OUTER AUTOMORPHISMS ON S_6

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ABSTRACT. The symmetric group S_n, consisting of all permutations on n elements, is a cornerstone of finite group theory. While the automorphism groups of most symmetric groups align precisely with their inner automorphisms, the case of S_6 stands as a unique exception. This paper investigates outer automorphisms in group theory, with a particular focus on the structure of $\mathrm{Aut}(S_6)$. We examine the definitions and consequences of group and symmetric group automorphisms, delve into the role of conjugacy classes, and explore several concrete constructions of the outer automorphism of S_6—including those arising from projective linear groups, coset actions, and the combinatorics of mystic pentagons.

1. Introduction

Group theory originated in the 19th century through the work of mathematicians such as Évariste Galois, who introduced the idea of a group as a way to study the solvability of polynomial equations. In this early framework, symmetric groups—denoted S_n, the groups of all permutations on n elements—emerged as fundamental objects because of their "universality" for finite groups.

The study of *automorphisms* of a group, or symmetries of the group itself, is a rich and central topic in group theory. In particular, we are interested in distinguishing between *inner* automorphisms—those arising from conjugation within the group—and *outer* automorphisms, which do not derive from conjugation. For many groups, all automorphisms are inner. This is notably true for most symmetric groups: it can be shown that $\mathrm{Aut}(S_n) \cong \mathrm{Inn}(S_n)$ for all $n \neq 2, 6$.

Yet the symmetric group on six elements, S_6, defies this pattern as it possesses a *outer automorphism* that does not arise from the conjugation by elements of S_6 itself. This singularity makes S_6 the only symmetric group with a nontrivial outer automorphism, and its study reveals an intersection between combinatorics, projective geometry over finite fields, and representation theory.

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The outer automorphism of S_6 has fascinated algebraists for over a century. Its construction involves deep concepts such as the classification of conjugacy classes, embeddings of projective linear groups like $\mathrm{PGL}_2(\mathbb{F}_5)$, and geometric symmetries such as those found in the "mystic pentagons". Understanding why S_6 is special provides both a case study in group theory and an accessible glimpse into more advanced topics such as group actions, exceptional isomorphisms, and the broader landscape of finite simple groups.

This paper aims to present a comprehensive exposition of the outer automorphism of S_6. We begin with the necessary background in group theory and symmetric groups, then move toward the formal definition of automorphisms and the distinction between inner and outer types. From there, we explore multiple concrete constructions of the outer automorphism of S_6.

2. What is a Group?

Before diving into the complexities of symmetric groups and their automorphisms, we must first establish the basic language and structure of group theory. The notion of a *group* is a fundamental concept in modern algebra, and it arises naturally in a wide variety of mathematical contexts—from solving polynomial equations to understanding symmetries in geometry and physics.

2.1. Definition of a Group.

Definition 1. *A* ***group*** *is a set G equipped with a binary operation $\cdot : G \times G \to G$ that satisfies the following four axioms:*

(1) ***Closure:*** *For all $a, b \in G$, the product $ab \in G$.*
(2) ***Associativity:*** *For all $a, b, c \in G$, we have $(ab)c = a(bc)$.*
(3) ***Identity:*** *There exists an element $e \in G$ such that for all $a \in G$, $ae = ea = a$.*
(4) ***Inverses:*** *For every $a \in G$, there exists an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$.*

2.2. **Examples and Motivation.** Groups arise in many natural settings. Here are a few classical examples to anchor the definition:

- **The integers under addition:** $(\mathbb{Z}, +)$ form a group. The identity is 0, and each element n has an inverse $-n$.
- **The nonzero real numbers under multiplication:** $(\mathbb{R}^\times, \cdot)$ form a group with identity 1 and inverses $1/x$.
- **Modular arithmetic:** $(\mathbb{Z}_n, +)$, the integers modulo n, form a group under addition mod n.

- **Matrices:** The set of invertible $n \times n$ matrices over $\mathbb{R}$, denoted $\mathrm{GL}_n(\mathbb{R})$, forms a group under matrix multiplication.

However, one of the most important families of groups in mathematics is the *symmetric groups*, which we now define.

