}^3$[Hal05]). *For every bounded set $E \subset \mathbb{R}^3$ the value $\lambda(E)$ is uniquely determined by the three properties*

(i) Translation invariance: $\lambda(E + x) = \lambda(E)$ for all vectors $x \in \mathbb{R}^3$.

(ii) Countable additivity: if $\{E_n\}_{n \geq 1}$ are pairwise disjoint, then

$$\lambda\Big(\bigsqcup_{n \geq 1} E_n\Big) = \sum_{n \geq 1} \lambda(E_n).$$

(iii) *Normalisation on boxes: for every axis-aligned box* $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$,

$$\lambda\big([a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]\big) = (b_1 - a_1)(b_2 - a_2)(b_3 - a_3).$$

Observe that the Banach-Tarski Paradox relies on splitting the sphere into pieces that aren't Lebesgue-measurable, for otherwise $\lambda(B^3)$ would equal $2\,\lambda(B^3)$. A concept tangent to unmeasurable sets in $\mathbb{R}^3$ are the Vitali Sets, which are unmeasurable sets in $\mathbb{R}$. In the next section, we will go over the construction of such a set in order to grasp the nature of unmeasurable sets in further detail.

4.2 Vitali Set

The Lebesgue measure λ on $\mathbb{R}$ is translation- invariant and countably additive on its domain of measurable sets. Vitali's construction shows that these two properties cannot be extended to *every* subset of an interval when the Axiom of Choice (AC) is available.

Definition 4.4. *A relation* $\sim$ *on a set* X *is an* equivalence relation *if it is reflexive* ($x \sim x$)*, symmetric* ($x \sim y \Rightarrow y \sim x$) *and transitive* ($x \sim y \wedge y \sim z \Rightarrow x \sim z$)*.*

Proposition 4.5. *The relation*

$$x \sim y \iff x - y \in \mathbb{Q}.$$

on the interval $[0, 1)$ *is an equivalence relation.*

Proof. We shall go through each property of Definition 4.4.

(i) *Reflexive:* For every $x \in [0, 1)$ one has $x - x = 0 \in \mathbb{Q}$, so $x \sim x$.

(ii) *Symmetric:* If $x \sim y$ then $x - y \in \mathbb{Q}$; taking negatives gives $y - x = -(x - y) \in \mathbb{Q}$, hence $y \sim x$.

(iii) *Transitive:* Assume $x \sim y$ and $y \sim z$. Then $x - y \in \mathbb{Q}$ and $y - z \in \mathbb{Q}$. Adding, $x - z = (x - y) + (y - z) \in \mathbb{Q}$, so $x \sim z$. ∎

Definition 4.6. *For* $x \in [0, 1)$ *write* $[x]$ *for its equivalence class* $[x] = \{y \in [0, 1) : x \sim y\}$*. The set of all classes is the* quotient $[0, 1)/\sim$*.*

For example, take the number $x = 0.2$ in $[0, 1)$.

$$[0.2] \;=\; \big\{\, 0.2 + q \pmod 1 \;:\; q \text{ is rational}\big\}.$$

A few concrete members are

$$0.2, \quad 0.2 + \tfrac{1}{3} = 0.533\overline{3}, \quad 0.2 + \tfrac{1}{2} = 0.7, \quad 0.2 - \tfrac{4}{5} = -0.6 \equiv 0.4 \pmod 1.$$

All of these differ from 0.2 by a rational number, so they belong to the same class. By contrast 0.25 is *not* in [0.2] because $0.25 - 0.2 = 0.05$ is irrational in lowest terms, so the two numbers fall into different classes.

Similarly, starting with $y = \sqrt{2} - 1 \approx 0.4142$ gives another class

$$[\sqrt{2} - 1] = \{\sqrt{2} - 1 + q \pmod 1 : q \in \mathbb{Q}\},$$

and no number from [0.2] can be in $[\sqrt{2}-1]$ because any two numbers chosen one from each set differ by an irrational amount.

