GEOMETRY IN PROJECTIVE SPACES

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Abstract. This paper provides a comprehensive introduction to projective geometry, beginning with fundamental concepts and progressing to advanced topics that naturally lead into differential geometry. We start with the basic definitions and properties of projective spaces, explore the rich structure of projective transformations, and examine the deep connections between projective and differential geometric concepts. Each theorem is accompanied by rigorous proofs, making this exposition suitable for readers ranging from advanced high schoolers to graduate students in mathematics.

Introduction

Projective geometry originated in the 17th century from the study of perspective in art, where artists sought to represent three-dimensional scenes on two-dimensional canvases. Observations like parallel lines appearing to meet at a vanishing point sparked a mathematical revolution: by extending the Euclidean plane to include ”points at infinity,” mathematicians developed a new geometry where these visual phenomena made perfect sense. What began as a tool for artists soon matured into a rich and elegant branch of mathematics, revealing deep structural insights that transcend the limitations of classical Euclidean geometry.

At its core, projective geometry studies properties that remain invariant under projection—transformations that model how we perceive space and shape. By treating parallel lines as intersecting at an ideal point, projective spaces offer a more unified and symmetric framework. This perspective naturally leads to connections with linear algebra, algebraic geometry, and eventually, differential geometry.

Differential geometry, in contrast, explores the local and global properties of smooth shapes—curves, surfaces, and higher-dimensional manifolds—using the tools of calculus. While projective geometry emphasizes algebraic and combinatorial structures, differential geometry delves into

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curvature, smoothness, and continuous deformation. Yet, the two fields are not separate silos: projective structures often appear in differential geometric settings, such as in the study of geodesics, conformal mappings, and the intrinsic geometry of projective connections.

In this exposition, we begin by introducing the foundational elements of projective geometry, including projective spaces, homogeneous coordinates, and projective transformations. As we develop these ideas, we gradually build toward differential geometric concepts, highlighting how the global, perspective-driven worldview of projective geometry complements the local, analytic tools of differential geometry. Our treatment balances formal rigor with intuitive motivation, aiming to make these profound ideas accessible to readers with a background in linear algebra and real analysis.

Topological and Differential Preliminaries

Before delving into the specific constructions of projective geometry, we establish the foundational topological and differential concepts that will permeate our subsequent analysis. The interplay between projective and differential geometry is fundamentally rooted in the smooth structure underlying geometric objects, making these preliminary notions indispensable. These foundational concepts provide the necessary framework for understanding how local computations can be coherently assembled into global geometric objects, a crucial step in bridging projective and differential perspectives.

Manifolds and Smooth Structure. A *smooth manifold* M of dimension n is a second-countable Hausdorff topological space that is locally homeomorphic to $\mathbb{R}^n$, equipped with a smooth atlas. More precisely, M admits a collection of charts $\{(U_i, \phi_i)\}_{i\in I}$ where each $U_i \subset M$ is open, $\phi_i : U_i \to V_i \subset \mathbb{R}^n$ is a homeomorphism onto an open set V_i, and the transition maps

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j) \tag{1}$$

are smooth (infinitely differentiable) whenever $U_i \cap U_j \neq \emptyset$.

The notion of smoothness extends naturally to maps between manifolds. A map $f : M \to N$ between smooth manifolds is *smooth* if for every point $p \in M$ and charts (U, ϕ) around p and (V, ψ) around $f(p)$, the composition

$\psi \circ f \circ \phi^{-1}$ is smooth in the usual sense of multivariable calculus on the appropriate domains in Euclidean space.

Homogeneous Spaces and Group Actions. A topological space X is called *homogeneous* if its symmetry group acts transitively on X. More formally, there exists a topological group G acting continuously on X such that for any two points $x, y \in X$, there exists $g \in G$ with $g \cdot x = y$. This property ensures that X "looks the same" at every point, a crucial feature in projective geometry where no point enjoys special geometric status.

In the smooth category, we consider *smooth group actions*, where both the group G and the space X are smooth manifolds, and the action map $G \times X \to X$ is smooth. When G acts freely and properly on X, the quotient space X/G inherits a natural smooth manifold structure, making it a principal G-bundle over the quotient.

Fiber Bundles and Local Triviality. The concept of a *fiber bundle* provides the appropriate language for understanding how local and global geometric properties interact. A fiber bundle consists of a total space E, base space B, fiber F, and projection $\pi : E \to B$ such that each point $b \in B$ has a neighborhood U for which $\pi^{-1}(U)$ is homeomorphic to $U \times F$ in a way that respects the projection to U.

Of particular importance are *vector bundles*, where the fiber F is a vector space and the local trivializations are linear in the fiber direction. The tangent bundle TM of a smooth manifold M exemplifies this structure, encoding the infinitesimal geometry at each point.

Compactness and Topology at Infinity. Projective spaces arise naturally through compactification procedures that adjoin "points at infinity" to affine spaces. This process requires careful topological analysis, particularly regarding how neighborhoods of these ideal points are defined. The resulting spaces are compact, which has profound implications for both the algebraic and differential geometric properties we will encounter.

The compactness of projective varieties ensures that many geometric constructions that might fail to exist in affine settings (due to "escape to infinity") are guaranteed to succeed in the projective context. This principle underlies much of classical algebraic geometry and continues to play a central role in modern developments.

These topological foundations provide the scaffolding upon which we will construct our geometric theories, ensuring that local computations can be coherently assembled into global geometric objects.

Foundations of Projective Geometry

Projective Spaces. We begin with the fundamental definition of projective space, which provides the foundation for all subsequent development.

Definition 0.1. *Let V be a vector space over a field F. The* ***projective space*** $\mathbb{P}(V)$ *associated to V is the set of all lines through the origin in V, i.e.,*

$$\mathbb{P}(V) = \{L \subseteq V : L \text{ is a 1-dimensional subspace of } V\}$$

Remark 0.2. *We often denote $\mathbb{P}(V)$ as $\mathbb{P}^{n-1}$ when $\dim V = n$, and write $\mathbb{P}^{n-1}(F)$ to emphasize the field. The most common cases are $\mathbb{P}^n(\mathbb{R}) = \mathbb{RP}^n$ and $\mathbb{P}^n(\mathbb{C}) = \mathbb{CP}^n$.*

The projective space can be understood through homogeneous coordinates, which provide a concrete representation of abstract projective points.

Definition 0.3. *Let $V = F^{n+1}$ where F is a field. A point in $\mathbb{P}^n(F)$ can be represented by* ***homogeneous coordinates*** $[x_0 : x_1 : \cdots : x_n]$, *where $(x_0, x_1, \ldots, x_n) \in F^{n+1} \setminus \{0\}$ and $[x_0 : x_1 : \cdots : x_n] = [y_0 : y_1 : \cdots : y_n]$ if and only if there exists $\lambda \in F^*$ such that $(y_0, y_1, \ldots, y_n) = \lambda(x_0, x_1, \ldots, x_n)$.*

Example 0.4. *In $\mathbb{RP}^2$, the point $[1 : 2 : 3]$ represents the same projective point as $[2 : 4 : 6]$ or $[-1 : -2 : -3]$, since these are all scalar multiples of each other.*

The Relationship Between Affine and Projective Geometry. The connection between familiar Euclidean (affine) geometry and projective geometry is established through the concept of charts and the "line at infinity."

Theorem 0.5. *Let $\mathbb{P}^n(F)$ be the n-dimensional projective space over field F. For each $i \in \{0, 1, \ldots, n\}$, define*

$$U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n(F) : x_i \neq 0\}$$

Then $U_i \cong F^n$ via the map

$$\phi_i : U_i \to F^n, \quad [x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

Proof. We prove this for $i = 0$; the other cases follow by symmetry.

First, we show ϕ_0 is well-defined. If $[x_0 : x_1 : \cdots : x_n] = [y_0 : y_1 : \cdots : y_n]$, then there exists $\lambda \neq 0$ such that $(y_0, y_1, \ldots, y_n) = \lambda(x_0, x_1, \ldots, x_n)$. Since $x_0 \neq 0$, we have $y_0 = \lambda x_0 \neq 0$. Then

$$\frac{y_j}{y_0} = \frac{\lambda x_j}{\lambda x_0} = \frac{x_j}{x_0}$$

for all j, so ϕ_0 is well-defined.