2.3. Symmetric Groups.

Definition 2. *The* ***symmetric group*** S_n *is the group of all permutations (i.e., bijective functions) from the set* $\{1, 2, \ldots, n\}$ *to itself. The group operation is function composition.*

This group captures the idea of *relabeling* or *shuffling* elements and it plays a central role in many branches of mathematics. Some key points:

- The order of S_n is $n!$, the number of ways to permute n elements.
- S_3 is the smallest non-abelian (non-commutative) group.
- Symmetric groups are prototypical examples in the classification of finite groups and in the study of group actions.

2.4. **Notation and Representations.** Permutations in S_n can be represented in several ways:

- **Two-line notation:** This notation lists the elements and their images. For example, the permutation σ such that $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$ is written:
$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$
- **Cycle notation:** This is a more compact and intuitive format expresses how the permutation cycles elements. The above permutation is written $(1\ 2\ 3)$, indicating that 1 maps to 2, 2 to 3, and 3 back to 1.

Cycle notation is especially helpful in analyzing the structure of permutations, particularly when studying conjugacy classes and automorphisms.

3. Group Actions and Transitivity

3.1. **Group Actions.** In abstract algebra, a *group action* is a way to relate a group to another mathematical object. The concept allows us to interpret the elements of a group as symmetries or transformations of a set, which helps us understand both the structure of the group and the geometry or combinatorics of the set being acted upon.

Definition 3. *Let G be a group and let X be a set. A* ***(left) group action*** *of G on X is a map*

$$G \times X \to X, \quad (g, x) \mapsto g \cdot x,$$

satisfying the following axioms for all $g, h \in G$ and all $x \in X$:

(1) ***Identity:*** *$e \cdot x = x$, where $e \in G$ is the identity element.*
(2) ***Compatibility (associativity):*** *$(gh) \cdot x = g \cdot (h \cdot x)$.*

This definition captures the idea that each group element acts as a transformation of the set X, and that the group structure is preserved under this action. In particular, the composition of group elements corresponds to the composition of their respective transformations on X.

3.2. Transitive Actions.

Definition 4. *A group action of G on a set X is called* ***transitive*** *if for every pair of elements $x, y \in X$, there exists $g \in G$ such that*

$$g \cdot x = y.$$

Intuitively, this means that there is a single orbit under the action: the group can move any point of X to any other point. Transitivity is an important property because it implies a high degree of symmetry in the group's action. For example, many classical geometric groups act transitively on certain sets of points, lines, or configurations.

3.3. **Examples of Group Actions.** To illustrate these ideas concretely, we now describe several examples of group actions. Each example demonstrates how group elements operate as symmetries or transformations of a given set.

Example 1 (Dihedral group acting on a square)**.** *Let D_4 denote the* ***dihedral group of order 8****, which consists of the symmetries of a square. This includes:*

- *4 rotations (including the identity),*
- *4 reflections (over horizontal, vertical, and diagonal axes).*

The group D_4 acts on the set of vertices of the square by permuting them according to these symmetries. This action satisfies the group action axioms and is transitive on the set of vertices.

Example 2 (The integers acting on the real line)**.** *Let $G = \mathbb{Z}$, the group of integers under addition, and let $X = \mathbb{R}$, the set of real numbers. Define the action by*

$$n \cdot x = x + n, \quad \text{for all } n \in \mathbb{Z},\ x \in \mathbb{R}.$$

This is an example of a translation action. Each integer shifts the real number line left or right by a fixed amount. The action is not transitive on all of $\mathbb{R}$, but it is transitive on the cosets $\mathbb{R}/\mathbb{Z}$, which plays a key role in modular arithmetic and torus geometry.

Example 3 (Conjugation action)**.** *Let G be any group. Then G acts on itself by **conjugation**, defined by*

$$g \cdot x = gxg^{-1}, \quad \textit{for } g, x \in G.$$

We will cover conjugation in depth later. The orbits under this action are called conjugacy classes*, and they partition the group. The stabilizer of an element $x \in G$ under this action is the* centralizer *of x, and the set of fixed points is the* center *$Z(G)$ of the group. classes.*

3.4. **Permutation Representations.** Every group action on a finite set naturally gives rise to a **group homomorphism** from G into a symmetric group. That is, the group G can be thought of as permuting the elements of X, so there is an associated map

$$\phi : G \to S_X,$$

where S_X is the symmetric group on the set X. If $|X| = n$, then $S_X \cong S_n$, and we can view G as a subgroup of S_n. This realization is the essence of *Cayley's Theorem*, which states that every group is isomorphic to a subgroup of some symmetric group.

