Incidence Geometry in $\mathbb{R}^3$ Via Polynomial Partitioning

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Preliminaries

Definition

A real polynomial $P(x_1, x_2, \dots x_n)$ of degree D is a continuous map of form $\sum_{\alpha_1+\alpha_2+\dots+\alpha_n \leq D} c_i x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ with $c_i \in \mathbb{R}$.

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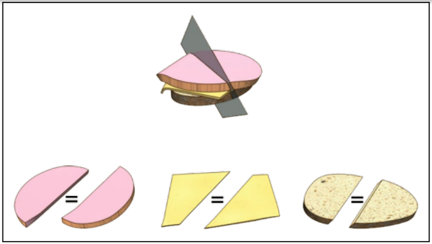
If P is a polynomial, let $Z(P)$ denote the zero set of of P (i.e. where P vanishes).

Background

General Ham Sandwich Theorem

- Let V be a vector space of continuous functions on $\mathbb{R}^n$.
- Let $U_1, \ldots, U_N \subset \mathbb{R}^n$ be finite-volume open sets
- Let $N < \dim V$.
- Suppose that for every nonzero $f \in V$, the zero set $Z(f)$ has Lebesgue measure zero.

Then there exists a nonzero function $f \in V \setminus \{0\}$ that bisects each set U_i.



Background

General Ham Sandwich Theorem

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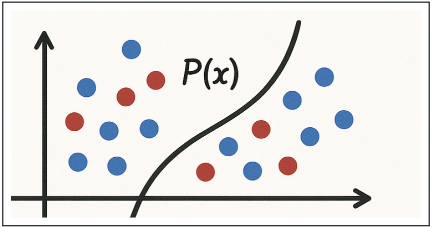
Corollary (Finite Polynomial Ham Sandwich)

Let $S_1, \ldots, S_N \subset \mathbb{R}^n$ be finite sets of points in $\mathbb{R}^n$ with

$$N < \binom{D+n}{n} = \dim \mathrm{Poly}_D(\mathbb{R}^n).$$

Then there exists a non-zero $P \in \mathrm{Poly}_D(\mathbb{R}^n)$ that bisects each set S_i.

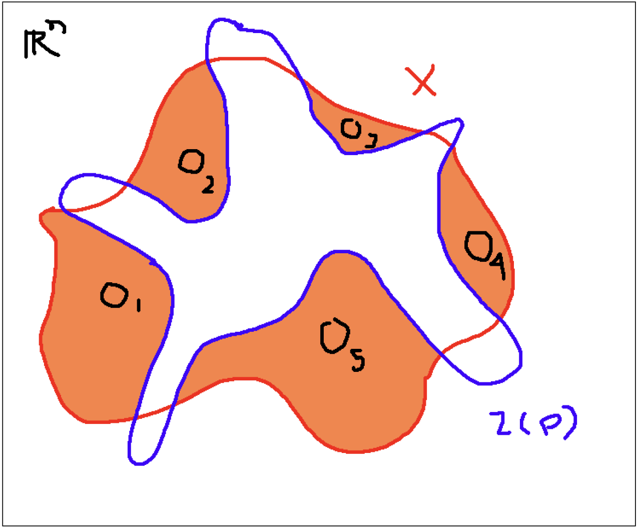


Polynomial Partitioning

Theorem (Polynomial Partitioning)

For any dimension n, we can choose $C(n)$ such that the following holds. If X is any finite subset of $\mathbb{R}^n$ and D is any degree, then there is a non-zero polynomial $P \in \text{Poly}_D(\mathbb{R}^n)$ such that $(\mathbb{R}^n \backslash Z(P)) \cap X = \bigcup O_i$ with $|O_i| \leq C(n)|X|D^{-n}$.

Polynomial Partitioning



Each O_i is bounded above by $C(n)|X|D^{-n}$.

Polynomial Partitioning

Proof of $|P_r(\mathfrak{L})| \leq C(n)|X|D^{-n}$

- Using the Polynomial Ham Sandwich Theorem, repeatedly bisect X with j polynomials $P_1, \ldots P_j$ of degree $\leq C(n)2^{j/n}$
- Pick j such that $\deg(P_1 \ldots P_j) \leq C(n) \sum_{i=1}^{j} 2^{j/n} \leq D$
- This yields 2^j different cells, with each containing at most $|X|/2^j \leq C(n)|X|D^{-n}$, as desired.

The Szemerédi-Trotter Bound

Definition

If $\mathfrak{L}$ is a set of lines, a point x is called an r-rich point of $\mathfrak{L}$ if x lies in at least r lines of L. The set of r-rich points of $\mathfrak{L}$ is denoted $P_r(\mathfrak{L})$.

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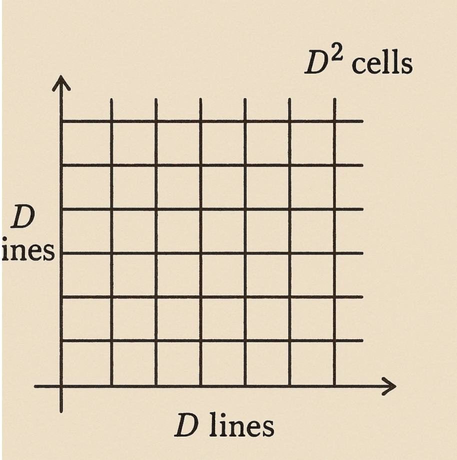
Theorem (Szemerédi-Trotter, 1983)

Suppose that $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$, $P \in \text{Poly}_D(\mathbb{R}^3)$, and that $Z(P)$ contains at most B lines of $\mathfrak{L}$. Then

$$|P_r(\mathfrak{L}) \cap Z(P)| \lesssim DLr^{-1} + B^2r^{-3}.$$

The Szemerédi-Trotter Bound



Expected for random lines.

