Projective Geometry

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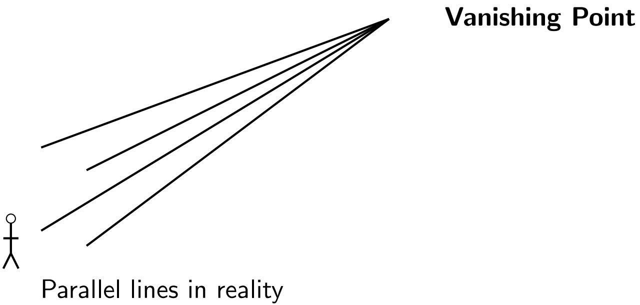
Introduction and Scope

- These slides serve as an **introduction and preface** to my expository paper on projective geometry
- The full paper contains detailed proofs, lemmas, and in-depth analysis
- Advanced topics like Grassmannians and Characteristic Classes are covered extensively in the paper
- **Here:** Focus on intuition, foundational concepts, and proof sketches
- **Goal:** Build geometric intuition before diving into technical details

The Genesis of Projective Geometry

- Born from art: Study of perspective in Renaissance paintings
- Key observation: Parallel lines appear to meet at a "vanishing point"
- Mathematical insight: Add "points at infinity" to unify geometry
- Goal: Study properties invariant under projection



Bridging Two Worlds

Projective Geometry

- Algebraic structures
- Global perspective
- Symmetries
- Invariants

Differential Geometry

- Smooth manifolds
- Local properties
- Curvature
- Calculus tools

Connection: Projective structures appear in differential settings (geodesics, conformal mappings, characteristic classes)

What is a Projective Space? The Intuition

Physical Intuition:

- Imagine looking at the world through your eye
- Every point you see lies on a ray from your eye
- Different points on the same ray look identical
- A projective space captures this: points are *directions*, not locations

Mathematical Intuition:

- Start with vector space V (like $\mathbb{R}^{n+1}$)
- Remove the origin (can't have zero direction)
- Identify points that differ by scaling: $v \sim \lambda v$ for $\lambda \neq 0$
- Result: $P(V) = (V \setminus \{0\})/\sim$

Projective Space: The Mathematical Definition

Definition

Projective space $P(V)$ is the set of all 1-dimensional subspaces of vector space V.

Equivalently: $P(V) = (V \setminus \{0\})/\mathbb{F}^*$

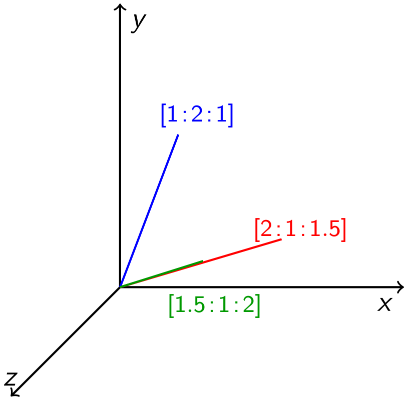
Homogeneous Coordinates:

- Points written as $[x_0 : x_1 : \cdots : x_n]$
- $[x_0 : x_1 : \cdots : x_n] = [\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n]$ for $\lambda \neq 0$
- Example: $[1 : 2 : 3] = [2 : 4 : 6] = [\lambda : 2\lambda : 3\lambda]$

Understanding $\mathbb{RP}^2$: The Real Projective Plane

Construction: Start with $\mathbb{R}^3$, remove origin, identify by scaling

Visualization: Points are lines through the origin in $\mathbb{R}^3$



Each colored line represents a point in $\mathbb{RP}^2$

Theorem: Structure of Real Projective Space

Theorem

$\mathbb{RP}^n$ can be covered by $n+1$ affine charts, each isomorphic to $\mathbb{R}^n$.

Proof Sketch:

- Define charts: $U_i = \{[x_0 : \cdots : x_n] : x_i \neq 0\}$
- Map $U_i \to \mathbb{R}^n$ by $[x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$
- This is well-defined: scaling doesn't change ratios
- Charts cover all of $\mathbb{RP}^n$: every point has some non-zero coordinate

Working Through the Chart Construction

Example: $\mathbb{RP}^2$ with three charts

Chart U_0: $x_0 \neq 0$

- $[x_0 : x_1 : x_2] \mapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) = (u, v)$
- Covers "finite" points - ordinary affine plane
- Inverse: $(u, v) \mapsto [1 : u : v]$

Chart U_1: $x_1 \neq 0$

- $[x_0 : x_1 : x_2] \mapsto \left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right)$
- Covers points where $x_1 \neq 0$

Chart U_2: $x_2 \neq 0$

- $[x_0 : x_1 : x_2] \mapsto \left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right)$
- Covers points where $x_2 \neq 0$

The Line at Infinity: Where Parallel Lines Meet

In $\mathbb{RP}^2$: "Line at infinity" is $H_\infty = \{[0 : x_1 : x_2]\}$
Why do parallel lines meet there?