Group actions are thus the key bridge between abstract groups and concrete permutation groups—a crucial step in understanding the internal symmetries of objects.

3.5. **Orbits and Stabilizers.** Every group action gives rise to important associated concepts:

- The **orbit** of an element $x \in X$ under G is the set:

$$\mathrm{Orb}(x) = \{g \cdot x : g \in G\}.$$

- The **stabilizer** of x is the subgroup:

$$\mathrm{Stab}_G(x) = \{g \in G : g \cdot x = x\}.$$

- The **orbit-stabilizer theorem** states that if G is finite, then

$$|G| = |\mathrm{Orb}(x)| \cdot |\mathrm{Stab}_G(x)|.$$

3.6. **Importance in the Study of S_n.** Group actions are central to understanding the symmetric groups. Not only do symmetric groups themselves arise as groups of permutations (actions on finite sets), but much of their structure—such as conjugacy classes, normal subgroups, and automorphisms—can be understood through their actions on sets, partitions, or cosets. In particular, later in this paper we will examine how S_6 acts on cosets of certain subgroups and how this leads to a construction of the outer automorphism.

4. Automorphisms of Groups

Understanding the symmetries within a group G is fundamental to group theory. These symmetries are captured by *automorphisms*, which are isomorphisms from the group to itself that preserve the group structure.

4.1. Automorphisms.

Definition 5. *An* ***automorphism*** *of a group G is a bijective group homomorphism*

$$f : G \to G$$

such that for all $a, b \in G$,

$$f(ab) = f(a)f(b).$$

The set of all automorphisms of G, denoted $\mathrm{Aut}(G)$, forms a group under composition. The identity automorphism acts as the identity element in $\mathrm{Aut}(G)$, and every automorphism has an inverse which is also an automorphism.
Example: Automorphisms of a Cyclic Group. Consider the cyclic group

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$$

with addition modulo n. Every automorphism of $\mathbb{Z}_n$ is determined by the image of the generator 1. Since the automorphism must preserve order, 1 is sent to an element coprime to n. Hence,

$$\mathrm{Aut}(\mathbb{Z}_n) \cong (\mathbb{Z}_n)^\times = \{k \in \mathbb{Z}_n : \gcd(k, n) = 1\},$$

the multiplicative group of units modulo n.

4.2. **Inner Automorphisms.** Among automorphisms, a particularly important subclass arises from conjugation by elements of G.

Definition 6. *For a fixed element $a \in G$, the map*

$$f_a : G \to G, \quad f_a(x) = a^{-1}xa$$

is called an ***inner automorphism****.*

Each f_a is an automorphism, and the collection of all such maps forms a subgroup of $\mathrm{Aut}(G)$, called the **inner automorphism group** $\mathrm{Inn}(G)$.

Proposition 1. *The map*

$$\varphi : G \to \mathrm{Inn}(G), \quad a \mapsto f_a$$

is a surjective group homomorphism with kernel equal to the center of G,

$$Z(G) := \{z \in G : zx = xz \textit{ for all } x \in G\}.$$

By the First Isomorphism Theorem, we have the important isomorphism:

$$\mathrm{Inn}(G) \cong G/Z(G).$$

Moreover, $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$:

$$\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G).$$

Examples.

- If G is abelian, then $Z(G) = G$, and so $\mathrm{Inn}(G)$ is trivial.
- For the symmetric group S_n with $n \geq 3$, the center is trivial, so

$$\mathrm{Inn}(S_n) \cong S_n.$$

4.3. **Outer Automorphisms.** Not all automorphisms are inner. Those which are not can be detected by considering the quotient group

Definition 7. *The* ***outer automorphism group*** *of G is the quotient*

$$\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G).$$

Elements of $\mathrm{Out}(G)$ represent automorphisms of G that cannot be realized by conjugation by any element of G. If

$$\mathrm{Out}(G) \neq \{e\},$$

then G has at least one *outer automorphism.*
Notable facts.