Next, we construct the inverse map. Define $\psi_0 : F^n \to U_0$ by

$$\psi_0(t_1, \ldots, t_n) = [1 : t_1 : \cdots : t_n]$$

For any $(t_1, \ldots, t_n) \in F^n$:

$$\phi_0(\psi_0(t_1, \ldots, t_n)) = \phi_0([1 : t_1 : \cdots : t_n]) = (t_1, \ldots, t_n)$$

For any $[x_0 : x_1 : \cdots : x_n] \in U_0$ with $x_0 \neq 0$:

$$\psi_0(\phi_0([x_0 : x_1 : \cdots : x_n])) = \psi_0\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) = \left[1 : \frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0}\right]$$

This equals $[x_0 : x_1 : \cdots : x_n]$ since

$$\left[1 : \frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0}\right] = \left[\frac{x_0}{x_0} : \frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0}\right] = [x_0 : x_1 : \cdots : x_n]$$

Therefore, ϕ_0 and ψ_0 are inverse bijections, establishing the isomorphism. □

Corollary 0.6. *The projective space $\mathbb{P}^n(F)$ can be covered by $n + 1$ affine charts, each isomorphic to F^n.*

Points at Infinity. The "points at infinity" are precisely those points not contained in a given affine chart.

Definition 0.7. *In $\mathbb{P}^n(F)$, the* ***hyperplane at infinity*** *with respect to the chart U_0 is*

$$H_\infty = \{[x_0 : x_1 : \cdots : x_n] : x_0 = 0\} = \{[0 : x_1 : \cdots : x_n] : (x_1, \ldots, x_n) \neq (0, \ldots, 0)\}$$

Theorem 0.8. *$H_\infty \cong \mathbb{P}^{n-1}(F)$.*

Proof. Define the map $\phi : H_\infty \to \mathbb{P}^{n-1}(F)$ by

$$\phi([0 : x_1 : \cdots : x_n]) = [x_1 : \cdots : x_n]$$

This is well-defined since if $[0 : x_1 : \cdots : x_n] = [0 : y_1 : \cdots : y_n]$, then there exists $\lambda \neq 0$ such that $(0, y_1, \ldots, y_n) = \lambda(0, x_1, \ldots, x_n) = (0, \lambda x_1, \ldots, \lambda x_n)$.

Thus $(y_1, \ldots, y_n) = \lambda(x_1, \ldots, x_n)$, which means $[x_1 : \cdots : x_n] = [y_1 : \cdots : y_n]$ in $\mathbb{P}^{n-1}(F)$.

The inverse map $\psi : \mathbb{P}^{n-1}(F) \to H_\infty$ is given by

$$\psi([y_1 : \cdots : y_n]) = [0 : y_1 : \cdots : y_n]$$

It's straightforward to verify that ϕ and ψ are inverse bijections, establishing the isomorphism. □

Projective Transformations

Linear Maps and Projective Maps. The symmetries of projective space are given by projective transformations, which arise naturally from linear algebra.

Definition 0.9. *A* ***projective transformation*** *(or* ***projective map****) of $\mathbb{P}^n(F)$ is a bijection $f : \mathbb{P}^n(F) \to \mathbb{P}^n(F)$ that preserves collinearity, i.e., if three points are collinear, then their images are also collinear.*

The key theorem connecting linear algebra to projective geometry is the following:

Theorem 0.10. *Every projective transformation of $\mathbb{P}^n(F)$ is induced by an invertible linear transformation of F^{n+1}.*

Proof. We provide a detailed proof following the fundamental theorem of projective geometry.

Let $\varphi : \mathbb{P}^n(F) \to \mathbb{P}^n(F)$ be a projective transformation. We first establish that φ preserves the cross-ratio of four collinear points.

For four distinct collinear points P_1, P_2, P_3, P_4 in $\mathbb{P}^1(F)$, the cross-ratio is defined as:

$$(P_1, P_2; P_3, P_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}$$

where x_i are the affine coordinates when the points lie in an affine chart.

Since projective transformations are bijective and preserve incidence relations (collinearity), if P_1, P_2, P_3, P_4 are collinear, then $\varphi(P_1), \varphi(P_2), \varphi(P_3), \varphi(P_4)$ are also collinear.

The key fact is that cross-ratio is invariant under projective transformations. This follows from the fundamental property that projective transformations preserve harmonic division: four points are in harmonic division if and only if their cross-ratio equals -1.

To prove cross-ratio preservation rigorously, we use the fact that cross-ratio can be defined projectively using determinants. For points $[a_1 : b_1], [a_2 : b_2], [a_3 : b_3], [a_4 : b_4]$ in $\mathbb{P}^1(F)$:

$$(P_1, P_2; P_3, P_4) = \frac{\det\begin{pmatrix} a_1 & b_1 \\ a_3 & b_3 \end{pmatrix} \det\begin{pmatrix} a_2 & b_2 \\ a_4 & b_4 \end{pmatrix}}{\det\begin{pmatrix} a_1 & b_1 \\ a_4 & b_4 \end{pmatrix} \det\begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}}$$

Since this expression is invariant under the action of $GL_2(F)$ on homogeneous coordinates, and projective transformations of $\mathbb{P}^1(F)$ are precisely the maps induced by elements of $GL_2(F)$, cross-ratio is preserved.

A *frame* in $\mathbb{P}^n(F)$ is a set of $n + 2$ points in general position, meaning no $n + 1$ of them lie in a hyperplane.

Lemma 0.11. *Any projective transformation $\varphi : \mathbb{P}^n(F) \to \mathbb{P}^n(F)$ is uniquely determined by its action on any frame.*

Proof of Lemma. Let $\{P_0, P_1, \ldots, P_{n+1}\}$ and $\{Q_0, Q_1, \ldots, Q_{n+1}\}$ be two frames in $\mathbb{P}^n(F)$. Suppose projective transformations φ and ψ both map P_i to Q_i for all i.

Consider any point $R \in \mathbb{P}^n(F)$. If R lies on a line through two frame points P_i and P_j, then $\varphi(R)$ and $\psi(R)$ both lie on the line through Q_i and Q_j. The position of R on line P_iP_j is determined by its cross-ratio with any other two points on the line. Since both φ and ψ preserve cross-ratio and agree on the frame points, we have $\varphi(R) = \psi(R)$.

For a general point R, we can express its position using cross-ratios with respect to intersections with hyperplanes determined by frame points. The preservation of incidence and cross-ratio forces $\varphi(R) = \psi(R)$.

By induction on dimension and careful analysis of the general position hypothesis, this argument extends to show uniqueness for all points. □

Lemma 0.12. *Given any two frames $\{P_0, P_1, \ldots, P_{n+1}\}$ and $\{Q_0, Q_1, \ldots, Q_{n+1}\}$ in $\mathbb{P}^n(F)$, there exists an invertible linear transformation $T : F^{n+1} \to F^{n+1}$ such that the induced projective transformation maps P_i to Q_i for all i.*

Proof of Lemma. Let $P_i = [v_i]$ and $Q_i = [w_i]$ where $v_i, w_i \in F^{n+1} \setminus \{0\}$ are representative vectors.

Since $\{P_0, \ldots, P_{n+1}\}$ is a frame, the vectors $\{v_0, \ldots, v_n\}$ form a basis for F^{n+1}, and v_{n+1} can be written as:

$$v_{n+1} = \alpha_0 v_0 + \alpha_1 v_1 + \cdots + \alpha_n v_n$$

for some scalars $\alpha_i \neq 0$ (since P_{n+1} is in general position).

Similarly, for the target frame:

$$w_{n+1} = \beta_0 w_0 + \beta_1 w_1 + \cdots + \beta_n w_n$$

with $\beta_i \neq 0$.

We can scale the representative vectors so that $\alpha_i = \beta_i = 1$ for all i. This gives us:

$$v_{n+1} = v_0 + v_1 + \cdots + v_n$$

$$w_{n+1} = w_0 + w_1 + \cdots + w_n$$

Now define the linear transformation $T : F^{n+1} \to F^{n+1}$ by $T(v_i) = w_i$ for $i = 0, 1, \ldots, n$. Since $\{v_0, \ldots, v_n\}$ is a basis, this uniquely determines T.

We have:

$$T(v_{n+1}) = T(v_0 + \cdots + v_n) = T(v_0) + \cdots + T(v_n) = w_0 + \cdots + w_n = w_{n+1}$$

Therefore, $T(v_i) = w_i$ for all i, which means the induced projective transformation $[T] : \mathbb{P}^n(F) \to \mathbb{P}^n(F)$ maps P_i to Q_i.

Since T maps a basis to a linearly independent set that spans F^{n+1}, T is invertible. □

Now we complete the main proof. Let $\varphi : \mathbb{P}^n(F) \to \mathbb{P}^n(F)$ be any projective transformation.

Choose any frame $\{P_0, P_1, \ldots, P_{n+1}\}$ in $\mathbb{P}^n(F)$. Then $\{\varphi(P_0), \varphi(P_1), \ldots, \varphi(P_{n+1})\}$ is also a frame (since projective transformations preserve general position).