Proposition 4.7 (Choice set)**.** *There exists a set $V \subset [0, 1)$ that contains exactly one element of every equivalence class $[x]$.*

Proof. Think of the interval $[0, 1)$ as being divided into disjoint groups, each group consisting of all numbers that differ by a rational. Call these groups the *classes.*

Apply the Axiom of Choice to the family of classes. The result is a "selection rule" that, for each class, designates one distinguished number in that class. Collect all the selected numbers into a single set and call it V. By construction V meets every class, and it meets each class only once, because the rule never chooses two different numbers from the same set. ■

Lemma 4.8. *For every rational q with $0 \leq q < 1$ set $V_q = V + q \pmod 1$. Then the family $\{V_q\}$ has two key features:*

(i) No overlaps: a point of $[0, 1)$ belongs to at most *one translate V_q*

(ii) Full coverage: every point of $[0, 1)$ belongs to at least *one translate V_q.*

Proof. Let us prove each property individually.

(i) *Why they cannot overlap:* Suppose that some number $x \in [0, 1)$ sat in two different translates, say V_{q_1} and V_{q_2} with $q_1 \neq q_2$. Then $x = v_1 + q_1 = v_2 + q_2$ (working "mod 1" if we leave the interval), where v_1 and v_2 are members of the choice set V. Rearranging gives $v_1 - v_2 = q_2 - q_1$. The right–hand side is rational, so v_1 and v_2 belong to the *same* equivalence class. But V was constructed to contain only one representative from each class, forcing $v_1 = v_2$ and hence $q_1 = q_2$. Our initial assumption that the two shifts were different fails; therefore no overlap is possible.

(ii) *Why they cover $[0, 1)$:* Take any number x in the interval. Look at the class to which x belongs and pick its unique representative v inside V. The difference $x - v$ is rational, and after reducing it modulo 1 we obtain a rational q with $0 \leq q < 1$. By construction $x = v + q \pmod{1}$, so x lies in the translate V_q. Thus every point of $[0, 1)$ falls into some V_q.

With no overlaps and no gaps, the family $\{V_q\}$ is both disjoint and has full coverage, as claimed. ■

Theorem 4.9 (Vitali)**.** *The choice set V described in Proposition 4.7 is* not *Lebesgue-measurable.*

Proof. For every rational $q \in [0, 1)$ the translate $V_q = V + q \pmod 1$ is just V slid along the line. Because Lebesgue measure is unchanged by such slides,

$$\lambda(V_q) = \lambda(V) \quad \text{for every } q.$$

Lemma 4.8 tells us two facts at once: no two different translates overlap, and taken together they fill the whole interval $[0, 1)$. With these facts in mind, consider the only three numerical possibilities for $\lambda(V)$.

1. $\lambda(V) = 0$**.** Adding up the measures of all the disjoint translates would give $\lambda([0, 1)) = 0 + 0 + 0 + \cdots = 0$, contradicting the known value $\lambda([0, 1)) = 1$.

2. $0 < \lambda(V) < \infty$**.** Every translate has the same positive length, so summing $\lambda(V) + \lambda(V) + \lambda(V) + \ldots$ over infinitely many rationals would blow past every finite bound. That would force $\lambda([0, 1))$ to be infinite—again contradicting $\lambda([0, 1)) = 1$.

3. $\lambda(V) = \infty$**.** Impossible, because $V \subset [0, 1)$ and the whole interval has length 1.

Every case leads to a contradiction, so the assumption that V is measurable must be false. ■

Thus, Vitali's example reveals a fundamental limit of measurability under the Axiom of Choice. His result parallels the Banach-Tarski Paradox, only that the latter shows unmeasurability in $\mathbb{R}^3$ instead of $\mathbb{R}$. Using the content so far, we will now attempt to prove the Banach-Tarski Paradox.

5 Proof of Banach-Tarski Paradox

We will follow [Rob14]'s transitive proof, which breaks the dramatic duplication of a sphere into more basic steps.

(i) Remove a set of points from B^3 to ultimately form L''.

(ii) Prove that $L'' \sim L'' \sqcup L''$.