- For most symmetric groups S_n, the outer automorphism group is trivial:

$$\mathrm{Out}(S_n) = \{e\} \quad \text{for } n \neq 6.$$

- The group S_6 is exceptional: it admits a nontrivial outer automorphism,

$$|\mathrm{Out}(S_6)| = 2,$$

making S_6 the only symmetric group with non-inner automorphisms.

5. Complete Groups and the Special Case of Symmetric Groups

5.1. **Definition of Complete Groups.** In the study of group automorphisms, a particularly important class of groups is that of *complete groups.*

Definition 8. *A group G is called* ***complete*** *if it satisfies two key properties:*

(1) *The center of G is trivial, i.e.,*
$$Z(G) = \{e\},$$
where
$$Z(G) = \{z \in G : zx = xz \text{ for all } x \in G\}.$$
(2) *Every automorphism of G is inner, that is,*
$$\text{Aut}(G) = \text{Inn}(G).$$

The triviality of the center means no nontrivial element commutes with every element of the group. The second condition ensures that all symmetries of the group arise from conjugation within the group itself.

5.2. **Completeness of Symmetric Groups.** The symmetric groups S_n, which are groups of all permutations on n elements, form a central object of study in group theory. Their automorphism groups have been well-characterized, yielding the following theorem about their completeness.

Theorem 1 (Completeness of Symmetric Groups)**.** *For all $n \neq 2, 6$, the symmetric group S_n is complete. That is,*
$$Z(S_n) = \{e\} \quad \text{and} \quad \text{Aut}(S_n) = \text{Inn}(S_n).$$

Remarks:

- For $n = 1$, S_1 is trivial.
- For $n = 2$, $S_2 \cong \mathbb{Z}_2$ is abelian, so the center is nontrivial, and the group is not complete.

5.3. **The Exceptional Case $n = 6$.** The case $n = 6$ is famously exceptional in the theory of symmetric groups.

- The order of S_6 is
$$|S_6| = 6! = 720.$$
- Since the center of S_6 is trivial (as for all $n \geq 3$), the inner automorphism group has order
$$|\text{Inn}(S_6)| = |S_6| = 720.$$

- However, it can be shown that the automorphism group of S_6 is strictly larger:

$$|\mathrm{Aut}(S_6)| = 1440 = 2 \times 720.$$

Thus,

$$\mathrm{Aut}(S_6) \neq \mathrm{Inn}(S_6),$$

and S_6 is *not* complete.

This indicates the existence of a nontrivial **outer automorphism** of S_6, a phenomenon unique to this group among the symmetric groups.

6. Proof: Conjugacy Classes and the Outer Automorphism of S_6

6.1. Conjugacy Classes of Products of Disjoint Transpositions.

Recall that for the symmetric group S_n, the conjugacy classes correspond to cycle types of permutations. One important family of conjugacy classes arises from products of disjoint transpositions.

Definition 9. *For each integer k with $1 \leq k \leq \lfloor n/2 \rfloor$, define*

$$T_k := \{\sigma \in S_n : \sigma \textit{ is a product of } k \textit{ disjoint transpositions}\}.$$

The conjugacy classes T_k are disjoint subsets of S_n, and each T_k consists of elements whose cycle decomposition has exactly k 2-cycles and $n - 2k$ fixed points.

The first few classes are:

$$T_1, T_2, T_3, \ldots$$

where

$$\begin{aligned} T_1 &= \text{transpositions}, \\ T_2 &= \text{products of two disjoint transpositions}, \\ T_3 &= \text{products of three disjoint transpositions}. \end{aligned}$$

6.2. Sizes of the Conjugacy Classes T_k.

The size of each conjugacy class T_k can be computed combinatorially as follows:

Proposition 2. *For $k \geq 1$,*

$$|T_k| = \frac{1}{k!} \prod_{i=0}^{k-1} \binom{n-2i}{2}.$$

Proof. To construct a product of k disjoint transpositions, select pairs sequentially:

- For the first transposition, choose 2 elements out of n, which can be done in $\binom{n}{2}$ ways.