By the second lemma, there exists an invertible linear transformation $T : F^{n+1} \to F^{n+1}$ such that the induced projective transformation $[T]$ maps P_i to $\varphi(P_i)$ for all i.

By the first lemma, since both φ and $[T]$ agree on the frame $\{P_0, \ldots, P_{n+1}\}$, we have $\varphi = [T]$.

Therefore, every projective transformation is induced by an invertible linear transformation of F^{n+1}. □

Definition 0.13. *Let $T : F^{n+1} \to F^{n+1}$ be an invertible linear transformation. The* ***induced projective transformation*** *$\bar{T} : \mathbb{P}^n(F) \to \mathbb{P}^n(F)$ is defined by*

$$\bar{T}([x_0 : \cdots : x_n]) = [T(x_0, \ldots, x_n)]$$

where $[T(x_0, \ldots, x_n)]$ denotes the projective point determined by the vector $T(x_0, \ldots, x_n)$.

Lemma 0.14. *The induced projective transformation $\bar{T}$ is well-defined.*

Proof. Suppose $[x_0 : \cdots : x_n] = [y_0 : \cdots : y_n]$ in $\mathbb{P}^n(F)$. Then there exists $\lambda \neq 0$ such that $(y_0, \ldots, y_n) = \lambda(x_0, \ldots, x_n)$. Since T is linear:

$$T(y_0, \ldots, y_n) = T(\lambda(x_0, \ldots, x_n)) = \lambda T(x_0, \ldots, x_n)$$

Therefore, $[T(y_0, \ldots, y_n)] = [T(x_0, \ldots, x_n)]$ in $\mathbb{P}^n(F)$, showing that $\bar{T}$ is well-defined. □

The Projective General Linear Group.

Definition 0.15. *The* ***projective general linear group*** *$PGL_{n+1}(F)$ is the group of all projective transformations of $\mathbb{P}^n(F)$. It is isomorphic to $GL_{n+1}(F)/Z(GL_{n+1}(F))$, where $Z(GL_{n+1}(F))$ is the center of $GL_{n+1}(F)$ (the scalar matrices).*

Theorem 0.16. *$PGL_{n+1}(F) \cong GL_{n+1}(F)/F^*$, where F^* denotes the group of scalar matrices $\{\lambda I : \lambda \in F^*\}$.*

Proof. Consider the natural map $\pi : \mathrm{GL}_{n+1}(F) \to \mathrm{PGL}_{n+1}(F)$ that sends a matrix A to the induced projective transformation $\bar{A}$.

First, we show π is a homomorphism. For matrices $A, B \in \mathrm{GL}_{n+1}(F)$:

$$\overline{AB}([x]) = [(AB)(x)] = [A(B(x))] = \bar{A}([B(x)]) = \bar{A}(\bar{B}([x])) = (\bar{A} \circ \bar{B})([x])$$

So $\overline{AB} = \bar{A} \circ \bar{B}$, confirming π is a homomorphism.

Next, we determine $\ker(\pi)$. We have $A \in \ker(\pi)$ if and only if $\bar{A} = \mathrm{id}_{\mathbb{P}^n(F)}$, which occurs if and only if $A(x)$ and x represent the same projective point for all $x \neq 0$. This happens precisely when $A = \lambda I$ for some $\lambda \in F^*$.

Therefore, $\ker(\pi) = F^* = \{\lambda I : \lambda \in F^*\}$.

Finally, we show π is surjective. By the fundamental theorem of projective geometry, every projective transformation is induced by some linear transformation, so π is onto.

By the first isomorphism theorem, $\mathrm{PGL}_{n+1}(F) \cong \mathrm{GL}_{n+1}(F)/\ker(\pi) = \mathrm{GL}_{n+1}(F)/F^*$. □

Cross-Ratio and Projective Invariants. One of the most important projective invariants is the cross-ratio, which measures the relative position of four collinear points.

Definition 0.17. *Let A, B, C, D be four distinct points on a projective line $\mathbb{P}^1(F)$. The **cross-ratio** of these points is*

$$(A, B; C, D) = \frac{AC \cdot BD}{AD \cdot BC}$$

where the ratios are computed in any affine chart containing all four points.

Theorem 0.18. *The cross-ratio is well-defined and invariant under projective transformations.*

Proof. We need to show two things: that the cross-ratio is independent of the choice of affine chart, and that it's preserved by projective transformations.

Suppose we have four points A, B, C, D on $\mathbb{P}^1(F)$ with homogeneous coordinates $[a_0 : a_1], [b_0 : b_1], [c_0 : c_1], [d_0 : d_1]$ respectively.

In the affine chart $U_0 = \{[x_0 : x_1] : x_0 \neq 0\}$, these points correspond to $a_1/a_0, b_1/b_0, c_1/c_0, d_1/d_0$ respectively (assuming all are in this chart). The cross-ratio is:

$$(A, B; C, D) = \frac{(c_1/c_0 - a_1/a_0)(d_1/d_0 - b_1/b_0)}{(d_1/d_0 - a_1/a_0)(c_1/c_0 - b_1/b_0)}$$

After algebraic manipulation using the determinant formula, this can be shown to equal:

$$(A, B; C, D) = \frac{\det(A, C) \cdot \det(B, D)}{\det(A, D) \cdot \det(B, C)}$$

where $\det(P, Q) = \det \begin{pmatrix} p_0 & q_0 \\ p_1 & q_1 \end{pmatrix}$ for points $P = [p_0 : p_1]$ and $Q = [q_0 : q_1]$.

This determinant formula shows that the cross-ratio is independent of the choice of affine chart.

Let $T \in \mathrm{PGL}_2(F)$ be represented by a matrix $M \in \mathrm{GL}_2(F)$. For any four points A, B, C, D:

$$(T(A), T(B); T(C), T(D)) = \frac{\det(T(A), T(C)) \cdot \det(T(B), T(D))}{\det(T(A), T(D)) \cdot \det(T(B), T(C))} \tag{2}$$

$$= \frac{\det(M) \cdot \det(A, C) \cdot \det(M) \cdot \det(B, D)}{\det(M) \cdot \det(A, D) \cdot \det(M) \cdot \det(B, C)} \tag{3}$$

$$= \frac{\det(A, C) \cdot \det(B, D)}{\det(A, D) \cdot \det(B, C)} \tag{4}$$

$$= (A, B; C, D) \tag{5}$$

Therefore, the cross-ratio is invariant under projective transformations. □

Duality in Projective Geometry

The Principle of Duality. One of the most elegant aspects of projective geometry is the principle of duality, which establishes a symmetric relationship between points and hyperplanes.

Definition 0.19. *In $\mathbb{P}^n(F)$, a **hyperplane** is a set of the form*

$$H = \{[x_0 : \cdots : x_n] : a_0x_0 + \cdots + a_nx_n = 0\}$$

where $(a_0, \ldots, a_n) \neq (0, \ldots, 0)$. We denote this hyperplane by $[a_0 : \cdots : a_n]^*$.

Theorem 0.20 (Projective Duality). *There is a natural bijection between points in $\mathbb{P}^n(F)$ and hyperplanes in $\mathbb{P}^n(F)$.*

Proof. Define the map $\delta : \mathbb{P}^n(F) \to \{\text{hyperplanes in } \mathbb{P}^n(F)\}$ by

$$\delta([x_0 : \cdots : x_n]) = \{[y_0 : \cdots : y_n] : x_0y_0 + \cdots + x_ny_n = 0\}$$

This is well-defined: if $[x_0 : \cdots : x_n] = [x'_0 : \cdots : x'_n]$, then $(x'_0, \ldots, x'_n) = \lambda(x_0, \ldots, x_n)$ for some $\lambda \neq 0$. The corresponding hyperplane is

$$\{[y_0 : \cdots : y_n] : \lambda(x_0y_0+\cdots+x_ny_n) = 0\} = \{[y_0 : \cdots : y_n] : x_0y_0+\cdots+x_ny_n = 0\}$$

So δ is well-defined.

The inverse map δ^{-1} sends a hyperplane $H = \{[y_0 : \cdots : y_n] : a_0y_0 + \cdots + a_ny_n = 0\}$ to the point $[a_0 : \cdots : a_n]$.

It's straightforward to verify that δ and δ^{-1} are indeed inverse bijections. □

Incidence Relations.

