INCIDENCE GEOMETRY IN $\mathbb{R}^3$ VIA POLYNOMIAL PARTITIONING

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ABSTRACT. We aim to study incidences in $\mathbb{R}^3$, and in particular bound them under various conditions. To do this, we use polynomial partitioning. We present two bounds on r-rich points, one by induction on the polynomial degree D, and one on the number of lines. More generally than r-rich points, we discuss the classical Szemerédi-Trotter theorem on incidences , as well as its extensions, much of which is the work of Guth and Katz from 2014. We also generalize this bound to algebraic curves in $\mathbb{R}^3$ using the work of Sharir and Solomon.

1. Introduction

The polynomial method in combinatorics is a relatively new technique, often used to describe the structure of combinatorial objects using the algebraic properties of polynomials. The Polynomial Method saw major development in the past two decades, especially after a series of groundbreaking results, after it trivialized the proofs of few infamous open problems. These problems tend to have a very geometric structure:

Theorem 1.1 (Finite Field Kakeya). *Define a Kakeya set in $\mathbb{R}^2$ to be any set which contains a unit line segment in each direction, and let $E \subset F^n$ be a Kakeya set. Then*

$$|E| > c(n)|F|^n.$$

In 2008, Dvir [3] proved this theorem in 4 pages, resolving a long-standing conjecture in finite field geometry. The Kakeya problem over $\mathbb{R}^n$ remains open in full generality, but Dvir's proof placed him as one of the pioneers of the Polynomial Method. Another seminal result was by Guth and Katz [5], who applied polynomial partitioning to the Erdös distinct distances problem. Erdös had conjectured that any set of n points in the plane determines at least $\Omega(n/\sqrt{\log n})$ distinct distances. Guth and Katz nearly resolved this by showing that the number of distinct distances is at least $n^{1-o(1)}$.

Prior to their work, the best known bound was only $\Omega(n^{0.8641})$ due to Solymosi and Töth [12]. Guth and Katz's approach introduced a geometric and algebraic perspective, interpreting the distances as incidences between points and algebraic surfaces in $\mathbb{R}^3$. This was one of the first major applications of the polynomial method.

Within the Polynomial Method, several tools exist. Arguably the most well known of these tools is Alon's Combinatorial Nullstellensatz [1], which gives conditions for the nonvanishing of a polynomial over a finite field. However, in this paper, we will discuss a different tool, namely, Polynomial Partitioning.

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Polynomial partitioning is especially effective for problems of this geometric type, where we focus on the relationships between sets of points. In the late 1900s, these kinds of problems were unclassified, but today are part of a field known as incidence geometry. Essentially, polynomial partitioning claims the existence of a polynomial whose zero set can split some subspace $X \subset \mathbb{R}^n$ into bounded components. Choosing a polynomial of degree D imposes roughly $\binom{D+n}{n}$ conditions so we can enforce up to $\binom{D+n}{n}$ constraints to balance point-sets or line-sets across cells. This is what makes polynomial partitioning so much more effective than random ad-hoc cuts.

Work of this type has led to many classical results such as the Szémerdi-Trotter theorem [14]. The theorem claims a bound on the number of points at the intersection of r lines. Before polynomial partitioning, one approach towards proving the Szemerédi-Trotter bound was the cutting method. Namely, given L lines and S points, we expect that cutting a space into D^2 random cells should yield about L/D lines and S/D^2 points per cell. However, this is far from true. The intuition behind polynomial partitioning is the desire to reduce a problem in X to smaller cells within X.

The paper is structured in the following way: in Section 2, relevant terminology and notation is introduced. Additionally, we state some key properties relating to points in projection theory, like critical or flat. In Section 3, we formalize and prove the idea of polynomial partitioning. In Section 4 and 5, we show two different incidence bounds on lines. In section 6, we provide a generalization to algebraic curves. One currently open conjecture is that the number of incidences given n points and E dimensional algebraic curves is bounded above by $m^{2/3}n^{2/3} + m + n$. We show some theorems that come close to this bound.

2. Definitions and Notation

First we introduce some notation that we use throughout the paper.

- For two quantities A, B, we write $A \lesssim B$ to mean there exists $C = C(n) > 0$ (depending only on the dimension n) so that $A \le C\,B$. We write $A \sim B$ if $A \lesssim B \lesssim A$.
- $\mathrm{Poly}_D(\mathbb{R}^n)$ denotes the vector space of real polynomials in n variables of total degree at most D.
- If $P \in \mathrm{Poly}_D(\mathbb{R}^n)$, its *zero set* is $Z(P) = \{x \in \mathbb{R}^n : P(x) = 0\}$, and $\deg P$ is the total degree of P.

Next we introduce basic definitions needed for incidence geometry.

Definition 2.1 (r-rich Points)**.** For $r \in \mathbb{N}$, the *r–rich points* of $\mathcal{L}$ are

$$P_r(\mathcal{L}) = \{\, x \in \mathbb{R}^n : x \text{ lies on at least } r \text{ lines of } \mathcal{L}\}.$$

Definition 2.2 (Incidence)**.** Given a finite point set $P \subset \mathbb{R}^n$ and a finite line set $\mathcal{L}$, the *incidence count* is

$$I(P, \mathcal{L}) = \left|\{(p, \ell) \in P \times \mathcal{L} : p \in \ell\}\right|.$$

Definition 2.3 (Algebraic surface)**.** An algebraic surface is a two-dimensional algebraic variety.

Definition 2.4 (Flecnodes)**.** Fix a constructible set $\mathcal{C}_0 \subset \mathbb{C}_E^3$ of irreducible curves of degree at most E in three-dimensional space, and let f be a trivariate polynomial. We call a point $p \in Z(f)$ a $(t, \mathcal{C}_0, r)$-*flecnode* if there exist at least t curves $\gamma_1, \ldots, \gamma_t \in \mathcal{C}_0$ such that, for each $i = 1, \ldots, t$:

(1) γ_i is incident to p,
(2) p is a non-singular point of γ_i, and
(3) γ_i osculates to $Z(f)$ to order r at p.

This is a generalization of the notion of a *flecnodal point*, by Salmon [9]. The main intuition is that a $(t, \mathcal{C}_0, r)$-flecnode is a marker where the algebraic surface aligns with many curves to a very high order of tangency.

When we work with algebraic curves, we need the notion of constructibility. Informally, a set $Y \subset \mathbb{C}$ is constructible if it is a Boolean combination of algebraic sets.

Formally, this means the following.

Definition 2.5 (Constructibility)**.** For $z \in \mathbb{C}$, define $v(z) = \begin{cases} 0, & z = 0, \\ 1, & z \neq 0. \end{cases}$ A subset $Y \subset \mathbb{C}^d$ is called a *constructible set* if there exist

- a finite collection of polynomials $f_j : \mathbb{C}^d \to \mathbb{C}$, for $j = 1, \ldots, J_Y$, and
- a subset $B_Y \subset \{0,1\}^{J_Y}$,

such that

$$x \in Y \quad \Longleftrightarrow \quad \big(v(f_1(x)), v(f_2(x)), \ldots, v(f_{J_Y}(x))\big) \in B_Y.$$

When we apply this to a set of curves, we think of them as points in some parametric (complex) d-space, where ds is the number of parameters needed to specify a curve.

Now we will define a few properties of points in projection theory.

Definition 2.6 (Critical Points)**.** A point x is a critical point of P if all the partial derivatives of P vanish at x.

Definition 2.7 (Flat Points)**.** A point x in a smooth surface in $\mathbb{R}^3$ is called a flat point if there is a plane that is tangent to the surface at x to second order.

We say a point $x \in Z(P)$ is special if x is critical or flat. The following lemma is well known.