- Consider parallel lines in affine plane: $y = mx + b_1$ and $y = mx + b_2$
- In homogeneous coordinates: $x_1 = mx_2 + b_1x_0$ and $x_1 = mx_2 + b_2x_0$
- As we go to infinity ($x_0 \to 0$): both approach $x_1 = mx_2$
- They meet at $[0 : m : 1]$ on the line at infinity!

Geometric Insight:

- Parallel lines have same "direction"
- This direction becomes their meeting point at infinity
- Different slopes m give different points $[0 : m : 1]$
- Vertical lines meet at $[0 : 1 : 0]$

Complex Projective Space: $\mathbb{CP}^n$

Construction: Same idea with $\mathbb{C}^{n+1}$

Theorem

$\mathbb{CP}^n$ has a natural structure as a compact complex manifold of complex dimension n.

Key Differences from $\mathbb{RP}^n$:

- Charts map to $\mathbb{C}^n$ instead of $\mathbb{R}^n$
- Transition maps are holomorphic, not just smooth
- Compactness: $\mathbb{CP}^n$ is compact (unlike $\mathbb{C}^n$)
- Rich complex structure enables powerful tools

Example: $\mathbb{CP}^1 \cong S^2$ (Riemann sphere)

- Chart U_0: $[z : w] \mapsto z/w$ (finite complex plane)
- Chart U_1: $[z : w] \mapsto w/z$ (neighborhood of infinity)
- Transition: $z/w \mapsto w/z = 1/(z/w)$

Projective Transformations: The Intuition

What should a "projective transformation" preserve?

- Lines should map to lines (perspective preserves straight edges)
- Collinearity: if points lie on a line, their images should too
- But distances and angles can change!

Linear Algebra Connection:

- Projective space $P(V)$ comes from vector space V
- Natural transformations of V: invertible linear maps
- These should induce transformations of $P(V)$

Key Insight: If $T : V \to V$ is linear and invertible, then T maps lines through origin to lines through origin

Definition of Projective Transformations

Definition

A projective transformation is a bijection $f : P^n(\mathbb{F}) \to P^n(\mathbb{F})$ that preserves collinearity.

Equivalently: Maps lines to lines

Matrix Representation:

- Given matrix $A \in GL_{n+1}(\mathbb{F})$
- Acts on $P^n(\mathbb{F})$ by: $[x_0 : \cdots : x_n] \mapsto [Ax]$
- Well-defined: $A(\lambda x) = \lambda(Ax)$, so $[A(\lambda x)] = [Ax]$

Scaling Invariance: A and λA ($\lambda \neq 0$) give same transformation

- Projective group: $PGL_{n+1}(\mathbb{F}) = GL_{n+1}(\mathbb{F})/\mathbb{F}^*$

Fundamental Theorem of Projective Geometry

Theorem (Fundamental Theorem)

Every projective transformation $f : P^n(\mathbb{F}) \to P^n(\mathbb{F})$ is induced by an invertible linear transformation of $\mathbb{F}^{n+1}$.

Proof Sketch:

1. **Step 1:** Show that f preserves linear subspaces
2. **Step 2:** Use preservation of lines to construct linear map
3. **Step 3:** Verify this map induces f

Significance:

- Completely classifies projective transformations
- Connects projective geometry to linear algebra
- Enables computational approaches

Proof Details: Step 1 - Preserving Subspaces

Goal: Show f maps k-dimensional subspaces to k-dimensional subspaces
Key Lemma: If f preserves collinearity, then it preserves linear dependence.
Proof of Lemma:

- Points $[v_1], [v_2], [v_3]$ are collinear $\Leftrightarrow$ v_1, v_2, v_3 are linearly dependent
- If v_1, v_2, v_3 are linearly dependent, then $[v_1], [v_2], [v_3]$ lie on a line
- Since f preserves collinearity: $[f(v_1)], [f(v_2)], [f(v_3)]$ lie on a line
- Therefore, $f(v_1), f(v_2), f(v_3)$ are linearly dependent

Consequence: f preserves all linear subspaces (by induction on dimension)

Proof Details: Step 2 - Constructing the Linear Map

Strategy: Use a basis to define linear map $T : \mathbb{F}^{n+1} \to \mathbb{F}^{n+1}$
Construction:

- Choose basis $\{e_0, e_1, \ldots, e_n\}$ of $\mathbb{F}^{n+1}$
- Choose "unit point" $u = e_0 + e_1 + \cdots + e_n$
- Apply f to get images $f([e_i])$ and $f([u])$
- Use the fact that f preserves linear relations to define T

Key Insight: The images $f([e_i])$ and $f([u])$ contain enough information to reconstruct T uniquely (up to scaling)
Technical Detail: Need to show T is well-defined and invertible

