The Outer Automorphism of S_6

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What is a Group?

Definition: Group

A **group** is a set G equipped with a binary operation $\cdot : G \times G \to G$ satisfying the following properties:

- **Closure:** For all $a, b \in G$, the combination $a \cdot b \in G$.
- **Associativity:** For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- **Identity:** There exists an element $e \in G$ such that for all $a \in G$, $e \cdot a = a \cdot e = a$.
- **Inverse:** For each $a \in G$, there exists $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Definition: Symmetric Group S_n

The **symmetric group** S_n is the group of all permutations (bijective functions) of the set $\{1, 2, \ldots, n\}$ with group operation given by composition.

Group Actions and Transitivity

Group Action

A group action is a map $G \times X \to X$, written $(g, x) \mapsto g \cdot x$, satisfying:

$$e \cdot x = x, \quad (gh) \cdot x = g \cdot (h \cdot x).$$

Transitive Action

The action is **transitive** if for all $x, y \in X$, there exists $g \in G$ such that $g \cdot x = y$.

Examples:

- D_4 acts on the vertices of a square by rotation and reflection.
- $\mathbb{Z}$ acts on $\mathbb{R}$ by $n \cdot x = x + n$.
- G acts on itself by conjugation: $g \cdot x = gxg^{-1}$.

Automorphisms and an Example

Definition

An **automorphism** of a group G is an isomorphism from G to itself:

$$f : G \to G,$$

- The set of all automorphisms, denoted $\mathrm{Aut}(G)$, forms a group under composition.
- **Example:** For the cyclic group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ under addition mod n:
  - Every automorphism is determined by the image of the generator $1 \in \mathbb{Z}_n$.
  - Since the map must preserve order, 1 must be sent to an element coprime to n.
  - Thus,

$$\mathrm{Aut}(\mathbb{Z}_n) \cong (\mathbb{Z}_n)^\times = \{k \in \mathbb{Z}_n : \gcd(k, n) = 1\},$$

  where $(\mathbb{Z}_n)^\times$ is the group of units modulo n.

Inner Automorphisms

Definition

An **inner automorphism** of a group G is a map of the form

$$f_a(x) = a^{-1}xa \quad \text{for fixed } a \in G \text{ and all } x \in G.$$

- The set of all inner automorphisms forms a subgroup of $\mathrm{Aut}(G)$, denoted $\mathrm{Inn}(G)$.
- **Key result:** $\mathrm{Inn}(G) \cong G/Z(G)$, where $Z(G)$ is the center of G.
- $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$:

$$\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G).$$

Outer Automorphisms

Definition

An **outer automorphism** of a group G is an element of the quotient group

$$\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G).$$

It represents an automorphism that is not inner—that is, not of the form $f_a(x) = a^{-1}xa$ for any $a \in G$.

- If $\mathrm{Out}(G) \neq \{e\}$, then G has at least one non-inner (i.e., *outer*) automorphism.
- For most symmetric groups, $\mathrm{Out}(S_n) = \{e\}$. But S_6 is special: it has a nontrivial outer automorphism.

Complete Groups and Symmetric Groups

Definition

A group G is called **complete** if:

- Its center is trivial: $Z(G) = \{e\}$
- Every automorphism is inner: $\mathrm{Aut}(G) = \mathrm{Inn}(G)$

Theorem

The symmetric group S_n is complete for all $n \neq 2, 6$.

The case $n = 6$ is exceptional:

- $|\mathrm{Inn}(S_6)| = |S_6| = 720$
- $|\mathrm{Aut}(S_6)| = 1440$
- So $\mathrm{Aut}(S_6) \neq \mathrm{Inn}(S_6)$, and S_6 is *not* complete

This makes S_6 the only symmetric group with a nontrivial **outer automorphism**.

Transpositions and Inner Automorphisms

Definition: Transposition

A **transposition** is a 2-cycle in S_n that swaps two elements and fixes the rest. For example, $(1\ 2) \in S_n$ sends $1 \mapsto 2$, $2 \mapsto 1$, and $k \mapsto k$ for all $k \neq 1, 2$.

- More generally, let T_k denote the conjugacy class of elements in S_n that are products of k disjoint transpositions.

Proposition

If an automorphism of S_n sends each transposition in T_1 to another transposition in T_1, then the automorphism is inner.