Lemma 2.8 (Plane detection)**.** *Plane detection lemma. For any polynomial $P \in \mathbb{R}[x_1, x_2, x_3]$, we can associate a list of polynomials S_P with the following properties.*

(1) *If $x \in Z(P)$ then $S_P(x) = 0$ iff x is critical or flat.*
(2) *If x is contained in three lines in $Z(P)$, then $S_P(x) = 0$.*
(3) $\deg S_P \leq 3 \deg P$.
(4) *If P is irreducible and S_P vanishes on $Z(P)$ and $Z(P)$ contains a regular point, then $Z(P)$ is a plane.*

This will become useful when we deal with the non-planar parts of a zero set $Z(P)$ in Theorem 5.5.

3. Polynomial Partitioning

Now we will introduce the idea of polynomial partitioning, which relies on a theorem about the existence of a polynomial whose zero set satisfies certain conditions. To prove this, we need some preliminary claims.

Lemma 3.1 ([2], Theorem 3)**.** *For every continuous map $f : S^n \to \mathbb{R}^n$, there is some x with $f(x) = f(-x)$.*

Proof. Assume for the sake of contradiction $f(x) \neq f(-x)$ for all x. Then

$$g(x) = \frac{f(x) - f(-x)}{||f(x) - f(-x)||}$$

defines a continuous odd map $g : S^n \to S^{n-1}$, which cannot exist. Thus there exists x such that $f(x) = f(-x)$. ∎

Lemma 3.2 ([13], Theorem 2.1)**.** *Let V be a vector space of continuous functions on $\mathbb{R}^n$. Let $U_1, \ldots, U_N \subset \mathbb{R}^n$ be finite-volume open sets with $N < \dim V$. Suppose that for every nonzero $f \in V$, the zero set $Z(f)$ has Lebesgue measure zero. Then there exists a nonzero function $f \in V \setminus \{0\}$ that bisects each set U_i.*

Proof. For each $i = 1$ to N define

$$\phi_i(F) = \mathrm{Vol}(\{x \in U_i : F(x) > 0\}) - \mathrm{Vol}(\{x \in U_i : F(x) < 0\}),$$

so ϕ_i is antipodal and continuous on $V \setminus \{0\}$.

Assemble them into

$$\Phi(F) = (\phi_1(F), \ldots, \phi_N(F)) : V \setminus \{0\} \to \mathbb{R}^N.$$

Since $\dim V = N + 1$, identify $S^N \subset V$ as the unit sphere. Then $\Phi : S^N \to \mathbb{R}^N$ is continuous and odd. By Lemma 3.1, there exists $F \in S^N$ with $\Phi(F) = 0$, hence each $\phi_i(F) = 0$. So F bisects each U_i. It remains to show ϕ is continuous.

Let $f_k \to f \in V$. Then $f_k \to f$ pointwise and uniformly off a set of arbitrarily small measure. Given $\varepsilon > 0$, choose $\delta > 0$ so that $\mathrm{Vol}(\{x \in U : |f(x)) < \delta\}) < \varepsilon/2$. Then for large k, $|f_k(x) - f(x)| < \delta$ outside a set of volume less than $\varepsilon/2$. Hence $\mathrm{Vol}(\{x \in U : f_k(x) > 0\}) - \mathrm{Vol}(\{x \in U : f(x) > 0\}) < \varepsilon$.

∎

In proving the polynomial partitioning theorem, will only employ a special case of Lemma 3.2, namely, the Finite Polynomial Ham Sandwich Theorem

Lemma 3.3 ([7], Theorem 2.5)**.** *Let $S_1, \ldots, S_N \subset \mathbb{R}^n$ be finite sets of points in $\mathbb{R}^n$ with*

$$N < \binom{D+n}{n} = \dim \mathrm{Poly}_D(\mathbb{R}^n).$$

Then there exists a non-zero $P \in \mathrm{Poly}_D(\mathbb{R}^n)$ that bisects each set S_i.

Proof. For each $\delta > 0$, let

$$U_{i,\delta} = \bigcup_{x \in S_i} B_\delta(x),$$

be the union of δ-balls around the points of S_i. By Lemma 3.3, there exists a nonzero polynomial P_δ of degree at most D that bisects each $U_{i,\delta}$. Rescaling, we may assume $P_\delta \in S^N \subset \mathrm{Poly}_D(\mathbb{R}^n) \setminus \{0\}$.

Since S^N is compact, choose a sequence $\delta_m \to 0$ so that $P_{\delta_m} \to P \in S^N$. Uniform convergence of coefficients implies $P_{\delta_m} \to P$ uniformly on compact sets.

We show P bisects each finite set S_i. Suppose for the sake of contradiction that for some i, the set

$$S_i^+ = \{x \in S_i : P(x) > 0\}$$

has more than half the points of S_i. Then pick $\varepsilon > 0$ small enough that

- $P > \varepsilon$ on the ε-ball around each $x \in S_i^+$, and these balls are disjoint;
- for large m, $|P_{\delta_m} - P| < \varepsilon$ on all such balls;
- also $\delta_m < \varepsilon$.

So for sufficiently large m, $P_{\delta_m} > 0$ on each of those balls, hence on more than half of U_{i,δ_m}. Contradiction. Therefore P bisects S_i, as claimed. ■

We will repeatedly apply this in the proof of Polynomial Partitioning.

Theorem 3.4 (Polynomial Partitioning)**.** *For any dimension n, we can choose $C(n)$ such that the following holds. If X is any finite subset of $\mathbb{R}^n$ and D is any degree, then there is a non-zero polynomial $P \in Poly_D(\mathbb{R}^n)$ such that $\mathbb{R}^n \backslash Z(P) = \bigcup_{i=1}^k O_i$ with*

$$|O_i \cap X| \leq C(n)|X|D^{-n}$$

Proof. To start, we find a polynomial P_1 of degree at most 1 that bisects S. That is, the number of points of S in the open sets $\{x : P_1(x) > 0\}$ and $\{x : P_1(x) < 0\}$ are each at most $|S|/2$. Define

$$S^+ = \{x \in S : P_1(x) > 0\}, \quad S^- = \{x \in S : P_1(x) < 0\}.$$

Now apply Lemma 3.3 to the sets S^+ and S^-: we obtain a polynomial P_2 of controlled degree that bisects both S^+ and S^-. This results in four open regions of $\mathbb{R}^n$ determined by the sign patterns of (P_1, P_2). Each region contains at most $|S|/4$ points of S, and together these regions cover $\mathbb{R}^n \setminus Z(P_1P_2)$.

Continuing inductively, at stage j, we have 2^{j-1} subsets of S, each associated with a distinct sign condition on $P_1, \dots, P_{j-1}$, and each with at most $|S|/2^{j-1}$ points. By Lemma 3.3, we can find a polynomial P_j of degree at most $C(n)2^{j/n}$ that simultaneously bisects each of these subsets. Then $\mathbb{R}^n \setminus Z(P_1 \cdots P_j)$ decomposes into 2^j open regions, and each region contains at most $|S|/2^j$ points of S.

We iterate this process J times, yielding polynomials $P_1, \dots, P_J$ and a final polynomial

$$P = P_1 P_2 \cdots P_J.$$

The zero set $Z(P)$ partitions $\mathbb{R}^n$ into 2^J open cells, each of which contains at most $|S|/2^J$ points of S.

To bound the degree of P, we see that

$$\deg P \leq \sum_{j=1}^{J} \deg P_j \leq \sum_{j=1}^{J} C(n)2^{j/n}.$$

This is a geometric sum whose final term is comparable to the whole sum, so

$$\deg P \leq C(n)2^{J/n}.$$

Solving for J such that $\deg P \leq D$ gives $2^J \leq (D/C(n))^n$. Hence, the number of points of S in each open cell is at most

$$|S|/2^J \leq C(n)|S|D^{-n}.$$

This completes the construction. ■

We may strengthen Theorem 3.4 by bounding k

Theorem 3.5 ([8] Theorem 2). *If $P \in Poly_D(\mathbb{R}^n)$, then $\mathbb{R}^n \backslash Z(P)$ has $O(D^n)$ connected components.*

Proof. We proceed by induction on n. Let $k = C(n, D)$ denote the maximum number of connected components of $\mathbb{R}^n \setminus Z(P)$ when $\deg P \leq D$.