Proof Details: Step 3 - Verification

Goal: Show that T induces f on projective space

Verification:

- For any $[v] \in P^n(\mathbb{F})$, need to show $f([v]) = [T(v)]$
- Use the fact that f preserves all linear relations
- Since T was constructed to preserve these relations, we get $f([v]) = [T(v)]$

Uniqueness: T is unique up to scaling

- If T_1 and T_2 both induce f, then $T_1 = \lambda T_2$ for some $\lambda \neq 0$
- This gives the quotient $PGL_{n+1}(\mathbb{F}) = GL_{n+1}(\mathbb{F})/\mathbb{F}^*$

Examples of Projective Transformations

Example 1: Translation in affine chart

- In $\mathbb{RP}^2$, chart U_0: $(x, y) \mapsto (x + a, y + b)$
- Matrix form: $\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$
- Maps $[1 : x : y] \mapsto [1 : x + a : y + b]$

Example 2: Inversion (swapping finite and infinite points)

- Matrix: $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- Maps $[x_0 : x_1 : x_2] \mapsto [x_1 : x_0 : x_2]$
- In chart U_0: $(x, y) \mapsto (1/x, y/x)$

Example 3: Perspective transformation

- General form: $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$
- Maps lines to lines, but changes angles and distances

The Cross-Ratio: A Projective Invariant

Definition

For four distinct collinear points A, B, C, D on $P^1(\mathbb{F})$:

$$(A, B; C, D) = \frac{\det(A, C) \cdot \det(B, D)}{\det(A, D) \cdot \det(B, C)}$$

Theorem

The cross-ratio is invariant under projective transformations.

Significance:

- Only invariant of four collinear points
- Fundamental tool in projective geometry
- Connects to complex analysis (cross-ratios of complex numbers)

Duality in Projective Geometry

Key Idea: Symmetric relationship between points and hyperplanes

Theorem (Principle of Duality)

There's a natural bijection between points and hyperplanes in $P^n(\mathbb{F})$.

Example in $\mathbb{RP}^2$:

- Point $[a : b : c] \leftrightarrow$ Line $ax + by + cz = 0$
- Incidence preserved: point lies on line $\Leftrightarrow$ dual line passes through dual point

Duality Principle: Every theorem has a dual where:

- Points $\leftrightarrow$ Lines (in plane)
- "Lies on" $\leftrightarrow$ "Passes through"

Conics in Projective Geometry

Definition

A conic in $P^2(\mathbb{F})$ is the zero set of a homogeneous quadratic:

$$x^T Q x = 0$$

Theorem

Every non-degenerate conic in $P^2(\mathbb{C})$ is projectively equivalent to

$$x_0^2 + x_1^2 + x_2^2 = 0$$

Example: Circle $x^2 + y^2 = z^2$ in $\mathbb{RP}^2$

- Homogeneous form: $x_0^2 + x_1^2 - x_2^2 = 0$
- Intersects line at infinity at $[1 : i : 0]$ and $[1 : -i : 0]$ (over $\mathbb{C}$)
- These are the "circular points at infinity"

Smooth Manifold Structure

Theorem

$P^n(\mathbb{R})$ has a natural n-dimensional smooth manifold structure compatible with the action of $PGL(n+1,\mathbb{R})$.

Construction:

- Atlas: $\{U_i : i = 0, \ldots, n\}$ where $U_i = \{[x_0 : \cdots : x_n] : x_i \neq 0\}$
- Charts: $\phi_i : U_i \to \mathbb{R}^n$
- Transition maps $\phi_j \circ \phi_i^{-1}$ are smooth rational functions

Example: On $\mathbb{RP}^1$, transition map $(x_1/x_0) \mapsto (x_0/x_1) = 1/(x_1/x_0)$

Significance: Bridges projective and differential geometry

Advanced Topics in the Full Paper

The complete paper includes detailed treatments of:

- **Grassmannians:** Moduli spaces of linear subspaces
- **Characteristic Classes:** Chern classes and vector bundles
- **Fubini-Study Metric:** Canonical Kähler metric on $\mathbb{CP}^n$
- **Moduli Spaces:** Geometric parameter spaces
- **Mirror Symmetry:** Duality in string theory
- **Geometric Invariant Theory:** Quotient constructions

All proofs, lemmas, and technical details are provided in the full paper.

Conclusion: The Unifying Power

What we've seen:

- Projective geometry transforms complex problems into elegant statements
- Natural bridge between algebra and geometry
- Seamless integration with differential geometry
- Foundation for modern geometric research

Key Takeaways:

- Projective spaces compactify and unify
- Homogeneous coordinates provide computational tools
- Duality reveals hidden symmetries
- Invariants like cross-ratio capture essential geometry

Future Directions: Moduli theory, representation theory, algebraic geometry, arithmetic applications

See the full paper for complete proofs and advanced applications!

Questions?