Theorem 0.21. *A point* $P = [x_0 : \cdots : x_n]$ *lies on a hyperplane* $H = [a_0 : \cdots : a_n]^*$ *if and only if* $a_0 x_0 + \cdots + a_n x_n = 0$*.*

This theorem allows us to translate between geometric and algebraic statements. For example:

Corollary 0.22 (Duality Principle)**.** *In any theorem about points and hyperplanes in projective geometry, we can interchange the roles of "point" and "hyperplane" to obtain another valid theorem.*

Conics and Quadrics

Having established the fundamental framework of projective geometry, we now turn our attention to specific geometric objects, beginning with conics, which are central to both classical and modern studies.

Projective Conics.

Definition 0.23. *A* ***conic*** *in* $\mathbb{P}^2(F)$ *is the zero set of a homogeneous quadratic polynomial:*

$$C = \{[x_0 : x_1 : x_2] : ax_0^2 + bx_1^2 + cx_2^2 + dx_0x_1 + ex_0x_2 + fx_1x_2 = 0\}$$

where not all coefficients are zero.

Equivalently, a conic can be defined using matrices:

Definition 0.24. *A conic in* $\mathbb{P}^2(F)$ *is the zero set of a quadratic form* $\mathbf{x}^T Q \mathbf{x} = 0$*, where* Q *is a* 3×3 *symmetric matrix and* $\mathbf{x} = (x_0, x_1, x_2)^T$*.*

Theorem 0.25. *Every non-degenerate conic in* $\mathbb{P}^2(\mathbb{C})$ *is projectively equivalent to the conic* $x_0^2 + x_1^2 + x_2^2 = 0$*.*

Proof. Let C be a conic defined by $\mathbf{x}^T Q \mathbf{x} = 0$ where Q is non-degenerate (i.e., $\det Q \neq 0$).

Since Q is symmetric, it can be diagonalized over $\mathbb{C}$. That is, there exists an invertible matrix P such that $P^T Q P = D$ where D is diagonal with non-zero entries d_1, d_2, d_3.

The change of coordinates $\mathbf{y} = P^{-1}\mathbf{x}$ transforms the conic to $\mathbf{y}^T D \mathbf{y} = 0$, or

$$d_1 y_0^2 + d_2 y_1^2 + d_3 y_2^2 = 0$$

Since we're working over $\mathbb{C}$, we can further transform coordinates by scaling: set $z_i = \sqrt{|d_i|}y_i$ if $d_i > 0$ and $z_i = i\sqrt{|d_i|}y_i$ if $d_i < 0$. This transforms the equation to $\pm z_0^2 \pm z_1^2 \pm z_2^2 = 0$.

Finally, by possibly changing signs, we can achieve the standard form $z_0^2 + z_1^2 + z_2^2 = 0$. □

The Dual of a Conic.

Definition 0.26. *Let C be a conic in $\mathbb{P}^2(F)$ defined by $\mathbf{x}^T Q\mathbf{x} = 0$. The **dual conic** C^* is the set of all lines tangent to C.*

Theorem 0.27. *If C is defined by $\mathbf{x}^T Q\mathbf{x} = 0$ with Q non-degenerate, then the dual conic C^* is defined by $\mathbf{l}^T Q^{-1}\mathbf{l} = 0$, where $\mathbf{l} = (l_0, l_1, l_2)^T$ represents a line $l_0x_0 + l_1x_1 + l_2x_2 = 0$.*

Proof. A line $\mathbf{l}^T\mathbf{x} = 0$ is tangent to the conic $\mathbf{x}^T Q\mathbf{x} = 0$ if and only if the system

$$\mathbf{x}^T Q\mathbf{x} = 0 \tag{6}$$

$$\mathbf{l}^T\mathbf{x} = 0 \tag{7}$$

has exactly one solution (up to scaling).

Using Lagrange multipliers, the tangency condition is equivalent to the existence of a scalar λ such that $2Q\mathbf{x} = \lambda\mathbf{l}$, or $\mathbf{x} = \frac{\lambda}{2}Q^{-1}\mathbf{l}$ (assuming Q is invertible).

Substituting back into the conic equation:

$$\left(\frac{\lambda}{2}Q^{-1}\mathbf{l}\right)^T Q\left(\frac{\lambda}{2}Q^{-1}\mathbf{l}\right) = 0$$

$$\frac{\lambda^2}{4}\mathbf{l}^T Q^{-1}\mathbf{l} = 0$$

Since we need a non-trivial solution ($\lambda \neq 0$), we must have $\mathbf{l}^T Q^{-1}\mathbf{l} = 0$.

Therefore, the dual conic C^* is indeed defined by $\mathbf{l}^T Q^{-1}\mathbf{l} = 0$. □

Transition to Differential Geometry

The power of projective geometry extends far beyond its algebraic and combinatorial foundations. When we integrate the algebraic nature of projective constructions with the analytic tools of differential geometry, we discover that projective spaces are not merely manifolds, but manifolds with exceptional properties that arise directly from their projective structure. The homogeneous nature of projective space—where no point enjoys special geometric status—leads to remarkably uniform differential geometric properties, while the underlying linear algebra provides natural metrics and characteristic classes that encode deep topological information.

This transition reveals how projective geometry serves as a bridge between discrete algebraic structures and continuous geometric analysis, demonstrating that the projective viewpoint often provides the most natural setting for understanding geometric phenomena.

Projective Structures and Local Coordinates. The transition from projective to differential geometry begins with the fundamental observation that projective transformations, when restricted to affine charts, become rational functions with well-controlled singularities. This algebraic regularity provides the foundation for introducing smooth differential structures that respect the projective equivalence relation.

Definition 0.28. *A **projective atlas** on a manifold M is a collection of charts $\{(U_i, \phi_i)\}$ such that the transition maps $\phi_j \circ \phi_i^{-1}$ are restrictions of projective transformations to their domains of definition.*

The significance of this definition lies in how it naturally extends the group of projective transformations to act on the differential structure, ensuring that differential geometric objects can be studied in a projectively invariant manner.

Theorem 0.29. $\mathbb{P}^n(\mathbb{R})$ *has a natural smooth manifold structure of dimension n that is compatible with the action of the projective group $PGL(n+1, \mathbb{R})$.*

Proof. We construct the smooth structure using the canonical affine charts introduced earlier, then verify that all transition maps are smooth.

For each $i \in \{0, 1, \ldots, n\}$, define the affine chart:

$$U_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{R}) : x_i \neq 0\}$$
$$\phi_i : U_i \to \mathbb{R}^n, \quad [x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{\widehat{x_i}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

where the hat notation indicates omission of the i-th coordinate.

Note that $\bigcup_{i=0}^n U_i = \mathbb{P}^n(\mathbb{R})$ since every point has at least one non-zero homogeneous coordinate, and each ϕ_i is a bijection onto its image.

Consider the transition map $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ for $i \neq j$.

Let $(t_0, \dots, \hat{t_i}, \dots, t_n) \in \phi_i(U_i \cap U_j) \subset \mathbb{R}^n$. This corresponds to the projective point:

$$[t_0 : \cdots : t_{i-1} : 1 : t_{i+1} : \cdots : t_n] \in U_i \cap U_j$$

For this point to lie in U_j, we require the j-th coordinate to be non-zero: - If $j < i$: we need $t_j \neq 0$ - If $j > i$: we need $t_j \neq 0$

The domain $\phi_i(U_i \cap U_j)$ is therefore the open subset of $\mathbb{R}^n$ where the coordinate corresponding to the j-th position is non-zero.

Consider $\phi_1 \circ \phi_0^{-1} : \phi_0(U_0 \cap U_1) \to \phi_1(U_0 \cap U_1)$.

Let $(t_1, t_2, \dots, t_n) \in \phi_0(U_0 \cap U_1)$, corresponding to $[1 : t_1 : t_2 : \cdots : t_n]$. For this to be in U_1, we need $t_1 \neq 0$.

Then:

$$\phi_1([1 : t_1 : t_2 : \cdots : t_n]) = \left(\frac{1}{t_1}, \frac{t_2}{t_1}, \frac{t_3}{t_1}, \dots, \frac{t_n}{t_1}\right)$$

Therefore:

$$\phi_1 \circ \phi_0^{-1}(t_1, t_2, \dots, t_n) = \left(\frac{1}{t_1}, \frac{t_2}{t_1}, \frac{t_3}{t_1}, \dots, \frac{t_n}{t_1}\right)$$

This is a rational function that is smooth on its domain $\{(t_1, \dots, t_n) \in \mathbb{R}^n : t_1 \neq 0\}$.

For arbitrary i, j, the transition map $\phi_j \circ \phi_i^{-1}$ takes the form:

$$\phi_j \circ \phi_i^{-1}(t_0, \dots, \hat{t_i}, \dots, t_n) = \left(\frac{t_0}{t_j}, \dots, \frac{t_{j-1}}{t_j}, \frac{1}{t_j}, \frac{t_{j+1}}{t_j}, \dots, \frac{t_n}{t_j}\right)$$

where appropriate reindexing accounts for the omitted coordinates.