If $n = 1$, then P is a real univariate polynomial of degree at most D, so has at most D real roots. Hence $\mathbb{R} \setminus Z(P)$ is a union of at most $D+1$ open intervals, and

$$C(1, D) \leq D + 1 = O(D).$$

Now assume the result holds in dimension $n-1$, so $C(n-1, D) \leq C_{n-1}D^{n-1}$. Let $P(x_1, \ldots, x_n)$ be any real polynomial of degree at most D. Fix a generic unit vector $v \in S^{n-1}$ and write $t = v \cdot x$ as the coordinate along v, and y for the orthogonal coordinates. Consider the one-dimensional restriction in the t-direction

$$f_t(y) = P(tv + y),$$

viewed as a family of polynomials in $y \in \mathbb{R}^{n-1}$ parametrized by $t \in \mathbb{R}$.

By generic choice of v, the projection of $Z(P)$ to the t-axis is a finite union of at most D points (since any line intersects a real algebraic hypersurface of degree D in at most D points). Let these critical values be $t_1 < t_2 < \cdots < t_m, \quad m \leq D$.

Between consecutive critical levels, i.e. for $t \in (t_j, t_{j+1})$, the slices $\{P(tv + y) = 0\}$ form a smooth hypersurface in each hyperplane $\{v \cdot x = t\}$, and therefore the number of connected components of its complement is constant on each open interval (t_j, t_{j+1}). Also, at each "critical" t_j, as t crosses t_j, new components can only be created or destroyed when $t = t_j$, and their number is controlled by the local topology near a real singularity. At most $C(n-1, D)$ new components appear or disappear at each crossing.

Therefore,

$$\begin{aligned} C(n, D) &\leq (m+1) \cdot C(n-1, D) + 2m \\ &\leq (D+1)C_{n-1}D^{n-1} + 2D \\ &\leq C_n D^n, \end{aligned}$$

for a suitable constant C_n depending only on C_{n-1} and n. So we are done.

■

4. Bounding r-rich Points on Lines

Using polynomial partitioning, we aim to study points in $\mathbb{R}^3$, and notably estimate the number of r-rich points in $Z(P)$ in two different ways. The first of these approaches employs the Szemerédi-Trotter theorem.

Theorem 4.1 ([14], Theorem 1.1). *Suppose that $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$, $P \in \mathrm{Poly}_D(\mathbb{R}^3)$, and that $Z(P)$ contains at most B lines of $\mathfrak{L}$. Then*

$$|P_r(\mathfrak{L}) \cap Z(P)| \lesssim DLr^{-1} + B^2r^{-3}.$$

Proof. Write
$$\mathfrak{L}_Z = \{\, \ell \in \mathfrak{L} : \ell \subset Z(P)\}, \quad \mathfrak{L}_{\neg Z} = \mathfrak{L} \setminus \mathfrak{L}_Z,$$
so $|\mathfrak{L}_Z| \le B$ and $|\mathfrak{L}_{\neg Z}| = L - |\mathfrak{L}_Z|$. Let $\mathcal{S} = \big\{\, x \in Z(P) : x$ lies on at least r lines of $\mathfrak{L}\big\}$.

Each $x \in \mathcal{S}$ lies on r lines, and so either

(1) x lies on at least $\frac{r}{2}$ lines of $\mathfrak{L}_{\neg Z}$
(2) x lies on at least $\frac{r}{2}$ lines of $\mathfrak{L}_Z$.

Case (1). A line not contained in $Z(P)$ meets it in at most D points, so the total number of intersections of $\mathfrak{L}_{\neg Z}$ with $Z(P)$ is at most $D\big|\mathfrak{L}_{\neg Z}\big| \le D\,L$. Since each "case (i)" point is counted at least $r/2$ times in that intersection count,
$$\big|\{\, x \in \mathcal{S} : \text{(i)}\}\big| \;\le\; \frac{D\,L}{r/2} \;=\; \frac{2D\,L}{r} \;\lesssim\; \frac{D\,L}{r}.$$

Case (2). There are at most $\binom{|\mathfrak{L}_Z|}{2} \le \frac{B^2}{2}$ intersection-points among the B lines in $\mathfrak{L}_Z$. On the other hand, each "case (ii)" point lies on at least $r/2$ of those lines, so it is counted by at least $\binom{r/2}{2} = \frac{(r/2)(r/2-1)}{2} \gtrsim \frac{r^2}{8}$ line pairs.

Hence
$$\big|\{\, x \in \mathcal{S} : \text{(ii)}\}\big| \;\le\; \frac{B^2/2}{r^2/8} \;=\; \frac{4B^2}{r^2} \;\lesssim\; \frac{B^2}{r^2} \;\lesssim\; \frac{B^2}{r^3}$$
after absorbing the constants.

Combining the two bounds yields
$$|\mathcal{S}| \;\le\; \frac{C_1 D\,L}{r} \;+\; \frac{C_2 B^2}{r^3} \;\lesssim\; D\,L\,r^{-1} \;+\; B^2\,r^{-3},$$
so we are done.

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Lemma 4.2 ([4], Proposition 2.2)**.** *If $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^n$ and $r > 2L^{1/2}$, then*
$$|\mathcal{P}_r(\mathfrak{L})| \le 2Lr^{-1}.$$

Proof. List the r-rich points as $\mathcal{P}_r(\mathfrak{L}) = \{x_1, x_2, \dots, x_M\}$, so $M = |\mathcal{P}_r(\mathfrak{L})|$. We will show $M < r/2$, and then the desired bound will follow.

Since x_1 lies on at least r lines, we mark those r lines as *used.* Then x_2 lies on at least r lines in total, but at most one of them might already be used (the one through x_1), so there are at least $r-1$ new lines to mark.

Continuing in this way, when we reach the jth point x_j, at most $(j-1)$ lines have been used up, so there are at least $r-(j-1)$ new lines through x_j that we have not yet counted. Since there are only L lines altogether,
$$L \;\ge\; \sum_{j=1}^{M} \max\big(r-(j-1),\, 0\big).$$

If $M \ge \frac{r}{2}$, then the first $\lfloor r/2 \rfloor$ terms in the sum are each at least $r-(j-1) \;\ge\; r-\big(\frac{r}{2}-1\big) \;=\; \frac{r}{2}+1$, so
$$L \;\ge\; \sum_{j=1}^{\lfloor r/2 \rfloor} (r-(j-1)) \;\ge\; \frac{r}{2}\left(\frac{r}{2}+1\right) \;>\; \frac{r^2}{4}.$$

But since $r > 2\sqrt{L}$, we have $r^2 > 4L$, a contradiction. Hence $M < r/2$.

From $M < r/2$ and the fact that each of the M points contributed at least $\frac{r}{2}$ new lines, we get

$$L \ \geq \ M \cdot \frac{r}{2} \quad \Longrightarrow \quad M \ \leq \ \frac{2L}{r}$$

so we are done. ■

Theorem 4.3 ([5], Theorem 1.2). *For any $\varepsilon > 0$, there exists a degree $D = D(\varepsilon)$ and a constant $C(\varepsilon)$ such that the following holds. Suppose that $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$ with at most B lines in any algebraic surface of degree $\leq D$. Then for any $2 \leq r \leq 2L^{1/2}$,*

$$|\mathcal{P}_r(\mathfrak{L})| \leq C(\varepsilon) B^{\frac{1}{2}-\varepsilon} L^{\frac{3}{2}+\varepsilon} r^{-2}.$$

Proof. We proceed by induction. As the base case, if D is so small that there is no nontrivial surface of degree $\leq D$, then no more than B lines can lie in any such surface. By Lemma 4.2,

$$\big|\mathcal{P}_r(\mathfrak{L})\big| \ \leq \ 2\, L\, r^{-1} \ \leq \ 4\, L^{3/2}\, r^{-2},$$

and since $B^{\frac{1}{2}-\varepsilon} L^{\frac{3}{2}+\varepsilon} \geq L^{3/2}$ this is of the desired form for $C(\varepsilon) \geq 4$.