(ii) Prove that $L'' \sim B^3$.

By Proposition 4.2, this is logically equivalent to proving $B^3 \sim B^3 \sqcup B^3$.

5.1 Removing Points

Let

$$L = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le 1\}, \qquad L' = L \setminus \{(0,0,0)\}.$$

Throughout $G = \langle A, B\rangle \subset SO(3)$ is the free subgroup already constructed; its elements act on L' by matrix multiplication.

Definition 5.1 (Orbit)**.** *For $p, q \in L'$ write $p \sim q$ if $q = g\,p$ for some $g \in G$. The equivalence class $G\,p = \{g\,p \mid g \in G\}$ is the* orbit *of p.*

Because $\sim$ is an equivalence relation, its easy to see:

$$L' = \bigsqcup_{O \in L'/\sim} O$$

as a disjoint union of orbits.

Proposition 5.2 (Choice set)**.** *There exists a subset $M \subset L'$ that contains exactly one point of each orbit.*

Proof. Apply the Axiom of Choice to the family $\{\, O \subset L' : O \text{ is an orbit} \,\}$. ∎

Fix one such M for the remainder of the proof. Because the Banach-Tarski Paradox relies on mapping each point of a sphere, we will have to exploit the one-to-one bijection between $m \iff pm$ for a point m and $p \in G$. However, this is not always the case for a subset of points, which we must accurately quantify.

Definition 5.3. *For a point $p \in L'$ the* stabiliser *is*

$$G_p = \{g \in G \mid g\,p = p\}.$$

If $G_p = \{e\}$ we say p has trivial stabiliser*.*

The matrices A and B each fix every point on one line through the origin: A fixes the x-axis, B fixes the z-axis. For example, take $p = (1,0,0)$ on the x-axis. If $m \in M$ and $\rho \in G$ sends m to p then $(A\rho)\,m = p$ as well, since A fixes p. The two words ρ and $A\rho$ give two different first letters, so uniqueness fails at p.

Moreover, any conjugate gAg^{-1} fixes the image of the x-axis under g, and similarly for B. Therefore the set of points with non-trivial stabiliser lies on the countable family of these axis lines.

For instance, let $p = (\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$, which is not on any axis. Write $p = g\,m$. If another pair (h, n) also satisfies $p = h\,n$ then $n = h^{-1}g\,m$. Since m, n belong to the same orbit and M chooses exactly one point per orbit, $m = n$. The stabiliser of p is trivial, so $h^{-1}g = 1$ and $h = g$. Thus point p exists in only one orbit.

Definition 5.4. *Let*

$$D = \{p \in L' \mid G_p \ne \{1\}\}.$$

Lemma 5.5. *D is contained in countably many straight lines; hence $\lambda(D) = 0$.*

Proof. Each non-identity element of G is a rotation about one line through the origin. Because G is countable, there are countably many such lines; a line has Lebesgue measure zero, and a countable union of measure-zero sets still has measure zero. ■

By definition, for every $p \in L' \setminus D$ there is $g \in G$ with $p = g\,m$ for a unique $m \in M$. Injectivity of the map

$$G \times M \longrightarrow L \setminus D', \qquad (g, m) \longmapsto g\,m,$$

holds away from the rotation axes; on an axis one finds distinct g_1, g_2 with $g_1\,p = g_2\,p$.

Proposition 5.6 (Disjoint cover by translates)**.** *The collection $\{\, gM : g \in G \,\}$ is pairwise disjoint and*

$$L' \setminus D = \bigcup_{g \in G} gM.$$

Proof. Surjectivity is immediate from the definition of M. If $g_1M \cap g_2M \neq \varnothing$ with $m_1, m_2 \in M$, then $g_1m_1 = g_2m_2$. Since M contains at most one point per orbit, $m_1 = m_2$ and so $g_1 = g_2$. ■

We now attempt to duplicate $L' \setminus D$ with our free group paradoxical logic.