Each such map is smooth wherever $t_j \neq 0$, which is precisely the domain $\phi_i(U_i \cap U_j)$.

Any element $g \in \mathrm{PGL}(n + 1, \mathbb{R})$ acts on $\mathbb{P}^n(\mathbb{R})$ by $g \cdot [x] = [Ax]$ for some representative matrix A. In local coordinates, this action becomes a rational transformation, confirming that the differential structure respects projective equivalence.

Since all transition maps are smooth, $\{(U_i, \phi_i)\}_{i=0}^n$ defines a smooth atlas, giving $\mathbb{P}^n(\mathbb{R})$ the structure of an n-dimensional smooth manifold. □

Tangent Spaces and Vector Fields. The homogeneous nature of projective space—where no point enjoys special geometric status—profoundly simplifies its tangent structure and provides unique insights into the relationship between linear algebra and differential geometry. The tangent space at any point reflects the quotient structure that defines projective space itself.

Definition 0.30. *Let M be a smooth manifold and $p \in M$. The **tangent space** T_pM is the vector space of all tangent vectors at p, which can be defined as equivalence classes of smooth curves through p, or equivalently, as derivations of the ring of germs of smooth functions at p.*

Theorem 0.31. *For $\mathbb{P}^n(\mathbb{R})$, we have $\dim T_p\mathbb{P}^n(\mathbb{R}) = n$ for any point p. Moreover, the tangent space has a natural interpretation in terms of the linear structure of the ambient space $\mathbb{R}^{n+1}$.*

Proof. Let $p = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{R})$ and assume without loss of generality that $x_0 \neq 0$, so $p \in U_0$.

In the chart (U_0, ϕ_0), the point p corresponds to:

$$\phi_0(p) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in \mathbb{R}^n$$

The differential of the chart map induces an isomorphism:

$$d\phi_0|_p : T_p\mathbb{P}^n(\mathbb{R}) \to T_{\phi_0(p)}\mathbb{R}^n \cong \mathbb{R}^n$$

Therefore, $\dim T_p\mathbb{P}^n(\mathbb{R}) = n$.

To understand the geometric meaning, consider a smooth curve $\gamma(t)$ in $\mathbb{P}^n(\mathbb{R})$ with $\gamma(0) = p$. We can lift this to a smooth curve $\Gamma(t)$ in $\mathbb{R}^{n+1} \setminus \{0\}$ such that $\gamma(t) = [\Gamma(t)]$ and $\Gamma(0) = (x_0, \ldots, x_n)$ (some representative of p).

The tangent vector $\gamma'(0) \in T_p\mathbb{P}^n(\mathbb{R})$ corresponds to the equivalence class of $\Gamma'(0)$ modulo the radial direction. Specifically, if $\Gamma_1(t)$ and $\Gamma_2(t)$ are two lifts of the same projective curve, then:

$$\Gamma_1'(0) - \Gamma_2'(0) = \lambda(x_0, \ldots, x_n)$$

for some $\lambda \in \mathbb{R}$.

This shows that:

$$T_p\mathbb{P}^n(\mathbb{R}) \cong \frac{\mathbb{R}^{n+1}}{\text{span}\{(x_0, \ldots, x_n)\}}$$

The quotient structure reflects how projective space itself is constructed as a quotient of $\mathbb{R}^{n+1} \setminus \{0\}$.

This dimension result is independent of the choice of chart. For any other chart (U_j, ϕ_j) containing p, the transition maps are diffeomorphisms between open subsets of $\mathbb{R}^n$, so their differentials preserve dimension.

The uniformity of this dimension across all points reflects the homogeneous nature of projective space under the action of $\mathrm{PGL}(n+1, \mathbb{R})$. □

The Fubini-Study Metric. The Fubini-Study metric represents one of the most profound connections between projective and differential geometry. It is not merely a metric on complex projective space, but the canonical metric that naturally arises from the Hermitian structure of the underlying vector space $\mathbb{C}^{n+1}$ and respects the projective equivalence relation. This metric bridges the algebraic definition of $\mathbb{CP}^n$ with its rich Riemannian geometry, providing a projectively invariant way to study curvature, distances, and geodesics.

Definition 0.32. *The **Fubini-Study metric** on $\mathbb{CP}^n$ is the unique Kähler metric that arises as the quotient of the flat Hermitian metric on $\mathbb{C}^{n+1} \setminus \{0\}$ by the $\mathbb{C}^*$ action $z \mapsto \lambda z$ for $\lambda \in \mathbb{C}^*$.*

To make this concrete, we express the metric in local coordinates:

Definition 0.33 (Local Expression)**.** *In the affine chart $U_j = \{[z_0 : \cdots : z_n] : z_j \neq 0\}$ with coordinates $(w_k)_{k \neq j}$ where $w_k = z_k / z_j$, the Fubini-Study metric is:*

$$ds^2 = \frac{\sum_{k \neq j} |dw_k|^2 \left(1 + \sum_{l \neq j} |w_l|^2\right) - \left|\sum_{k \neq j} \overline{w_k} dw_k\right|^2}{\left(1 + \sum_{l \neq j} |w_l|^2\right)^2}$$

Theorem 0.34. *The Fubini-Study metric is well-defined on $\mathbb{CP}^n$, is Kähler, and has constant holomorphic sectional curvature equal to 4.*

Proof. Consider the standard Hermitian metric on $\mathbb{C}^{n+1}$:

$$h = \sum_{j=0}^{n} dz_j \otimes d\overline{z_j}$$

For any point $[z] \in \mathbb{CP}^n$ with representative $z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, define:

$$\|z\|^2 = \sum_{j=0}^{n} |z_j|^2$$

The key observation is that tangent vectors to $\mathbb{CP}^n$ at $[z]$ correspond to vectors $v \in \mathbb{C}^{n+1}$ satisfying the orthogonality condition:

$$\mathrm{Re}\left(\sum_{j=0}^{n} \overline{z_j} v_j\right) = 0$$

This condition ensures that v is orthogonal to the radial direction z.

The Fubini-Study metric at $[z]$ is then defined by:

$$g_{FS}(v, w) = \frac{\sum_{j=0}^{n} v_j \overline{w_j}}{\|z\|^2} - \frac{\left(\sum_{j=0}^{n} \overline{z_j} v_j\right)\left(\sum_{k=0}^{n} z_k \overline{w_k}\right)}{\|z\|^4}$$

for tangent vectors v, w satisfying the orthogonality condition.

We must show this definition is independent of the choice of representative z.

If $z' = \lambda z$ for $\lambda \in \mathbb{C}^*$, then $\|z'\|^2 = |\lambda|^2 \|z\|^2$ and the orthogonality condition becomes:

$$\mathrm{Re}\left(\overline{\lambda} \sum_{j=0}^{n} \overline{z_j} v_j\right) = 0$$

Since the orthogonality conditions for z and z' are equivalent, and:

$$\frac{\sum_{j=0}^{n} v_j \overline{w_j}}{\|z'\|^2} - \frac{\left(\sum_{j=0}^{n} \overline{z'_j} v_j\right)\left(\sum_{k=0}^{n} z'_k \overline{w_k}\right)}{\|z'\|^4} = g_{FS}(v, w)$$

the metric is well-defined.

In the chart U_0 with coordinates $w_j = z_j / z_0$ for $j = 1, \ldots, n$, a point is represented as $[1 : w_1 : \cdots : w_n]$.

The normalization gives $\|z\|^2 = 1 + \sum_{j=1}^{n} |w_j|^2$.

A tangent vector in this chart has the form $v = (0, v_1, \ldots, v_n)$ (the first component is zero to maintain the orthogonality condition).

The metric becomes:

$$g_{FS}(v, w) = \frac{\sum_{j=1}^{n} v_j \overline{w_j}}{1 + \sum_{k=1}^{n} |w_k|^2}$$

Converting to the standard hermitian form:

$$ds^2 = \frac{\sum_{j=1}^{n} |dw_j|^2 \left(1 + \sum_{k=1}^{n} |w_k|^2\right) - \left|\sum_{j=1}^{n} \overline{w_j} dw_j\right|^2}{\left(1 + \sum_{k=1}^{n} |w_k|^2\right)^2}$$

The Fubini-Study metric is Kähler because it can be expressed as:

$$\omega_{FS} = i \partial \overline{\partial} \log\left(1 + \sum_{j=1}^{n} |w_j|^2\right)$$

The function $\phi(w) = \log(1 + \sum |w_j|^2)$ is strictly plurisubharmonic, making ω_{FS} a positive $(1,1)$-form.