Now assume the bound holds for degree $D-1$. Let $\mathfrak{L}$ satisfy the hypotheses at degree D.By Theorem 3.4, there is a nonzero $P \in \mathrm{Poly}_D(\mathbb{R}^3)$ such that $\mathbb{R}^3 \setminus Z(P)$ splits into $O(D^3)$ open cells O_i, each containing at most $C'(3)\, L\, D^{-3}$ of the lines.

Every line not contained in $Z(P)$ meets it in $\leq D$ points, hence lies in at most $D+1$ cells. If $\mathfrak{L}_i$ is the set of lines meeting O_i, then $\sum_i |\mathfrak{L}_i| \leq (D+1)L$. By the inductive hypothesis on each cell,

$$\big|\mathcal{P}_r(\mathfrak{L}_i)\big| \leq C(\varepsilon)\, B^{\frac{1}{2}-\varepsilon}\, |\mathfrak{L}_i|^{\frac{3}{2}+\varepsilon}\, r^{-2}.$$

Summing it all up gives

$$\sum_i \big|\mathcal{P}_r(\mathfrak{L}_i)\big| \lesssim C(\varepsilon)\, D^{\frac{1}{2}-\varepsilon}\, B^{\frac{1}{2}-\varepsilon}\, L^{\frac{3}{2}+\varepsilon}\, r^{-2}.$$

Let $\mathfrak{L}_Z \subset \mathfrak{L}$ be those lines lying entirely in $Z(P)$. Then $|\mathfrak{L}_Z| \leq B$. If we apply Lemma 4.2 on the surface, we get

$$\big|\mathcal{P}_r(\mathfrak{L}_Z)\big| \leq 2\, B\, r^{-1} \leq 2\, B^{\frac{1}{2}-\varepsilon}\, L^{\frac{3}{2}+\varepsilon}\, r^{-2}.$$

Adding the contributions from the cells and from $Z(P)$ shows

$$I(B, D, L, r) \ \leq \ C(\varepsilon)\big[D^{\frac{1}{2}-\varepsilon} + 1\big]\, B^{\frac{1}{2}-\varepsilon}\, L^{\frac{3}{2}+\varepsilon}\, r^{-2}.$$

Choosing $D(\varepsilon)$ large enough so $D^{\frac{1}{2}-\varepsilon} \leq 2$ completes the induction. ■

5. Bounding Through Induction on L

When B is small (i.e. $\lesssim \log L$), Theorem 4.3 is close to the best known bound. But for larger B, it it quite loose. We will first show this best known bound, then prove a bound on the total number of incidences.

5.1. **The Best Known Bound on $|P_r(\mathfrak{L})|$.**

We first state a preliminary lemma.

Lemma 5.1 (Bézout's Inequality)**.** *Let $V \subset \mathbb{C}^N$ be a nonempty locally closed (constructible) set, and let $H_1, \ldots, H_r \subset \mathbb{C}^N$ be algebraic hypersurfaces of degrees* $\deg(H_i)$. *Then*

$$\deg\bigl(V \cap H_1 \cap \cdots \cap H_r\bigr) \;\le\; \deg(V) \prod_{i=1}^{r} \deg(H_i).$$

Proof. We proceed by induction on r. Write

$$W \;=\; V \cap H_1 \cap \cdots \cap H_{r-1},$$

so that we must show $\deg(W \cap H_r) \le \deg(W)\,\deg(H_r)$.

We take $r = 1$ as the base case. If V is irreducible of dimension d, then we see

$$\deg(V \cap H_1) \;\le\; \deg(V)\,\deg(H_1).$$

If $V = V_1 \cup \cdots \cup V_s$ is the decomposition into irreducible components of dimension d, then $\deg(V) = \sum_{j=1}^{s} \deg(V_j)$ and $\deg(V \cap H_1) \le \sum_{j=1}^{s} \deg(V_j \cap H_1) \le \sum_{j=1}^{s} \deg(V_j)\,\deg(H_1) = \deg(V)\,\deg(H_1)$.

Now assume the bound holds for $r-1$ hypersurfaces. Then

$$\begin{aligned}\deg\bigl(V \cap H_1 \cap \cdots \cap H_r\bigr) &= \deg\bigl(W \cap H_r\bigr)\\ &\le\; \deg(W)\,\deg(H_r)\\ &\le\; \Bigl[\deg(V) \prod_{i=1}^{r-1} \deg(H_i)\Bigr]\,\deg(H_r),\end{aligned}$$

where the first inequality is the base-case result applied to the pair (W, H_r), and the second is the inductive hypothesis applied to V and $H_1, \ldots, H_{r-1}$. So we are done ∎

Using Lemma 5.1, we bound the number of surfaces that contain many lines.

Lemma 5.2 ([4], Lemma 3.2)**.** *Let $\mathfrak{L}$ be a set of L lines in $\mathbb{R}^3$. Suppose Z_j are irreducible algebraic surfaces of degree at most D, each containing at least A lines of $\mathfrak{L}$. If $A > 2DL^{1/2}$, then the number of surfaces Z_j is at most $2L/A$.*

Proof. We use a double counting argument on the incidence between our surfaces and lines. Label the surfaces $Z_1, Z_2, \ldots, Z_J$. By hypothesis, Z_1 contains at least A lines of $\mathfrak{L}$, so it contributes A new lines. When we add Z_2, Lemma 5.1 guarantees their intersection can contain at most D^2 lines. Hence among the A lines on Z_2, at most D^2 were already counted on Z_1, and at least $A - D^2$ are actually new.

In general, when we reach Z_j, each previous surface Z_i $(i < j)$ can share at most D^2 lines with Z_j, and so the union of the first $j-1$ surfaces covers at most $(j-1)D^2$ of the lines on Z_j. Therefore Z_j adds at least $A \;-\; (j-1)D^2$ new lines.

Since there are only L lines total in $\mathfrak{L}$, summing over all j gives

$$L \;\ge\; \sum_{j=1}^{J} \bigl[\text{new lines from } Z_j\bigr] \;=\; \sum_{j=1}^{J} \max\bigl(A - (j-1)D^2,\; 0\bigr).$$

Now we extract the bound on J. Assume for the sake of contradiction, that $J \;>\; \frac{2L}{A}$.