- For the second transposition, choose 2 elements out of the remaining $n-2$ elements, giving $\binom{n-2}{2}$ choices.
- Continue similarly until the k-th transposition is chosen from $n-2(k-1)$ elements, with $\binom{n-2(k-1)}{2}$ choices.

Since the order in which the k transpositions are chosen does not matter, divide by $k!$ to account for permutations of these transpositions.

Hence the formula follows. □

In particular, for $k = 1$,

$$|T_1| = \binom{n}{2} = \frac{n(n-1)}{2}.$$

6.3. **Automorphisms and Conjugacy Classes.** Since any automorphism $f \in \mathrm{Aut}(S_n)$ must map conjugacy classes to conjugacy classes (because conjugacy is preserved under group isomorphisms), we have

$$f(T_1) = T_k \quad \text{for some } k,$$

meaning the image of the conjugacy class of transpositions must be some other conjugacy class T_k.

However, because conjugacy classes of symmetric groups are specifically identified by their cycle structure and size, f can only send T_1 to a conjugacy class of the same size.

6.4. **Uniqueness of T_1 for $n \neq 6$.** For most n, the sizes $|T_k|$ are all distinct. Thus:

$$|T_1| \neq |T_k| \quad \text{for all } k \neq 1,$$

which implies that f must send the transpositions to transpositions.

However, when $n = 6$, an exceptional coincidence occurs:

$$|T_1| = |T_3| = 15.$$

This equality allows the possibility of an automorphism that sends transpositions to triple disjoint transpositions.

6.5. **Completeness of S_n for $n \neq 6$.** By the proposition from the previous section, any automorphism that preserves T_1 (the transpositions) is inner. Hence for $n \neq 6$, all automorphisms are inner.

Moreover, the center of S_n is trivial for $n \geq 3$:

$$Z(S_n) = \{e\}.$$

This implies that for $n \neq 6$, S_n is a *complete* group, meaning:

$$\mathrm{Aut}(S_n) = \mathrm{Inn}(S_n).$$

6.6. **The Special Case $n = 6$: Existence of an Outer Automorphism.** The coincidence

$$|T_1| = |T_3| = 15$$

permits the construction of a non-inner automorphism:

Theorem 2. *There exists an automorphism $\varphi \in \mathrm{Aut}(S_6)$ such that*

$$\varphi : T_1 \longleftrightarrow T_3,$$

exchanging transpositions and triple disjoint transpositions. This automorphism is not inner and is the outer automorphism of S_6.

Idea of construction. Since T_1 and T_3 have the same size, one can define a bijection $\varphi : T_1 \to T_3$. Extending this map to the whole group by homomorphism property, and verifying it respects group operations, yields an automorphism.

Because inner automorphisms preserve cycle structure (and thus cannot interchange transpositions and triple transpositions), this φ is an outer automorphism. □

7. Constructing the Outer Automorphism via $\mathrm{PGL}_2(\mathbb{F}_5)$

7.1. **The Projective Line over $\mathbb{F}_5$.** Consider the finite field $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ with arithmetic modulo 5. The *projective line* over $\mathbb{F}_5$ is defined as

$$\mathbb{P}^1(\mathbb{F}_5) := \mathbb{F}_5 \cup \{\infty\} = \{0, 1, 2, 3, 4, \infty\}.$$

This set can be interpreted as the set of one-dimensional linear subspaces of the two-dimensional vector space $\mathbb{F}_5^2$. The addition of the point at infinity ∞ "completes" the line, making it a natural domain for projective transformations.

7.2. **The Group $\mathrm{PGL}_2(\mathbb{F}_5)$.** The group $\mathrm{PGL}_2(\mathbb{F}_5)$, or the *projective general linear group*, is defined as the quotient

$$\mathrm{PGL}_2(\mathbb{F}_5) := \mathrm{GL}_2(\mathbb{F}_5)/Z,$$

where $\mathrm{GL}_2(\mathbb{F}_5)$ is the group of invertible 2×2 matrices over $\mathbb{F}_5$, and Z is its center consisting of scalar multiples of the identity matrix. Concretely, $\mathrm{PGL}_2(\mathbb{F}_5)$ acts faithfully on $\mathbb{P}^1(\mathbb{F}_5)$ by Möbius transformations:

$$x \mapsto \frac{ax + b}{cx + d},$$

where $a, b, c, d \in \mathbb{F}_5$ satisfy $ad - bc \neq 0$, and by convention

$$\frac{a \cdot \infty + b}{c \cdot \infty + d} = \frac{a}{c}, \quad \text{and} \quad \frac{ax + b}{0 \cdot x + d} = \frac{ax + b}{d}.$$