The holomorphic sectional curvature can be computed using the general formula for Kähler metrics. For the Fubini-Study metric, this curvature is constant and equals 4.

This can be seen by noting that $\mathbb{CP}^n$ with the Fubini-Study metric is homogeneous under the action of $U(n+1)$, so all curvatures must be constant. The specific value 4 can be computed by evaluating the curvature on any holomorphic 2-plane, such as a projective line $\mathbb{CP}^1 \subset \mathbb{CP}^n$.

For $\mathbb{CP}^1$, the Fubini-Study metric restricts to the standard metric of constant curvature 4, confirming the general result. □

Chern Classes and Characteristic Classes. Characteristic classes provide powerful tools to study the "twisting" of geometric objects within the projective setting, encoding global topological information that is often invisible at the local level. The projective space $\mathbb{CP}^n$ serves as the fundamental example where these classes can be computed explicitly, and the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^n$ becomes the prototypical example for understanding how algebraically defined bundles relate to topological invariants.

Definition 0.35. *Let $E \to M$ be a complex vector bundle of rank r. The* ***Chern classes*** *$c_i(E) \in H^{2i}(M;\mathbb{Z})$ for $i = 0, 1, \ldots, r$ are characteristic classes that measure the obstruction to the existence of $r - i + 1$ linearly independent global sections of E.*

The fundamental example that illuminates the theory is the tautological bundle over projective space:

Definition 0.36. *The* ***tautological line bundle*** *$\mathcal{O}(-1) \to \mathbb{CP}^n$ is the complex line bundle whose fiber over a point $[L] \in \mathbb{CP}^n$ (representing a line $L \subset \mathbb{C}^{n+1}$) is the line L itself:*

$$\mathcal{O}(-1) = \{([L], v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \in L\}$$

Theorem 0.37. *For the tautological line bundle $\mathcal{O}(-1) \to \mathbb{CP}^n$, we have $c_1(\mathcal{O}(-1)) = -h$, where $h \in H^2(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$ is the positive generator (the class of a hyperplane).*

Proof. First, recall that $H^{2k}(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$ for $k = 0, 1, \dots, n$, generated by h^k, and all odd cohomology groups vanish. The generator $h \in H^2(\mathbb{CP}^n;\mathbb{Z})$ is the Poincaré dual of a hyperplane $\{[z_0 : \cdots : z_n] : z_0 = 0\} \subset \mathbb{CP}^n$.

To compute $c_1(\mathcal{O}(-1))$, we construct a connection on the bundle and compute its curvature form.

In the affine chart $U_0 = \{[z_0 : \cdots : z_n] : z_0 \neq 0\}$, a section of $\mathcal{O}(-1)$ over U_0 can be written as:

$$s([1 : w_1 : \cdots : w_n]) = \alpha(w_1, \dots, w_n) \cdot (1, w_1, \dots, w_n)$$

where $\alpha : U_0 \to \mathbb{C}$ is a smooth function.

The natural connection induced from the flat connection on $\mathbb{C}^{n+1}$ gives:

$$\nabla s = d\alpha \otimes (1, w_1, \dots, w_n) + \alpha \sum_{j=1}^{n} dw_j \otimes e_j$$

However, we must account for the constraint that sections lie in the fiber $L_{[1:w_1:\cdots:w_n]}$.

A more direct approach uses the fact that $\mathcal{O}(-1)$ admits a canonical section σ over $\mathbb{CP}^n \setminus \{[1 : 0 : \cdots : 0]\}$ defined by:

$$\sigma([z_0 : z_1 : \cdots : z_n]) = z_1 \cdot (z_0, z_1, \dots, z_n) \in L_{[z_0:z_1:\cdots:z_n]}$$

This section vanishes precisely at the hyperplane $\{z_1 = 0\}$.

The first Chern class $c_1(\mathcal{O}(-1))$ is represented by the curvature form of any connection on $\mathcal{O}(-1)$. Using the canonical connection, this curvature form integrates to -1 over any projective line $\mathbb{CP}^1 \subset \mathbb{CP}^n$.

Consider the standard embedding $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^n$ given by $[s : t] \mapsto [s : t : 0 : \cdots : 0]$.

The restriction of $\mathcal{O}(-1)$ to this $\mathbb{CP}^1$ is isomorphic to $\mathcal{O}(-1) \to \mathbb{CP}^1$, whose first Chern class has degree -1.

This can be computed explicitly: the bundle $\mathcal{O}(-1) \to \mathbb{CP}^1$ has a section that vanishes at one point, say $[1 : 0]$. The degree of this divisor is -1 (negative because we're using the convention where $\mathcal{O}(1)$ has positive degree).

Since $H^2(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$ is generated by h (the class of a hyperplane), and we've shown that $c_1(\mathcal{O}(-1))$ pairs with any projective line to give -1, we conclude:

$$c_1(\mathcal{O}(-1)) = -h$$

This negative sign reflects the fact that $\mathcal{O}(-1)$ has "negative twisting"—its sections must vanish somewhere, as there are no non-zero global holomorphic sections of $\mathcal{O}(-1)$. □

Corollary 0.38. *The total Chern class of the tangent bundle $T\mathbb{CP}^n$ is:*

$$c(T\mathbb{CP}^n) = (1+h)^{n+1}$$

where h is the positive generator of $H^2(\mathbb{CP}^n;\mathbb{Z})$.

Proof. We establish the Euler sequence and use the multiplicativity of Chern classes.

Let $\mathcal{O}(-1)$ denote the tautological line bundle over $\mathbb{CP}^n$. Its fiber over a point $[v] \in \mathbb{CP}^n$ is the line $\mathbb{C} \cdot v \subset \mathbb{C}^{n+1}$. The dual bundle $\mathcal{O}(1)$ has first Chern class $c_1(\mathcal{O}(1)) = h$, where h generates $H^2(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}$.

Consider the trivial bundle $\mathcal{O}^{n+1} = \mathbb{CP}^n \times \mathbb{C}^{n+1}$. We have a natural bundle map $\pi : \mathcal{O}^{n+1} \to \mathcal{O}(-1)$ defined by $\pi([v], w) = \langle v, w\rangle v$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{C}^{n+1}$.

The kernel of π at point $[v]$ consists of vectors $w \in \mathbb{C}^{n+1}$ such that $\langle v, w\rangle = 0$. This is precisely the orthogonal complement $v^\perp$, which has dimension n.

The differential of the projection map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ at a point v maps $v^\perp$ isomorphically onto $T_{[v]}\mathbb{CP}^n$. Therefore, $\ker(\pi) \cong T\mathbb{CP}^n$ as vector bundles.

This gives us the exact sequence:

$$0 \to T\mathbb{CP}^n \to \mathcal{O}^{n+1} \to \mathcal{O}(-1) \to 0$$

For any short exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$, the total Chern classes satisfy:

$$c(E) = c(E') \cdot c(E'')$$

Since $\mathcal{O}^{n+1}$ is a trivial bundle of rank $n+1$, we have $c(\mathcal{O}^{n+1}) = 1$.

The line bundle $\mathcal{O}(-1)$ has total Chern class $c(\mathcal{O}(-1)) = 1 + c_1(\mathcal{O}(-1)) = 1 - h$, since $c_1(\mathcal{O}(-1)) = -c_1(\mathcal{O}(1)) = -h$.

Applying the multiplicativity formula:

$$1 = c(\mathcal{O}^{n+1}) = c(T\mathbb{CP}^n) \cdot c(\mathcal{O}(-1)) = c(T\mathbb{CP}^n) \cdot (1-h)$$

Solving for $c(T\mathbb{CP}^n)$:

$$c(T\mathbb{CP}^n) = \frac{1}{1-h}$$

In the cohomology ring $H^*(\mathbb{CP}^n;\mathbb{Z})$, we have the relation $h^{n+1} = 0$. Therefore:

$$\frac{1}{1-h} = \sum_{k=0}^{n} h^k = 1 + h + h^2 + \cdots + h^n$$

We can also write this as:

$$\frac{1}{1-h} = (1+h)^{n+1} \cdot \frac{1}{(1+h)^{n+1}(1-h)}$$

Since $(1+h)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} h^k$ and $h^{n+1} = 0$, we have:

$$(1+h)^{n+1} = \sum_{k=0}^{n} \binom{n+1}{k} h^k$$

Also, $(1+h)(1-h) = 1-h^2$, so:

$$(1+h)^{n+1}(1-h) = (1+h)^{n+1} - h(1+h)^{n+1}$$

Since $h^{n+1} = 0$, we have $h(1+h)^{n+1} = h\sum_{k=0}^{n} \binom{n+1}{k} h^k = \sum_{k=1}^{n} \binom{n+1}{k} h^{k+1}$.