Set $K = \lceil 2L/A \rceil$, so $K \le J$. Then for every $1 \le j \le K$, we have

$$(j-1)D^2 \;\le\; (K-1)D^2 \;<\; \frac{2L}{A}\,D^2 \;<\; \frac{A}{2},$$

with the last strict inequality following from $A > 2D\sqrt{L}$. Hence each of the first K terms in our sum satisfies

$$A-(j-1)D^2 \;>\; \frac{A}{2},$$

and so

$$L \;\ge\; \sum_{j=1}^{J} \max\bigl(A-(j-1)D^2, 0\bigr) \;\ge\; \sum_{j=1}^{K}\bigl(A-(j-1)D^2\bigr) \;>\; K\cdot\frac{A}{2} \;\ge\; \frac{2L}{A}\cdot\frac{A}{2} \;=\; L,$$

a contradiction. Therefore $J \le K \le 2L/A$, and we are done. ∎

Now we are ready to work on the main bound below

Theorem 5.3 ([4], Theorem 0.2)**.** *For any $\varepsilon > 0$, there are constants $D(\varepsilon)$ and $C(\varepsilon)$ such that the following holds. If $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$ with at most $L^{1/2+\varepsilon}$ lines in any algebraic surface of degree $\le D(\varepsilon)$, then*

$$|\mathcal{P}_r(\mathfrak{L})| \;\le\; C(\varepsilon)L^{3/2+\varepsilon}r^{-2} + 2Lr^{-1}.$$

To show this, we will use induction on L and our method from the previous section. However, we run into an issue. After partitioning, we collect the lines in each cell into a subfamily $\mathfrak{L}_i$. Since $\mathfrak{L}_i \subset L$, we know that at most $L^{(1/2)+\epsilon}$ lines of $\mathfrak{L}_i$ lie in any algebraic surface of degree at most D, but we don't know that at most $|\mathfrak{L}_i|^{(1/2)+\epsilon}$ lines of $\mathfrak{L}_i$ lie in any algebraic surface of degree at most D. So we cannot directly apply induction to $\mathfrak{L}_i$. We have to somehow deal with low degree algebraic surfaces that contain more than $|\mathfrak{L}_i|^{(1/2)+\epsilon}$ lines of $\mathfrak{L}_i$. The majority of the proof is in resolving this.

To make the induction work better, we will actually show a stronger theorem, below.

Theorem 5.4 ([4], Theorem 2.1)**.** *For any $\varepsilon > 0$, there are $D(\varepsilon)$, and $K(\varepsilon)$ such that the following holds. For any $r \ge 2$, let $r' = \lceil r/2 \rceil$. If $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$, and if $2 \le r \le 2L^{1/2}$, then there is a set $\mathcal{Z}$ of algebraic surfaces such that*

- *Each surface $Z \in \mathcal{Z}$ is an irreducible surface of degree at most D.*
- *Each surface $Z \in \mathcal{Z}$ contains more than $L^{1/2+\varepsilon}$ lines of $\mathfrak{L}$.*
- *$|\mathcal{Z}| \le 2L^{1/2-\varepsilon}$.*
- *$\bigl|\mathcal{P}_r(\mathfrak{L}) \setminus \cup_{Z\in\mathcal{Z}}\mathcal{P}_{r'}(\mathfrak{L}_Z)\bigr| \;\le\; K(\varepsilon)\,L^{3/2+\varepsilon}\,r^{-2}$.*

This clearly implies Theorem 5.3. If $r > L^{1/2}$, we know $|P_r(\mathfrak{L})| \le 2Lr^{-1}$ by double counting. If $r \le 2L^{1/2}$, then Theorem 5.2 says the set Z must be empty and so $|P_r(\mathfrak{L})| \le KL^{3/2+\epsilon}r^{-2}$

We are now ready to prove Theorem 5.4.

Proof. Fix $\epsilon \in (0, \frac{1}{2}]$ and choose constants

- $D = D(\epsilon)$ so large that $D^{-\epsilon} \le 10^{-3}$ and $D \ll K$,
- $K = K(\epsilon, D)$ sufficiently large for all upcoming inequalities.

We proceed by induction on $L = |\mathfrak{L}|$. For the base case $L \le (2D)^{1/\epsilon}$, let $\mathcal{Z} = \emptyset$; then trivially $|\mathcal{Z}| = 0 \le 2L^{1/2-\epsilon}$ and

$$|P_r(\mathfrak{L})| \le L^2 \le KL^{3/2+\epsilon}r^{-2}.$$

Assume thes statement holds for all smaller L, and set $S := P_r(\mathfrak{L})$.

Now we apply Theorem 3.4 with degree D to the set S. It yields a real polynomial P of degree $\le D$, whose zero set $Z(P)$ partitions $\mathbb{R}^3$ into $O(D^3)$ open cells $\{O_i\}$, each containing at most $CD^{-3}|S|$ points of S. Lines not in $Z(P)$ intersect at most $D+1$ cells, so

$$\sum_i |\mathfrak{L}_i| \le (D+1)L \le 2DL,$$

where $\mathfrak{L}_i = \{\ell \in \mathfrak{L} : \ \ell \cap O_i \ne \emptyset\}$.

Call a cell O_i β-*good* if $|\mathfrak{L}_i| \le \beta D^{-2}L$, and choose a fixed constant β so that the total number of lines in *bad* cells is at most $\frac{1}{100}|S|$. For each good cell we have $|\mathfrak{L}_i| \le \frac{1}{2}L$ (by choosing D large), and either

$$r \le 2|\mathfrak{L}_i|^{1/2},$$

or we bound $|S \cap O_i| \le 4Lr^{-1} \le 4L^{3/2}r^{-2} \le C_1KD^{-3-2\epsilon}L^{3/2+\epsilon}r^{-2}$.

For the first case $r \le 2|\mathfrak{L}_i|^{1/2}$, apply the inductive hypothesis to $\mathfrak{L}_i$ and its rich set $S_i := \mathrm{Pr}_r(\mathfrak{L}_i) \supseteq S \cap O_i$. There is a collection $\mathcal{Z}_i$ of at most $2|\mathfrak{L}_i|^{1/2-\epsilon} \le 2(\beta D^{-2}L)^{1/2-\epsilon}$ surfaces, each of degree $\le D$, so that

$$|S_i \setminus \bigcup_{Z\in\mathcal{Z}_i} \mathop{\mathrm{Pr}}_r(\mathfrak{L}_Z)| \le K|\mathfrak{L}_i|^{3/2+\epsilon}r^{-2} \le C_2K(D^{-2}L)^{3/2+\epsilon}r^{-2}.$$

Summing over all good cells and adding the bad-cell contribution gives

$$\big|S \setminus \bigcup_{i, Z\in\mathcal{Z}_i} \mathop{\mathrm{Pr}}_r(\mathfrak{L}_Z)\big| \le \tfrac{1}{400}KL^{3/2+\epsilon}r^{-2}.$$

Let $\{Z_j\}$ be the irreducible components of $Z(P)$. Each line not contained in Z_j meets it in at most $\deg Z_j$ points, so

$$|S \cap Z_j \setminus \mathop{\mathrm{Pr}}_r(\mathfrak{L}_{Z_j})| \le 10r^{-1}(\deg Z_j)L,$$

and summing over j yields

$$\big|S \cap Z(P) \setminus \bigcup_j \mathop{\mathrm{Pr}}_r(\mathfrak{L}_{Z_j})\big| \le 10DLr^{-1} \le \tfrac{1}{400}KL^{3/2+\epsilon}r^{-2}.$$

Combining these two inequalities shows that outside the surfaces in

$$\mathcal{Z}' := \bigcup_i \mathcal{Z}_i \cup \{Z_j\},$$

all but at most $\frac{1}{200}KL^{3/2+\epsilon}r^{-2}$ points of S lie on some surface of degree $\le D$. Moreover,

$$|\mathcal{Z}'| \le \sum_i 2(\beta D^{-2}L)^{1/2-\epsilon} + D \le CD^3L^{1/2-\epsilon}.$$

Call these surfaces *preliminary*.