- For $n \neq 6$, the sizes of the conjugacy classes T_k are all distinct. In particular, $|T_1| \neq |T_k|$ for any $k \neq 1$.
- Therefore, any automorphism of S_n must preserve T_1, and by the lemma, it must be inner.

Proof Outline: Conjugacy Classes in S_n

- Recall: For each k, define T_k as the conjugacy class of elements in S_n that are products of k disjoint transpositions.

  $$T_1 = \text{transpositions}, \quad T_2 = \text{products of two disjoint transpositions}, \ldots$$

- If $f \in \mathrm{Aut}(S_n)$, then f sends conjugacy classes to conjugacy classes:

  $$f(T_1) = T_k \quad \text{for some } k.$$

- The size of T_1 is

  $$|T_1| = \binom{n}{2} = \frac{n(n-1)}{2}.$$

- For $k \geq 1$,

  $$|T_k| = \frac{1}{k!} \prod_{i=0}^{k-1} \binom{n-2i}{2}.$$

  These sizes count elements that are products of k disjoint transpositions.

Proof Outline: Uniqueness of T_1 and Completeness of S_n

- For $n \neq 6$, the sizes $|T_k|$ are all distinct, so

$$|T_1| \neq |T_k| \quad \text{for any } k \neq 1.$$

- But for $n = 6$, there is a unique coincidence:

$$|T_1| = |T_3| = 15,$$

  allowing a non-inner automorphism that maps transpositions to triple transpositions.

- Thus, for $n \neq 6$, automorphisms must preserve T_1.
- By the proposition, any automorphism that preserves T_1 is inner.
- Since the center $Z(S_n) = \{e\}$ for $n \geq 3$, S_n is **complete** for $n \neq 6$.

Constructing the Outer Automorphism via Conjugacy Classes

- Because $|T_1| = |T_3| = 15$ in S_6, there is a bijection:

$$\varphi : T_1 \longleftrightarrow T_3,$$

  swapping transpositions with triple disjoint transpositions.
- Extend φ to an automorphism of S_6 by defining its action on generators (transpositions).
- Since inner automorphisms preserve cycle structure, φ cannot be inner.
- Thus, φ is the unique outer automorphism of S_6.

Constructing the Outer Automorphism via The Group $\mathrm{PGL}_2(\mathbb{F}_5)$

- Consider the projective line over $\mathbb{F}_5$:

$$\mathbb{P}^1(\mathbb{F}_5) = \{0, 1, 2, 3, 4, \infty\}.$$

- $\mathrm{PGL}_2(\mathbb{F}_5)$ consists of Möbius transformations:

$$x \mapsto \frac{ax+b}{cx+d}, \quad \text{with } ad - bc \neq 0.$$

- The order of $\mathrm{PGL}_2(\mathbb{F}_5)$ is 120.

Sharp 3-Transitivity on 6 Points

- $\mathrm{PGL}_2(\mathbb{F}_5)$ acts on the 6 points of $\mathbb{P}^1(\mathbb{F}_5)$.
- The action is **sharply 3-transitive**:

  For any two triples of distinct points, there is exactly one transformation

- This gives an embedding:

$$\mathrm{PGL}_2(\mathbb{F}_5) \hookrightarrow S_6.$$

Realizing the Outer Automorphism

- Consider the subgroup $H := \mathrm{PGL}_2(\mathbb{F}_5) \cong S_5 \subset S_6$.
- The action of S_6 on the coset space S_6/H defines a homomorphism:

$$f : S_6 \to S_6.$$

- The image of f is isomorphic to S_5, but H is not conjugate to the standard $S_5 \subset S_6$.
- This homomorphism f induces the **outer automorphism** of S_6.

- **1. Conjugacy Class Sizes**
- **2. $\mathbf{PGL}_2(\mathbb{F}_5)$ Action**
- **3. Coset Action Representation:** Construct a homomorphism from S_6 acting on cosets of a subgroup $H \cong S_5$ of order 120
- **4. Mystic Pentagons / Geometric Construction:** Use the combinatorial structure of the six "mystic pentagons" related to S_5
- **5. Automorphisms of A_6:** Use the automorphism group structure of the alternating group A_6
- **6. Sylow Subgroups:** Analyze Sylow p-subgroups of S_6 to identify special subgroup embeddings that lead to non-conjugate S_5 subgroups

Thank You!

Thank You for Listening!

Questions?