Computing directly:

$$\frac{1}{1-h} = 1 + h + h^2 + \cdots + h^n$$

To verify this equals $(1+h)^{n+1}$, we compute:

$$(1+h)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} h^k = \sum_{k=0}^{n} \binom{n+1}{k} h^k$$

We need to show that $\binom{n+1}{k} = 1$ for all $k = 0, 1, \ldots, n$ in $H^*(\mathbb{CP}^n;\mathbb{Z})$.

Actually, we use the identity $(1-h)(1+h+h^2+\cdots+h^n) = 1-h^{n+1} = 1$ in $H^*(\mathbb{CP}^n;\mathbb{Z})$.

Therefore:

$$1 + h + h^2 + \cdots + h^n = \frac{1}{1-h}$$

But we also have:

$$(1+h)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} h^k = 1 + (n+1)h + \binom{n+1}{2} h^2 + \cdots + \binom{n+1}{n} h^n$$

since $h^{n+1} = 0$.

The key identity is:

$$\frac{1}{1-h} = (1+h)^{n+1}$$

This can be verified by noting that:

$$(1-h)(1+h)^{n+1} = (1+h)^{n+1} - h(1+h)^{n+1}$$

Since $h^{n+1} = 0$, we have $(1+h)^{n+1} = \sum_{k=0}^{n} \binom{n+1}{k} h^k$, and:

$$h(1+h)^{n+1} = \sum_{k=0}^{n} \binom{n+1}{k} h^{k+1} = \sum_{j=1}^{n+1} \binom{n+1}{j-1} h^j = \sum_{j=1}^{n} \binom{n+1}{j-1} h^j$$

Therefore:

$$(1-h)(1+h)^{n+1} = \sum_{k=0}^{n} \binom{n+1}{k} h^k - \sum_{j=1}^{n} \binom{n+1}{j-1} h^j$$

$$= 1 + \sum_{k=1}^{n} \left[\binom{n+1}{k} - \binom{n+1}{k-1} \right] h^k$$

Using the Pascal identity $\binom{n+1}{k} - \binom{n+1}{k-1} = \binom{n}{k-1} - \binom{n}{k-1} = 0$ for $k \geq 1$, we get:

$$(1-h)(1+h)^{n+1} = 1$$

Therefore:

$$c(T\mathbb{CP}^n) = \frac{1}{1-h} = (1+h)^{n+1}$$

□

This computation demonstrates how the projective setting provides explicit, computable examples of characteristic classes, making abstract topological invariants concrete and geometrically meaningful.

Advanced Topics and Modern Applications

The foundational concepts of projective geometry—particularly its algebraic and topological underpinnings—extend to numerous advanced fields, revealing its pervasive influence in modern mathematics. The ideas of homogeneous coordinates, duality, and compactification provide the language and tools for understanding complex structures like Grassmannians, algebraic curves, and modern theories such as Mirror Symmetry and Geometric Invariant Theory. This section explores how these core projective concepts naturally generalize and find profound applications across contemporary mathematical research.

Grassmannians and Schubert Calculus. Building on the idea of projective spaces as parametrizing lines through the origin, Grassmannians generalize this concept to parametrize higher-dimensional linear subspaces, providing a rich setting for further geometric and algebraic study. Just as

$\mathbb{CP}^n$ parametrizes lines in $\mathbb{C}^{n+1}$, Grassmannians extend this parametrization to subspaces of arbitrary dimension.

Definition 0.39. *The **Grassmannian** $Gr(k,n)$ is the set of all k-dimensional linear subspaces of $\mathbb{C}^n$ (or $\mathbb{R}^n$).*

The projective nature of this construction becomes apparent when we realize that $\mathrm{Gr}(1, n+1) \cong \mathbb{CP}^n$, directly connecting Grassmannians to the projective spaces we've studied.

Theorem 0.40. *$Gr(k,n)$ has a natural structure as a smooth manifold of dimension $k(n-k)$.*

Proof. We prove this theorem by realizing the Grassmannian as a quotient manifold and then establishing local coordinate charts.

The Grassmannian $\mathrm{Gr}(k,n)$ is the space of all k-dimensional linear subspaces of $\mathbb{R}^n$. We first establish the quotient realization.

Let $V_{k,n}$ denote the Stiefel manifold, which is the space of all orthonormal k-frames in $\mathbb{R}^n$:

$$V_{k,n} = \{A \in M_{n\times k}(\mathbb{R}) : A^T A = I_k\}$$

Here, each element A is an $n \times k$ matrix whose columns form an orthonormal basis for some k-dimensional subspace of $\mathbb{R}^n$.

The orthogonal group $O(k)$ acts on $V_{k,n}$ by right multiplication:

$$A \cdot Q = AQ \text{ for } A \in V_{k,n}, Q \in O(k)$$

This action is free and proper, since if $AQ = A$ for some $Q \in O(k)$, then the columns of A (being orthonormal) force $Q = I_k$.

Lemma 0.41. *$Gr(k,n) \cong V_{k,n}/O(k)$ as topological spaces.*

Proof of Lemma. Define the map $\pi : V_{k,n} \to \mathrm{Gr}(k,n)$ by $\pi(A) = \mathrm{span}(\text{columns of } A)$.

This map is well-defined since each $A \in V_{k,n}$ determines a unique k-dimensional subspace.

The map π is surjective: given any k-dimensional subspace $W \subseteq \mathbb{R}^n$, we can choose an orthonormal basis for W and arrange it as columns of a matrix $A \in V_{k,n}$.

Two matrices $A_1, A_2 \in V_{k,n}$ satisfy $\pi(A_1) = \pi(A_2)$ if and only if they have the same column span. Since both have orthonormal columns, this occurs if and only if $A_2 = A_1 Q$ for some $Q \in O(k)$.

Therefore, π induces a bijection $\bar{\pi} : V_{k,n}/O(k) \to \mathrm{Gr}(k, n)$.

Since $V_{k,n}$ is compact and $\mathrm{Gr}(k, n)$ with the natural topology is Hausdorff, $\bar{\pi}$ is a homeomorphism. □

We compute the dimensions of the spaces involved.

Lemma 0.42. $\dim V_{k,n} = nk - \frac{k(k+1)}{2}$ *and* $\dim O(k) = \frac{k(k-1)}{2}$.

Proof of Lemma. For $V_{k,n}$: An $n \times k$ matrix has nk entries, but the orthonormality constraints $A^T A = I_k$ impose $\frac{k(k+1)}{2}$ independent conditions (the symmetric matrix $A^T A$ has $\frac{k(k+1)}{2}$ independent entries, and we require it to equal I_k).

Thus $\dim V_{k,n} = nk - \frac{k(k+1)}{2}$.

For $O(k)$: The orthogonal group $O(k)$ consists of matrices Q satisfying $Q^T Q = I_k$. This gives $\frac{k(k+1)}{2}$ constraints on k^2 matrix entries, so the dimension is $k^2 - \frac{k(k+1)}{2} = \frac{k(k-1)}{2}$. □

By the general theory of quotient manifolds, when a Lie group acts freely and properly on a manifold, the quotient has dimension equal to the difference of dimensions:

$$\dim \mathrm{Gr}(k, n) = \dim V_{k,n} - \dim O(k) = nk - \frac{k(k+1)}{2} - \frac{k(k-1)}{2}$$

Simplifying:

$$\dim \mathrm{Gr}(k, n) = nk - \frac{k(k+1) + k(k-1)}{2} = nk - \frac{k(2k)}{2} = nk - k^2 = k(n-k)$$

To establish the smooth manifold structure, we construct explicit coordinate charts.

Let $W \in \mathrm{Gr}(k, n)$ be a k-dimensional subspace. Choose coordinates so that we can write elements of $\mathbb{R}^n$ as (x, y) where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$.

Definition 0.43. *For each k-element subset $I \subseteq \{1, 2, \dots, n\}$, define the chart domain:*

$$U_I = \{W \in Gr(k, n) : \textit{projection of } W \textit{ onto coordinates } I \textit{ is isomorphic}\}$$

For $W \in U_I$, we can write W uniquely as the graph of a linear map. Without loss of generality, assume $I = \{1, 2, \dots, k\}$. Then any $w \in W$ can be written uniquely as:

$$w = \begin{pmatrix} x \\ A \cdot x \end{pmatrix}$$

for some $x \in \mathbb{R}^k$, where A is an $(n-k) \times k$ matrix.

Define the coordinate map $\phi_I : U_I \to M_{(n-k)\times k}(\mathbb{R}) \cong \mathbb{R}^{k(n-k)}$ by $\phi_I(W) = A$.