Set $S_1 = S$ and iterate. Remove from S_j all points lying on any surface of $\mathcal{Z}'_j$, producing S_{j+1}. Then reapply everything and to obtain a fresh list $\mathcal{Z}'_j$. After $J = 1000 \log L$ iterations we see that

$$|S_{J+1}| \le (1/100) K L^{3/2+\epsilon} r^{-2},$$

and the union $\widetilde{\mathcal{Z}} = \bigcup_{j<J} \mathcal{Z}'_j$ has size

$$|\widetilde{\mathcal{Z}}| \le C D^3 L^{1/2-\epsilon} \log L.$$

Finally, prune to

$$\mathcal{Z} := \{\, Z \in \widetilde{\mathcal{Z}} : \ |\mathfrak{L}_Z| \ge L^{1/2+\epsilon}\},$$

and apply Lemma 5.1 with $A = L^{1/2+\epsilon}$ to get

$$|\mathcal{Z}| \le 2L^{1/2-\epsilon},$$

and redistribute the remaining few rich points by Theorem 3.4. We get

$$\big|S \setminus \bigcup_{Z \in \mathcal{Z}} P_r(\mathfrak{L}_Z)\big| \le K L^{3/2+\epsilon} r^{-2}.$$

So the induction is done. ∎

5.2. **A Bound on Total Incidences.**

For the remainder of this paper, we work on bounding incidences in general, and shift from r-rich points. The rest of this section will show an incidence bound for lines. Like Szemerédi-Trotter and our r-rich bound, we induct on L. When we carry out the argument, there end up being many terms, making it complicated. For some intuition, we discuss the thought process beforehand.

Basically, we use Theorem 3.4 to control the incidences in the cells outside of $Z(P)$. We divide the surface $Z(P)$ into planar parts and non-planar parts. The contribution from the planar parts is controlled using the fact that there are at most B lines in any plane. The contribution from the non-planar parts of $Z(P)$ is controlled using the theory of flat points and lines.

Theorem 5.5 ([10], Theorem 1)**.** *There exists an absolute constant $C_0 > 0$ such that the following holds. Let $S \subset \mathbb{R}^3$ be a set of $|S|$ points and $\mathfrak{L}$ a set of L lines, with at most B lines of $\mathfrak{L}$ contained in any single plane and with $B \ge L^{1/2}$. Then the number of incidences satisfies*

$$I(S, L) \ \le \ C_0\Big(|S|^{1/2} L^{3/4} + B^{1/3} L^{1/3} |S|^{2/3} + L + |S|\Big).$$

Proof. We proceed by induction on L. For small L or extreme S, trivial counts $I(S, \mathfrak{L}) \le L + S^2$, and Theorem 5.5 via projection show the claim holds with a suitable C_0. Henceforth assume

$$10L^{1/2} \le S \le \tfrac{1}{10} L^2.$$

Let $D \ge 1$ be a parameter chosen later. By Theorem 3.4, choose $P \in \mathbb{R}[x, y, z]$ of degree $\le D$ so each connected component of $\mathbb{R}^3 \setminus Z(P)$ contains $\le S/D^3$ points of S. Write

$$S_{\text{alg}} = S \cap Z(P), \qquad S_{\text{cell}} = S \setminus S_{\text{alg}},$$
$$L_{\text{alg}} = \{\ell \in \mathfrak{L} : \ell \subset Z(P)\}, \quad L_{\text{cell}} = \mathfrak{L} \setminus L_{\text{alg}}.$$

Decompose $\mathbb{R}^3 \setminus Z(P) = \bigcup_i O_i$. In each cell O_i, set

$$S_i = S \cap O_i, \quad L_i = \{\ell \in \mathfrak{L} : \ell \cap O_i \neq \emptyset\}.$$

Then $|S_i| \leq S/D^3$ and $\sum_i |L_i| \leq DL$. Applying Theorem 4.1 in each cell and summing gives

$$\sum_i I(S_i, L_i) \leq C\Big(D^{-1/3} S^{2/3} L^{2/3} + D\,L + |S_{\text{cell}}|\Big).$$

Each line in L_{cell} meets $Z(P)$ in at most D points, so

$$I(S_{\text{alg}}, L_{\text{cell}}) \leq D\,L.$$

Combining these two contributions,

$$I(S, \mathfrak{L}) \leq C\big(D^{-1/3} S^{2/3} L^{2/3} + DL + S\big) + I(S_{\text{alg}}, L_{\text{alg}}).$$

Next we bound $I(S_{\text{alg}}, L_{\text{alg}})$ by splitting into planar, multiple-planar, and non-planar pieces.

First the planar part. Let L_{plan} be lines in some plane of $Z(P)$, further split into $L_{\text{uni}} \subset L_{\text{plan}}$ lying in exactly one such plane, and $L_{\text{multi}} \subset L_{\text{plan}}$ lying in two or more planes. Similarly define $S_{\text{uni}}, S_{\text{multi}}$. Since each plane contains $\leq B$ lines, applying Theorem 3.4 in each plane yields

$$I(S_{\text{alg}}, L_{\text{plan}}) = I(S_{\text{alg}}, L_{\text{uni}}) + I(S_{\text{multi}}, L_{\text{multi}}) \leq C\big(B^{1/3} L^{1/3} S^{2/3} + D\,L + |S_{\text{uni}}|\big) + I(S_{\text{multi}}, L_{\text{multi}}),$$

and we show $|L_{\text{multi}}| \leq D^2$ and handles it by induction.

Now for the non-planar algebraic part. Define special points S_{spec} (critical or flat) and special lines L_{spec} on $Z(P)$. Set $S_{\text{nonspec}} = S_{\text{alg}} \setminus S_{\text{spec}}$, $L_{\text{nonspec}} = L_{\text{alg}} \setminus (L_{\text{plan}} \cup L_{\text{spec}})$. Now, by Lemma 2.8,

$$I(S_{\text{nonspec}}, L_{\text{alg}}) \leq C|S_{\text{nonspec}}|, \quad I(S_{\text{spec}}, L_{\text{nonspec}}) \leq C D\,L,$$

and bounds the remaining $I(S_{\text{spec}}, L_{\text{spec}} \setminus L_{\text{plan}})$ by noting

$$|L_{\text{spec}} \setminus L_{\text{plan}}| \leq 4D^2$$

and inducting on this small set.

Collecting all contributions, we see

$$I(S, \mathfrak{L}) \leq C\Big(D^{-1/3} S^{2/3} L^{2/3} + DL + B^{1/3} L^{1/3} S^{2/3} + L + S\Big) + I(S', L'),$$

where $|L'| \leq 10D^2 \leq L/2$, so the induction applies to $I(S', L')$. Finally choose $D \sim S^{1/2} L^{-1/4}$ in the range $1 \leq D \leq (1/10)L^{1/2}$. We check

$$D^{-1/3} S^{2/3} L^{2/3} + DL \lesssim S^{1/2} L^{3/4} + B^{1/3} L^{1/3} S^{2/3},$$

so the induction is complete. We are done. ∎

6. Sufficiently Rich Points on Algebraic Curves

In this section we present the two main incidence bounds for points on constant-degree algebraic curves in $\mathbb{R}^3$.

To do this, we revisit Lemma 5.1. One direct consequence of Bézout is that a constructible set $\mathcal{C}$ is a union of locally closed sets. Also, we can decompose $\mathcal{C}$ uniquely as the union of irreducible locally closed sets. We state an important corollary of Lemma 5.1.

Corollary 6.1 (Consequence of Bézout for Constructible Sets)**.** *Let $\mathcal{C} \subset \mathbb{C}^N$ be a constructible set and write it as the union of locally closed sets $\bigcup_{i=1}^t X_i$, where*

$$X_i = \{p \in \mathbb{C}^N \mid f_1^i(p) = 0, \ \ldots, \ f_{r_i}^i(p) = 0, \ g^i(p) \neq 0\},$$

for polynomials $f_1^i, \ldots, f_{r_i}^i, g^i \in \mathbb{C}[x_1, \ldots, x_N]$.

If $\mathcal{C}$ contains more than $\sum_{i=1}^t \deg(f_1^i) \cdots \deg(f_{r_i}^i)$ points, then $\mathcal{C}$ is infinite.