5.2 Duplicating an Incomplete Sphere

We work inside the full–measure set $L'' = L' \setminus D$; on L'' each point carries a unique label $g\,m$ with $g \in G,\ m \in M$. To group points by the first letter of g introduce

Definition 5.7. *For $\sigma \in \{A^{\pm 1}, B^{\pm 1}\}$ define*

$$S(\sigma) = \big\{ g \in G \bigm| g = \sigma\, w \text{ with } w \text{ a (possibly empty) reduced word} \big\}.$$

The family $\{S(A), S(A^{-1}), S(B), S(B^{-1})\}$ is a disjoint cover of $G \setminus \{e\}$. In the free–group paradox one additional singleton $\{\varepsilon\}$ was needed; here that role is played by an infinite set built from negative powers of A:

$$X \;=\; \bigcup_{i=1}^{\infty} A^{-i} M \quad \text{(points reached from } M \text{ by repeated } A^{-1}\text{)}.$$

Because $A^{-i}M \cap M = \varnothing$ for $i \geq 1$, X is disjoint from M.

Lemma 5.8. *The sets M and X are disjoint and $X \cap A^{-1}M = \varnothing$.*

Proof. Disjointness from M follows by construction. If $x \in X \cap A^{-1}M$ then $x = A^{-i}m_1 = A^{-1}m_2$ for some $i \geq 1$ and $m_1, m_2 \in M$. Freeness of G and uniqueness of labels imply $i = 1$ and $m_1 = m_2$, hence $x = A^{-1}m_1 \notin X$ (contradiction).

For example, let $m_0 \in M$ and consider $p = A^{-2}m_0 \in X \subset P_1$. Left multiplication by A sends p to $A^{-1}m_0 \in P_2$; a second multiplication by A reaches $m_0 \in P_1$. Thus the three consecutive points $\{m_0, A^{-1}m_0, A^{-2}m_0\}$ occupy P_1, P_2, and P_1 respectively, illustrating how the definition of X eliminates what would otherwise be a leftover "fifth piece" analogous to the identity element in the free–group model. ∎

We now split L'' into 4 pieces.

$$P_1 = S(A)M \ \cup \ M \ \cup \ X, \qquad P_2 = S(A^{-1})M \setminus X,$$

$$P_3 = S(B)M, \qquad P_4 = S(B^{-1})M.$$

Lemma 5.9. *The sets P_1, P_2, P_3, P_4 are pairwise disjoint and*

$$L'' \ = \ P_1 \sqcup P_2 \sqcup P_3 \sqcup P_4.$$

Proof. Every $g \neq 1$ lies in exactly one $S(\sigma)$; the label representation $p = g\,m$ therefore places p in exactly one of the four sets. Lemma 5.8 guarantees that inserting M and X into P_1 does not create overlaps with P_2. ∎

Proposition 5.10 (Two disjoint unions each equal L'')**.**

$$A\,P_2 = P_2 \cup P_3 \cup P_4, \qquad B\,P_4 = P_1 \cup P_2 \cup P_4.$$

Proof. Take $p = g\,m \in P_2$ so the reduced word for g begins with A^{-1}. Multiplying by A on the left deletes that first letter; the new word therefore begins with A, B or B^{-1}, placing $A\,p$ into P_2, P_3 or P_4 respectively. The mapping $p \mapsto A\,p$ is injective on P_2, so the three images are disjoint. A similar argument holds for $B\,P_4$. ∎

Theorem 5.11 (Duplication of L'')**.** $L'' = P_1 \cup A\,P_2 = P_3 \cup B\,P_4$.

Proof. Using Lemma 5.9 and Proposition 5.10:

$$P_1 \cup A\,P_2 = P_1 \cup P_2 \cup P_3 \cup P_4 = L'',$$

$$P_3 \cup B\,P_4 = P_3 \cup P_1 \cup P_2 \cup P_4 = L''.$$

The unions are disjoint because all sets involved are disjoint. ∎

With L'' now duplicated by the four sets $\{P_j\}_{j=1}^4$ and their rotated images, we must now prove the equidecomposability between L'' and L.To show this, we shall first prove $L'' \sim L'$ and then $L' \sim B^3$.