Bounding $|\mathrm{P}_r(\mathfrak{L})|$

Lemma

If $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^n$ and $r > 2L^{1/2}$, then

$$|\mathcal{P}_r(\mathfrak{L})| \leq 2Lr^{-1}.$$

Bounding $|\mathrm{P}_r(\mathfrak{L})|$

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Theorem (Guth-Katz, 2014)

For any $\varepsilon > 0$, there exists a degree $D = D(\varepsilon)$ and a constant $C(\varepsilon)$ such that the following holds. Suppose that $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$ with at most B lines in any algebraic surface of degree $\leq D$. Then for any $2 \leq r \leq 2L^{1/2}$,

$$|\mathcal{P}_r(\mathfrak{L})| \leq C(\varepsilon) B^{\frac{1}{2}-\varepsilon} L^{\frac{3}{2}+\varepsilon} r^{-2}.$$

A Potential Better Bound

Theorem (Guth-Katz, 2014)

For any $\varepsilon > 0$, there exists a degree $D = D(\varepsilon)$ and a constant $C(\varepsilon)$ such that the following holds. Suppose that $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$ with at most B lines in any algebraic surface of degree $\leq D$. Then for any $2 \leq r \leq 2L^{1/2}$,

$$|\mathcal{P}_r(\mathfrak{L})| \leq C(\varepsilon) B^{\frac{1}{2}-\varepsilon} L^{\frac{3}{2}+\varepsilon} r^{-2}.$$

- Small B (i.e. $B \lesssim \log L$) yields $|P_r(\mathfrak{L})| \lesssim_\varepsilon L^{3/2+\varepsilon} r^{-2}$, suggesting a sharper bound.

A Shaper Bound By Induction on L

Theorem (Guth-Katz, 2014)

For any $\varepsilon > 0$, there are constants $D(\varepsilon)$ and $C(\varepsilon)$ such that the following holds. If $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$ with at most $L^{1/2+\varepsilon}$ lines in any algebraic surface of degree $\leq D(\varepsilon)$, then

$$|\mathcal{P}_r(\mathfrak{L})| \leq C(\varepsilon)L^{3/2+\varepsilon}r^{-2} + 2Lr^{-1}.$$

A Shaper Bound By Induction on L

Proof Sketch of $|\mathcal{P}_r(\mathfrak{L})| \leq C(\varepsilon)L^{3/2+\varepsilon}r^{-2} + 2Lr^{-1}$

- Induct on L, partition with polynomials, and collect the lines into subfamilies $\mathfrak{L}_i$ by cell.
- We run into an issue: even though no low-degree surface carries too many lines overall, it's possible that one cell's lines all lie on a single small-degree surface.
- To resolve this, we first peel off and control any such "bad" surfaces before applying the inductive bound to the remaining lines.

Sufficiently Rich Points on Algebraic Curves

Theorem (Sharir-Solomon, 2018)

Let P be a set of m points and C a set of n irreducible algebraic curves of constant degree E, from a constructible family $\mathcal{C}_0$ with k degrees of freedom and multiplicity μ in $\mathbb{R}^3$, such that no surface infinitely ruled by curves of $\mathcal{C}_0$ contains more than q curves of C, and assume $\mathcal{C}_0$ has reduced dimension s. Then,

$$I(P,C) = O\left(\frac{m^k n^{3k-3}}{(3k-2)} + m^{2/3}n^{1/3}q^{1/3} + \frac{m^{2s}n^{3s-4}q^{2s-2}}{5s-4} + \varepsilon + m + n\right),$$

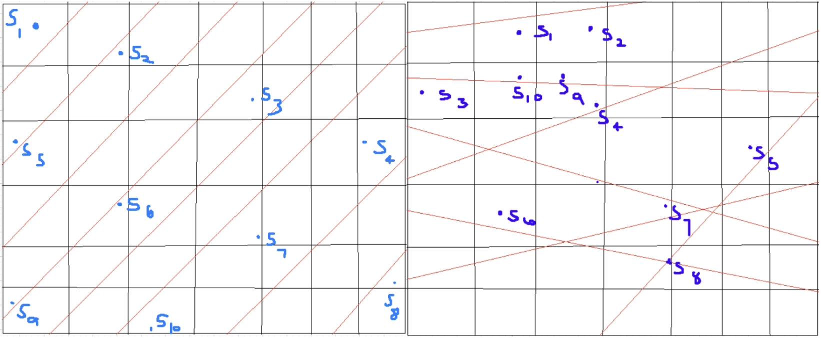
for any $\varepsilon > 0$.

- Aligns with Guth-Katz ($E = 1, k = 2, \mu = 1, s = 4$)

Takeaways

A Special Case (The Cutting Method)

Split the plane by D auxiliary lines and reduce the problem to smaller cells.



Lines per cell $\sim \log L$, not L/D

- Polynomial Partitioning and induction are often useful when intuition about the cutting method fails.

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