Proof. Assume, for the sake of contradiction, that $\mathcal{C}$ is finite. Then each X_i is also finite, and

$$|\mathcal{C}| \ = \ \sum_{i=1}^t |X_i| \ \leq \ \sum_{i=1}^t \deg(V_i),$$

where $V_i := \{p \in \mathbb{C}^N \mid f_1^i(p) = \cdots = f_{r_i}^i(p) = 0\}$ is the affine variety defined by the common zeros of $f_1^i, \ldots, f_{r_i}^i$. By Theorem 5.1,

$$\deg(V_i) \ \leq \ \prod_{j=1}^{r_i} \deg\big(f_j^i\big).$$

Hence

$$|\mathcal{C}| \ \leq \ \sum_{i=1}^t \deg(V_i) \ \leq \ \sum_{i=1}^t \prod_{j=1}^{r_i} \deg\big(f_j^i\big).$$

Contradiction. Therefore $\mathcal{C}$ must be infinite. ∎

Theorem 6.2 ([11], Theorem 1.4)**.** *Let P be m points and let $\mathcal{C}$ be n irreducible degree-E curves from a constructible family $\mathcal{C}_0$ of complexity $O(1)$ and k degrees of freedom in $\mathbb{R}^3$. Suppose no surface infinitely ruled by curves of $\mathcal{C}_0$ contains more than $q < n$ curves of $\mathcal{C}$. Then*

$$I(P, \mathcal{C}) \ = \ O\Big(m^{\frac{k}{3k-2}}\, n^{\frac{3k-3}{3k-2}} \ + \ m^{\frac{k}{2k-1}}\, n^{\frac{k-1}{2k-1}}\, q^{\frac{k-1}{2k-1}} \ + \ m \ + \ n\Big),$$

where the hidden constant depends on k, E and the complexity of $\mathcal{C}_0$.

Proof. We double induct on the number of curves n and the number of points m that for any sets of points P and curves C satisfying Theorem 6.2, there exists a constant A

$$I(P, C) \ \leq \ A\Big(m^{\frac{k}{3k-2}}\, n^{\frac{3k-3}{3k-2}} \ + \ m^{\frac{k}{2k-1}}\, n^{\frac{k-1}{2k-1}}\, q^{\frac{k-1}{2k-1}} \ + \ m + n\Big).$$

As the base cases, when $n \leq n_0$, choosing $A \geq n_0$ makes the bound trivial. When $m \leq a'\, n^{1/k}$, the naive bound $I(P, \mathcal{C}) = O(n)$ implies the bound upon taking A sufficiently large.

Assume now that the bound holds for all smaller pairs (m', n') in the lexicographic order, and let $m > a'\, n^{1/k}$ and $n > n_0$.

Now we apply partitioning. Choose constants $a, a', c > 0$ and define

$$D = \begin{cases} c\dfrac{m^{\frac{k}{3k-2}}}{n^{\frac{1}{3k-2}}}, & a'\, n^{1/k} \le m \le a\, n^{3/2}, \\ c\, n^{1/2}, & m > a\, n^{3/2}. \end{cases}$$

By Theorem 3.4, there is a real polynomial f of degree at most D whose complement has $O(D^3)$ connected cells, each crossed by at most $O(n/D^2)$ curves from C. Label these cells $\tau_1, \ldots, \tau_u$, and set

$$P_i = P \cap \tau_i, \quad \mathcal{C}_i = \{\gamma \in C : \gamma \cap \tau_i \neq \emptyset\}, \quad m_i = |P_i|, \quad n_i = |\mathcal{C}_i| = O\big(n/D^2\big).$$

For each cell τ_i, the trivial bound gives

$$I(P_i, \mathcal{C}_i) = O\big(m_i^{1-1/k}\, n_i + n_i\big).$$

Summing over all $O(D^3)$ cells, we get

- If $m \le a\, n^{3/2}$:
$$\sum I(P_i, \mathcal{C}_i) = O\Big(\tfrac{m^{\frac{k}{3k-2}}\, n^{\frac{3k-3}{3k-2}}}{D^{2(1-1/k)}} + nD^3\Big) = O\big(m^{\frac{k}{3k-2}} n^{\frac{3k-3}{3k-2}}\big).$$
- If $m > a\, n^{3/2}$: here $n_i = O(1)$, so
$$\sum I(P_i, \mathcal{C}_i) = O\big(\sum m_i\big) = O(m).$$

In either case,

$$\sum_i I(P_i, \mathcal{C}_i) = O\big(m^{\frac{k}{3k-2}} n^{\frac{3k-3}{3k-2}} + m\big).$$

Let $P^* = P \cap Z(f)$ and write $C = C^* \cup C'$, where C^* are the curves fully contained in $Z(f)$ and $C' = C \setminus C^*$. Then

$$I(P^*, C') = O(nD) = O\big(m^{\frac{k}{3k-2}} n^{\frac{3k-3}{3k-2}} + m\big).$$

It remains to handle $I(P^*, C^*)$. Factor $f = \prod_{i=1}^t f_i$ into irreducibles of degrees D_i, so $\sum D_i = D$. Assign each $p \in P^*$ and each $\gamma \in C^*$ to the first f_i that vanishes on it, yielding sets $P_i, \mathcal{C}_i$ with sizes m_i, n_i.

If $Z(f_i)$ carries at most q curves from C, projecting to the plane gives

$$I(P_i, \mathcal{C}_i) = O\big(m_i^{\frac{k}{2k-1}} n_i^{\frac{k-1}{2k-1}} q^{\frac{k-1}{2k-1}} + m_i + n_i\big).$$

Summing with Hölder's inequality yields

$$\sum_{i:\,\text{ruled}} I(P_i, \mathcal{C}_i) = O\Big(m^{\frac{k}{2k-1}} n^{\frac{k-1}{2k-1}} q^{\frac{k-1}{2k-1}} + m + n\Big).$$

Each component supports at most $O(D_i^2)$ "exceptional" curves. We can handle them by the outer induction on n to get a contribution bounded by

$$A\Big(m^{\frac{k}{3k-2}} n^{\frac{3k-3}{3k-2}} + m^{\frac{k}{2k-1}} n^{\frac{k-1}{2k-1}} q^{\frac{k-1}{2k-1}} + m + n\Big).$$

The non-exceptional curves contribute $O(m_i + n_i D_i)$, summing to $O(m + nD) = O(m^{\frac{k}{3k-2}} n^{\frac{3k-3}{3k-2}} + m + n)$.

Combining the contribution with the bounds for C', and choosing A large enough, we get the desired bound. So we are done. ■

This is a generalization of [5] for lines. The bound in Theorem 6.2 can be further improved if we also analyze the dimensionality of $\mathcal{C}_0$.

For example, we can take $\mathcal{C}_0$ to be the set of all circles in $\mathbb{R}^3$. Then, since the only surfaces that are infinitely ruled by circles are spheres and planes, the subfamily of all circles that are contained in some sphere or plane is only 3-dimensional (while $\mathcal{C}_0$ is 6 dimensional).

Formally, we say $\mathcal{C}_0$ is a family of reduced dimension s if, for each surface V that is infinitely ruled by curves of $\mathcal{C}_0$, the subfamily of the curves of $\mathcal{C}_0$ that are fully contained in V is s-dimensional. This gives us the following bound.

Lemma 6.3 ([11], Theorem 1.13)**.** *For given integer parameters c, E, there exist constants $C_1 = C_1(c, E)$, $r = r(c, E)$, and $t = t(c, E)$ such that the following holds. Let f be a complex irreducible polynomial of degree $D \gg E$, and let $\mathcal{C}_0 \subset \mathbb{C}^3_E$ be a constructible set of complexity at most c. If there exist at least $C_1 D^2$ curves of $\mathcal{C}_0$ such that each of them is contained in $Z(f)$ and contains at least $C_1 D$ points on $Z(f)$ that are $(t, \mathcal{C}_0, r)$-flecnodes, then $Z(f)$ is infinitely ruled by curves from $\mathcal{C}_0$.*

In particular, if $Z(f)$ is not infinitely ruled by curves from $\mathcal{C}_0$, then, except for at most $C_1 D^2$ exceptional curves, every curve in $\mathcal{C}_0$ that is fully contained in $Z(f)$ is incident to at most $C_1 D$ points that are incident to at least t curves in $\mathcal{C}_0$ that are also fully contained in $Z(f)$.