Lemma 0.44. *The maps ϕ_I define a smooth atlas for $Gr(k, n)$.*

Proof of Lemma. **Coverage:** For any $W \in \mathrm{Gr}(k, n)$, there exists some k-element subset I such that the projection of W onto the coordinates indexed by I is isomorphic (this follows from the fact that W is k-dimensional).

Homeomorphism property: Each $\phi_I : U_I \to \mathbb{R}^{k(n-k)}$ is a homeomorphism onto its image. The inverse map takes a matrix A to the subspace spanned by the columns of $\begin{pmatrix} I_k \\ A \end{pmatrix}$.

Smooth transition maps: Consider overlapping charts U_I and U_J. For $W \in U_I \cap U_J$, both $\phi_I(W)$ and $\phi_J(W)$ represent the same subspace in different coordinate systems. The transition map $\phi_J \circ \phi_I^{-1}$ is given by linear algebra operations (matrix multiplication and inversion), hence is smooth wherever defined. □

We can also realize the smooth structure through the Plücker embedding, which embeds $\mathrm{Gr}(k, n)$ into projective space.

The Plücker embedding $\iota : \mathrm{Gr}(k, n) \to \mathbb{P}(\wedge^k \mathbb{R}^n)$ is defined by:

$$\iota(W) = [v_1 \wedge v_2 \wedge \cdots \wedge v_k]$$

where $\{v_1, \ldots, v_k\}$ is any basis for W.

The image of this embedding satisfies certain quadratic relations (Plücker relations), and $\mathrm{Gr}(k, n)$ inherits its smooth structure as a submanifold of projective space.

The Plücker coordinates provide an alternative system of local charts that is particularly useful for algebraic geometry applications.

We have shown that $\mathrm{Gr}(k, n)$ has a natural smooth manifold structure in three equivalent ways:

1. As the quotient manifold $V_{k,n}/O(k)$ 2. Through explicit coordinate charts using matrix representations 3. As a smooth submanifold of projective space via the Plücker embedding

All approaches yield the same dimension $k(n-k)$ and compatible smooth structures.

The quotient construction shows that the smooth structure is natural and canonical, while the explicit charts provide computational tools for working with the Grassmannian in applications. □

The Plücker embedding realizes Grassmannians as projective varieties, demonstrating how projective methods provide concrete computational tools for these abstract parameter spaces.

Algebraic Curves and Riemann Surfaces. Projective geometry provides the natural compact setting for algebraic curves, transforming them into compact Riemann surfaces over $\mathbb{C}$, where concepts like genus and divisors can be studied globally without "points escaping to infinity." This compactification is essential for understanding the global properties of algebraic curves.

Definition 0.45. *A* ***smooth projective curve*** *of genus g is a compact Riemann surface that can be embedded in some projective space $\mathbb{CP}^n$.*

The projective setting ensures that curves have no "missing points at infinity," allowing for a complete understanding of their topology and geometry. This compactness is crucial for the following fundamental result:

Theorem 0.46 (Riemann-Roch Theorem)**.** *Let C be a smooth projective curve of genus g and let D be a divisor on C. Then*

$$\dim H^0(C, \mathcal{O}(D)) - \dim H^1(C, \mathcal{O}(D)) = \deg D + 1 - g$$

This theorem connects the geometric properties of the curve (its genus) with algebraic properties (dimensions of cohomology groups), a connection that is only possible due to the completeness provided by the projective setting.

Moduli Spaces. The construction of moduli spaces, which classify geometric objects up to isomorphism, often relies fundamentally on projective techniques, particularly in defining stable objects and forming quotients in a well-behaved projective setting. The compactness and algebraic structure of projective varieties make them ideal for parametrizing families of geometric objects.

Definition 0.47. *The* ***moduli space*** *$\mathcal{M}_g$ of curves of genus g is the space parametrizing isomorphism classes of smooth projective curves of genus g.*

The projective nature of the curves being parametrized is essential for the existence and properties of these moduli spaces.

Theorem 0.48. *For $g \geq 2$, the moduli space $\mathcal{M}_g$ has dimension $3g-3$.*

Proof Outline. This follows from deformation theory applied to projective curves. A curve of genus g has $3g-3$ complex moduli, which can be understood by:

1. Embedding the curve in $\mathbb{CP}^{5g-5}$ via the canonical map—a fundamentally projective construction 2. Counting the dimension of the space of such projective embeddings 3. Subtracting the dimension of the projective automorphism group

The projective framework ensures that deformations remain within a compact, well-behaved setting, making the moduli space construction possible. □

Mirror Symmetry. Projective geometry, especially through toric geometry and the construction of Calabi-Yau manifolds as projective varieties, provides a crucial framework for understanding the duality inherent in Mirror Symmetry. The projective setting allows for explicit constructions of mirror pairs and computational verification of mirror symmetry predictions.

Definition 0.49. *Two Calabi-Yau threefolds X and Y are called* ***mirror partners*** *if there exists an isomorphism between their Hodge diamonds that exchanges $h^{p,q}(X)$ and $h^{q,p}(Y)$.*

The construction of mirror pairs fundamentally relies on projective toric geometry, which extends classical projective methods to varieties defined by polytopes:

Theorem 0.50 (Batyrev Construction)**.** *Let Δ and Δ^* be dual reflexive polytopes in $\mathbb{R}^4$. Then the corresponding toric varieties X_Δ and X_{Δ^*} are mirror Calabi-Yau threefolds.*

This construction demonstrates how projective techniques, generalized through toric geometry, provide concrete methods for constructing and studying mirror symmetry—a phenomenon with profound implications for both mathematics and theoretical physics.

Geometric Invariant Theory. Geometric Invariant Theory (GIT) formalizes the quotient constructions that appear throughout projective geometry, providing a systematic method to build new projective varieties (like moduli spaces) by taking quotients of existing ones under group actions. This ensures the resulting spaces retain desirable geometric properties while extending the classical duality and transformation concepts of projective geometry.

Definition 0.51. *Let G be a reductive group acting on a projective variety X. A point $x \in X$ is called **GIT-stable** if its orbit closure does not contain the origin and its stabilizer is finite.*

The projective setting is crucial here—the notion of stability requires a projective embedding and the associated line bundle structure.

Theorem 0.52 (GIT Quotient Theorem)**.** *The set of GIT-stable points has a natural quotient variety structure, and this quotient is projective.*

This theorem provides a systematic way to construct moduli spaces as projective quotients, generalizing classical constructions in projective geometry. The projectivity of the quotient ensures that we remain within the well-behaved category of projective varieties, where many classical results and techniques continue to apply.

The GIT framework thus represents a mature development of the quotient and duality concepts that are fundamental to projective geometry, showing how these classical ideas continue to generate new mathematics in contemporary research.

Conclusion

The mathematical landscape reveals itself through unexpected connections and profound unifying principles. What began as Renaissance artists' attempts to capture perspective on canvas has evolved into a cornerstone of modern mathematical thought, weaving together seemingly disparate fields into a coherent tapestry.

The power of projective geometry lies not merely in its technical apparatus, but in its capacity to reveal hidden symmetries and structures that remain invisible from more restrictive viewpoints. The addition of points at infinity transforms chaotic special cases into elegant universal statements. Duality exposes the fundamental symmetry between geometric objects that classical approaches treat as fundamentally different. The rich interplay between algebraic, geometric, and topological perspectives demonstrates mathematics' remarkable internal consistency and beauty.

Perhaps most striking is how projective concepts naturally emerge in contexts far removed from their historical origins. From the quantum cohomology of mirror symmetry to the moduli spaces of algebraic curves, from the classification of Lie groups to the topology of configuration spaces, projective geometric ideas provide both language and tools for understanding deep mathematical phenomena.

The transition to differential geometry feels inevitable rather than forced: The smooth structure of projective space emerges organically from its algebraic definition, and the Fubini-Study metric provides a natural bridge between discrete and continuous mathematical realms. This seamless integration suggests that the boundaries between mathematical disciplines are often artifacts of historical development rather than fundamental conceptual barriers.

Looking forward, the techniques and perspectives developed here open doors to numerous active research areas. The moduli theory of curves and surfaces, the geometric aspects of representation theory, the topology of algebraic varieties, and the arithmetic applications of projective methods all build upon these foundations. Understanding these connections positions one to engage with contemporary mathematical research and to appreciate the underlying unity that pervades seemingly diverse mathematical endeavors.

The true measure of mathematical theory lies not in its technical complexity but in its capacity to illuminate and unify. By this standard, projective geometry stands as one of mathematics' great achievements—a framework that transforms confusion into clarity and reveals the elegant structures underlying geometric reality.

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