5.3 Proving $L'' \sim L'$

Definition 5.12 (Null set)**.** *A subset* $N \subset \mathbb{R}^3$ *is called* null *(or* measure-zero*) if its Lebesgue measure satisfies* $\lambda(N) = 0$*.*

Definition 5.13 (Null Swap)**.** *A* null swap *is a rigid motion that moves a measure-zero set disjointly off itself while leaving its complement unchanged.*

Proposition 5.14. *Because* D *has Lebesgue measure* 0*, we can perform a null swap.*

Proof. Choose a line ℓ through the origin that meets none of the axis lines; such an ℓ exists because only countably many directions are forbidden. Rotate about ℓ through an angle θ whose ratio with π is irrational, and denote this rotation by ρ. Irrationality implies that the sets $\rho^n(D)$ $(n \in \mathbb{Z})$ are pairwise disjoint: the only points fixed by a power of ρ lie on ℓ, which avoids D.

Define

$$E = \bigcup_{n \geq 0} \rho^n(D), \qquad \rho(E) = E \setminus D.$$

Both E and $\rho(E)$ are still null sets, and they are congruent via the single motion ρ. ■

A reader may wonder why the rotation ρ must have an irrational angle. If θ/π were rational, some power of ρ would be the identity, and the corresponding iterate $\rho^n(D)$ would land back on D, defeating the disjointness required for the swap. The irrational choice guarantees that no such overlap occurs.

Take a point p_0 on an axis of A, e.g. $p_0 = (1, 0, 0)$. Under successive applications of ρ the point traces a countable spiral of locations, none of which lie on a rotation axis because ℓ misses every axis. The union of these images is part of the set E. After the swap p_0 is replaced by $\rho(p_0)$, freeing p_0 from any stabiliser without changing the shape of L'.

Theorem 5.15. *The punctured ball without axes,* $L'' = L' \setminus D$*, is equidecomposable with the full punctured ball* L'*.*

Proof. Write

$$L' = E \sqcup (L' \setminus E) \quad \text{and} \quad L' = \rho(E) \sqcup (L' \setminus E).$$

The set $L' \setminus E$ appears unchanged in both decompositions, while E and $\rho(E)$ are rigid-motion copies of each other. By the definition of equidecomposable sets this suffices to conclude that $(L' \setminus E) \sqcup E = L'$ and $(L' \setminus E) \sqcup \rho(E) = L''$ stand in the same equidecomposability class, hence $L' \sim L''$. ■

Since Theorem 5.15 identifies L'' with L' in the equidecomposability sense, the four-piece duplication previously constructed for L'' now applies to L' as well.

5.4 Proving $L' \sim B^3$

The origin is the final point still outside our duplication. Its situation is analogous to the axis set D: both are null and both must be translated into the partition without disturbing any other piece. For D we performed a *null swap* by rotating the whole ball about a carefully chosen line. Here we carry out the same idea in two dimensions and then promote it to three.

Draw any circle C of radius $r < 1$ lying in the horizontal plane $z = z_0$ and passing through the origin. Removing the center leaves $C \setminus \{0\}$. The next lemma provides a planar null–swap for that missing point.

Lemma 5.16 (Circle minus a point)**.** *A circle S^1 is equidecomposable with $S^1 \setminus \{p\}$ for any point p on the circle.*

Proof. Use the unit circle and take $p = (1, 0)$. Define $A = \big\{(\cos n, \sin n) : n \in \mathbb{N}\big\}$. Because π is irrational, the points of A are distinct, so A is countably infinite and does not contain p. Let $B = S^1 \setminus \{p\} \setminus A$.