Proof. We present a sketch of the proof and machinery needed to show this lemma.

For the moment, fix an arbitrary integer r. By Lemma 8.3 and Equation (8.1) in [6], since f is irreducible there exist r polynomials

$$h_j(\alpha, p) \in \mathbb{C}[\alpha, x, y, z], \qquad j = 1, \ldots, r,$$

each of degree at most b_j in the parameter α (where b_j depends only on j and E) and of degree $O(D)$ in the point coordinates $p = (x, y, z)$, with the following characterization: if γ is an irreducible curve, p is a non-singular point of γ, and α encodes γ, then γ osculates to $Z(f)$ to order r at p exactly when

$$h_j(\alpha, p) = 0 \quad \text{for all } j = 1, \ldots, r.$$

(Geometrically, each h_j represents one of the first r coefficients of the Taylor expansion of f along γ at p. See Section 6.2 of [6], and the classical line-specific analyses in [10].

Now, treating p as fixed, the equations

$$h_j(\alpha, p) = 0, \quad j = 1, \ldots, r,$$

together with the condition $\alpha \in \mathcal{C}_0$, define a constructible set $\mathcal{C}_p$. By definition its degree satisfies

$$d(\mathcal{C}_p) \leq \left(\prod_{j=1}^{r} b_j\right) \cdot d(\mathcal{C}_0),$$

which is a constant depending only on r and E. Hence, by Corollary 6.1, $\mathcal{C}_p$ is either infinite or contains at most $d(\mathcal{C}_p) = O(1)$ points.

Also , Corollary 12.1 of [6] guarantees the existence of a Zariski-open subset $\Omega \subset Z(f)$ and a sufficiently large threshold r_0 (determined by $\mathcal{C}_0$ and E via Theorem 8.1 of [6] such that, whenever $p \in \Omega$ is a $(t, \mathcal{C}_0, r)$-flecnode with $r \geq r_0$, there are at least t curves from our family passing through p and lying entirely in $Z(f)$. By hypothesis we have at least C_1D^2 such curves, each carrying at least C_1D $(t,\mathcal{C}_0,r)$-flecnodes, so Proposition 10.2 of [6] further implies that Ω is entirely composed of $(t,\mathcal{C}_0,r)$-flecnodes. Applying Theorem 8.1 of [6] again, we deduce that every point of Ω lies on at least t degree-$\leq E$ curves contained in $Z(f)$.

Finally, if we choose $t \geq \left(\prod_{j=1}^r b_j\right) d(\mathcal{C}_0)$—a constant depending only on $\mathcal{C}_0$ and E—then $Z(f)$ must be infinitely ruled by $\mathcal{C}_0$ on Ω, and this ruling extends over all of $Z(f)$. So we are done.

■

This is a result of the analysis of Guth and Zahl [6]. It's novelty is that it addresses surfaces that are infinitely ruled by certain families of curve, instead of only doubly ruled.

Theorem 6.4 ([11], Theorem 1.5)**.** *Under the same hypotheses as Theorem 6.2, assume further that for every surface V infinitely ruled by curves of $\mathcal{C}_0$, the curves of $\mathcal{C}_0$ contained in V form an s-dimensional family. Then for any $\epsilon > 0$,*

$$I(P,\mathcal{C}) \;=\; O\Big(m^{\frac{k}{3k-2}}\, n^{\frac{3k-3}{3k-2}} \;+\; m^{2/3}n^{1/3}q^{1/3} \;+\; m^{\frac{2s}{5s-4}}\, n^{\frac{3s-4}{5s-4}}\, q^{\frac{2s-2}{5s-4}}\, n^{\epsilon} \;+\; m \;+\; n\Big),$$

where the constant depends on k,μ,s,E,ϵ and the family complexity.

Theorem 6.4 is an improvement of Theorem 6.2 when $s \leq k$ and $m > n^{1/k}$, in cases where q is sufficiently large.

Proof. We again proceed by double induction on n and m.

As the base cases, if $n \leq n_0$, choose $A \geq n_0$ so that the bound holds trivially. If $m \leq a'\, n^{1/k}$, then the naive bound $I(P,\mathcal{C}) = O(n)$ suffices, upon taking A large.

Assume now the bound holds for all pairs (m',n') with $n' < n$ or $n' = n$, $m' < m$, and let $n > n_0$, $m > a'\, n^{1/k}$.

Fix constants $a, a', c > 0$ and set

$$D = \begin{cases} c\dfrac{m^{\frac{k}{3k-2}}}{n^{\frac{1}{3k-2}}}, & a'\, n^{1/k} \leq m \leq a\, n^{3/2}, \\ c\, n^{1/2}, & m > a\, n^{3/2}. \end{cases}$$

By Theorem 3.4, there is a real polynomial f of degree $\leq D$ partitioning $\mathbb{R}^3$ into $O(D^3)$ cells, each met by $O(n/D^2)$ curves of $\mathcal{C}$. Denote the cells by τ_i, and write

$$P_i = P \cap \tau_i, \quad \mathcal{C}_i = \{\gamma \in \mathcal{C} : \gamma \cap \tau_i \neq \emptyset\}, \quad m_i = |P_i|,\; n_i = |\mathcal{C}_i| = O(n/D^2).$$

Using the trivial estimate $I(P_i,\mathcal{C}_i) = O(m_i^{1-1/k} n_i + n_i)$ and summing over $O(D^3)$ cells yields

$$\sum_i I(P_i,\mathcal{C}_i) = O\Big(m^{\frac{k}{3k-2}} n^{\frac{3k-3}{3k-2}} + m\Big).$$

Let $P^* = P \cap Z(f)$ and split $\mathcal{C} = \mathcal{C}^* \cup \mathcal{C}'$, with $\mathcal{C}^*$ the curves lying entirely in $Z(f)$. Then

$$I(P^*,\mathcal{C}') = O(nD) = O\big(m^{\frac{k}{3k-2}} n^{\frac{3k-3}{3k-2}} + m\big).$$

Factor $f = \prod_{i=1}^t f_i$ into irreducibles of degrees D_i, and assign each point in P^* and each curve in $\mathcal{C}^*$ to the first component f_i vanishing on it, creating sets $P_i, \mathcal{C}_i$ of sizes m_i, n_i.

If $Z(f_i)$ is infinitely ruled by the family $\mathcal{C}_0$, then at most q curves of $\mathcal{C}^*$ lie on it. Projecting points P_i and curves $\mathcal{C}_i$ to a generic plane, we get

$$I(P_i, \mathcal{C}_i) = O\Big(m_i^{\frac{2s}{5s-4}} n_i^{\frac{3s-6}{5s-4}} + m_i^{2/3} n_i^{2/3} + m_i + n_i\Big).$$

By Hölder's inequality and tracking the q curves, we get

$$\sum_{i:\,\text{ruled}} I(P_i, \mathcal{C}_i) = O\Big(m^{\frac{2s}{5s-4}} n^{\frac{3s-4}{5s-4}} q^{\frac{2s-2}{5s-4}} + \epsilon\, m^{2/3} n^{1/3} q^{1/3} + m + n\Big).$$

By Lemma 6.3, each $Z(f_i)$ contains $O(D_i^2)$ exceptional curves, which are handled by induction on n. The remaining curves contribute $O(m_i + n_i D_i)$, summing to $O(m + nD) = O(m^{\frac{k}{3k-2}} n^{\frac{3k-3}{3k-2}} + m + n)$.

Collecting contributions from the cells, from $\mathcal{C}'$, and from both cases on $Z(f)$, and choosing A large, yields the bound. So we are done. ∎

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