The Lemke-Howson Algorithm

Finding Nash Equilibria in Bimatrix Games

Amélie Mendeleev

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Historical Background of Game Theory

Origins (1944)

Game theory began with von Neumann and Morgenstern's work on two-player zero-sum games.

Nash's Revolution (1950s)

- Extended theory to non-zero-sum games
- Proved every finite game has at least one equilibrium

The Computational Challenge

Nash's proof didn't show *how* to find equilibria. This led to computational methods like Lemke-Howson.

Applications

Markets, negotiations, biology - situations where systematic solution methods became essential.

What is a Game?

Definition: Game Theory

A mathematical framework for analyzing strategic interactions where players make decisions that affect both their own and others' outcomes, with each player aiming to maximize their own payoff.

Example: Rock-Paper-Scissors

Two players each choose Rock, Paper, or Scissors. If I play Rock and you play Scissors, I win (+1) and you lose (-1). If we both play Rock, we tie (0, 0).

Key Components

- **Players** (decision makers)
- **Strategies** (available actions)
- **Payoffs** (outcomes/utilities)

Pure vs Mixed Strategies

Definition: Pure Strategy

A specific action chosen with certainty.

Example: Pure Strategy

Always play Rock in Rock-Paper-Scissors (predictable and exploitable).

Definition: Mixed Strategy

A probability distribution over pure strategies, allowing for unpredictability and strategic randomization.

Example: Mixed Strategy

Play Rock 40%, Paper 30%, Scissors 30%. This prevents opponents from exploiting predictable patterns.

Payoffs and Bimatrix Games

Definition: Payoffs

Numerical values representing players' utilities, where higher values indicate better outcomes for that player.

Definition: Bimatrix Game

A two-player game represented by two matrices: Matrix A contains the row player's payoffs, and Matrix B contains the column player's payoffs.

Example: Rock-Paper-Scissors Matrices

Matrix A (Row player): $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

Matrix B (Column player): $\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$

Best Response

Definition: Best Response

The set of all optimal strategies for a player given the opponent's strategy. It maximizes expected payoff against the opponent's mixed strategy.

Example: Best Response in RPS

When your opponent plays $(1/3, 1/3, 1/3)$, ALL of your strategies give expected payoff of 0, so they're all best responses.

Mathematical Definition

For player 1: $BR_1(\sigma_2) = \arg\max_{\sigma_1} \sigma_1^T A \sigma_2$
For player 2: $BR_2(\sigma_1) = \arg\max_{\sigma_2} \sigma_1^T B \sigma_2$

Nash Equilibrium

Definition: Nash Equilibrium

A strategy profile where no player can improve their payoff by unilaterally changing their strategy. Each player is playing a best response to what the other player is doing.

Example: RPS Nash Equilibrium

In Rock-Paper-Scissors, $\left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right)$ is a Nash equilibrium.

Formal Definition

(σ_1^*, σ_2^*) is a Nash equilibrium if:

- $\sigma_1^* \in BR_1(\sigma_2^*)$
- $\sigma_2^* \in BR_2(\sigma_1^*)$

John Nash's Contribution

Every finite game has at least one Nash equilibrium, though it might involve mixed strategies.

Rock-Paper-Scissors Setup

The Game

	Paper	Rock	Scissors
Paper	0,0	1,-1	-1,1
Rock	-1,1	0,0	1,-1
Scissors	1,-1	-1,1	0,0

Payoff Matrices

Matrix A (Row player): $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

Matrix B (Column player): $\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$

Why Lemke-Howson?

The Challenge

- Finding Nash equilibria analytically can be difficult
- Pure strategy equilibria may not exist
- Mixed strategy equilibria require solving complex systems

Lemke-Howson Algorithm

- Systematic method to find Nash equilibria
- Uses complementary pivoting
- Guaranteed to find at least one equilibrium

The Complementary Slackness Idea

Definition: Complementary Slackness

In equilibrium, if a player uses a pure strategy with positive probability, that strategy must be optimal against the opponent's mixed strategy. Conversely, if a strategy is not optimal, it must be used with zero probability.

Example: Strategic Logic

If you're using Rock with positive probability in equilibrium, Rock must give you the highest possible expected payoff. If Paper were better, you'd want to play Paper more and Rock less.

Complementarity Condition

Either a strategy is used with positive probability AND it's optimal, OR it's used with zero probability AND it's not optimal.

Setting Up the Linear System

For Rock-Paper-Scissors

Let (p_1, p_2, p_3) be row player's mixed strategy
Let (q_1, q_2, q_3) be column player's mixed strategy

Equilibrium Conditions

- Expected payoffs from each strategy must be equal (if used)
- Probabilities sum to 1: $p_1 + p_2 + p_3 = 1$, $q_1 + q_2 + q_3 = 1$
- All probabilities ≥ 0

The Lemke-Howson Process

Algorithm Steps

1. Start with completely mixed strategies
2. Check complementary slackness conditions
3. If violated, pivot to restore complementarity
4. Continue pivoting until equilibrium found

For RPS

Due to symmetry, we expect equal probabilities
Nash equilibrium: $\left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right)$

Verification of RPS Solution

Check

Is $\left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right)$ a Nash equilibrium?

Expected payoffs for row player

$$\text{Playing Paper:} \quad \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot (-1) = 0 \tag{1}$$

$$\text{Playing Rock:} \quad \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1 = 0 \tag{2}$$

$$\text{Playing Scissors:} \quad \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 0 = 0 \tag{3}$$

All strategies yield same payoff $\Rightarrow$ Equilibrium confirmed!

Why This Works

Lemke-Howson guarantees

- Finds at least one Nash equilibrium
- Systematic approach (no guessing)
- Works for any 2-player game

Key advantages

- Constructive proof method
- Computationally feasible
- Provides insight into equilibrium structure

Summary

Key Takeaways

1. Game theory provides framework for strategic analysis
2. Nash equilibria represent stable outcomes
3. Lemke-Howson algorithm systematically finds equilibria
4. Rock-Paper-Scissors illustrates the concepts clearly

Applications

Economics, computer science, biology, politics

The End

Thank you for your attention!