Rotate S^1 anticlockwise by one radian; call the rotation ρ and set $A' = \rho(A)$. Then ρ sends p to $\rho(p) = (\cos 1, \sin 1)$ and permutes A. Hence $S^1 = A \cup B$ and $S^1 \setminus \{p\} = A' \cup B$, with A congruent to A' via ρ. The two partitions use the same finite family $\{A, B\}$, establishing equidecomposability. ■

Proposition 5.17 (Ball without center versus full ball)**.** *The punctured ball $L' = B^3 \setminus \{0\}$ is equidecomposable with the full ball B^3.*

Proof. Choose a Euclidean circle $C \subset B^3$ that passes through the center and lies entirely inside the ball.

Remove the center to obtain $C \setminus \{0\}$. Lemma 5.16 tells us that C is equidecomposable with $C \setminus \{0\}$: the circle can swap its missing point with a rotated copy of a countable set of points without changing the rest of the circle.

View the circle's two pieces—as given by the lemma—as lying inside the ball. Everything *outside* the circle stays fixed. Replace the missing center by the rotated copy of those circle–pieces; no other part of the ball is modified. Thus one decomposition of the ball contains the center, the other does not, yet both decompositions use the same finite family of sets (the two circle–pieces and the ball minus the circle). By definition of equidecomposability these two versions of the ball are equivalent. Hence $B^3 \sim B^3 \setminus \{0\} = L'$. ■

Now that we have proved $L' \sim B^3$, we have all the steps to prove the Banach-Tarski Paradox.

5.5 Putting It All Together

Theorem 5.18 (Banach–Tarski Paradox)**.** *Let $B^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 1\}$ be the solid unit ball. There exist finitely many pairwise disjoint sets $E_1, \dots, E_k \subset B^3$ and rigid motions $g_1, \dots, g_k, h_1, \dots, h_k \in \mathrm{Iso}(\mathbb{R}^3)$ such that*

$$B^3 = \bigsqcup_{i=1}^{k} E_i, \qquad B^3 = \bigsqcup_{i=1}^{k} g_i E_i, \qquad B^3 = \bigsqcup_{i=1}^{k} h_i E_i.$$

Equivalently, the unit ball is equidecomposable with two disjoint copies of itself by rotations (and, if desired, translations).

Proof. Let D be the countable union of rotation axes of the free subgroup $G = \langle A, B\rangle$. Removing both the center and the axes leaves the set $L'' = B^3 \setminus \big(D \cup \{0\}\big)$. The four–piece construction in Proposition 5.10 duplicates precisely this regular region: L'' can be partitioned into finitely many pieces that rotate, under elements of G, into two full copies of L''.

Although L'' omits the axis lines, the null–swap Theorem 5.15 shows that inserting those lines does not change the equidecomposability type; more concretely L'' is equidecomposable with the punctured ball $L' = B^3 \setminus \{0\}$. Applying the duplication obtained for L'' therefore yields a duplication of L' itself.

The only omission now is the center point. Proposition 5.17 observes that one may choose a circle inside the ball passing through the center, replace the missing point by a congruent planar arc, and thereby swap the center back in without altering any other piece; this establishes that L' and B^3 are equidecomposable.

Using Proposition 4.2, since $B^3 \sim L'$ and $L' \sim L' \sqcup L'$ (by the duplication just obtained), it follows that $B^3 \sim B^3 \sqcup B^3$. In other words, the solid unit ball can be cut into finitely many disjoint pieces which, when reassembled by rigid motions, form two balls each congruent to the original. ■

6 Conclusion

The Banach-Tarski Paradox is a bizarre consequence of the Axiom of Choice. Without it, both this paradox and the Vitali Set would crumble.

For readers who were wondering if the Banach-Tarski Paradox works in $\mathbb{R}$ or $\mathbb{R}^2$, the answer is no, as there are no 2 or more generators that can be used to split the figure into more than 2 parts, which is a necessity to exploit the paradoxical nature of free groups. However, this paradox does work in any dimension higher than $\mathbb{R}^3$ because it has more than 1 generator.

The journey from axioms to paradoxes reminds us that rigorous reasoning can lead to results at odds with everyday models. In this case, we exploited the fact that a sphere had infintely many points however that isn't true in a real real-life model.

As a closing remark, in the words of my mentor, Ethan Martirosyan: *"Don't trust math!"*.

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