The cardinality of $\mathrm{PGL}_2(\mathbb{F}_5)$ can be computed using the formula for $\mathrm{GL}_2(\mathbb{F}_5)$:

$$|\mathrm{GL}_2(\mathbb{F}_5)| = (5^2 - 1)(5^2 - 5) = 24 \times 20 = 480,$$

and the center $Z \cong \mathbb{F}_5^\times$ has order 4 (all nonzero scalars mod 5), so

$$|\mathrm{PGL}_2(\mathbb{F}_5)| = \frac{480}{4} = 120.$$

Remarkably, this order coincides with that of the symmetric group S_5, hinting at a deep connection between these groups.

7.3. **Sharp 3-Transitivity on Six Points.** The action of $\mathrm{PGL}_2(\mathbb{F}_5)$ on $\mathbb{P}^1(\mathbb{F}_5)$ is *sharply 3-transitive*, meaning:

Definition 10. *An action of a group G on a set X is* sharply k-transitive *if for any two ordered k-tuples of distinct elements $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ in X, there exists a* unique *group element $g \in G$ such that $g \cdot x_i = y_i$ for all $1 \leq i \leq k$.*

For $\mathrm{PGL}_2(\mathbb{F}_5)$ acting on the six points $\mathbb{P}^1(\mathbb{F}_5)$, it can be shown that this action is sharply 3-transitive:

- **Transitivity:** Given any two triples of distinct points, there is some Möbius transformation sending the first triple to the second.
- **Uniqueness:** This transformation is unique, which means the group action is "as transitive as possible" for triples.

This symmetry property allows us to embed $\mathrm{PGL}_2(\mathbb{F}_5)$ into the symmetric group on 6 points:

$$\mathrm{PGL}_2(\mathbb{F}_5) \hookrightarrow S_6,$$

via its natural action on $\mathbb{P}^1(\mathbb{F}_5)$.

7.4. **Realizing the Outer Automorphism.** Inside S_6, consider the subgroup

$$H := \mathrm{PGL}_2(\mathbb{F}_5) \cong S_5.$$

Note that this embedding is not the "standard" one given by fixing one point and permuting the remaining five elements. Instead, it corresponds to a more subtle embedding arising from the projective geometry of $\mathbb{P}^1(\mathbb{F}_5)$.

The symmetric group S_6 acts on the set of left cosets S_6/H, a set of size

$$|S_6/H| = \frac{|S_6|}{|H|} = \frac{720}{120} = 6.$$

This induces a group homomorphism:

$$f : S_6 \to S_6,$$

defined by the action on these cosets. The image of f is isomorphic to S_6, but the crucial point is that H is *not* conjugate to the "standard" $S_5 \subset S_6$. Thus, this conjugation action yields a non-inner automorphism of S_6.

More precisely, f induces an automorphism of S_6 which is *not* given by conjugation by an element of S_6. By definition, this automorphism represents the (up to inner automorphisms) **outer automorphism** of S_6.

8. Mystic Pentagons: A Combinatorial Realization of the Outer Automorphism

One of the most elegant and historically rich constructions of the outer automorphism of S_6 arises from a 19th-century discovery by J. J. Sylvester, who introduced a set of six combinatorial configurations called the *mystic pentagons*. This geometric insight reveals an unexpected symmetry hidden in the structure of the symmetric group on six elements.

8.1. **Definition of Mystic Pentagons.** Let the six elements of the set be $\{1, 2, 3, 4, 5, 6\}$. A **mystic pentagon** is defined as a configuration consisting of a 5-cycle — that is, five elements arranged cyclically — and one isolated vertex. There are precisely six such configurations, corresponding to the six choices of which vertex is left out of the pentagon.

Each mystic pentagon can be viewed as a cyclic ordering of five points:

$$(1 \to 2 \to 3 \to 4 \to 5 \to 1) \quad \text{with vertex 6 excluded,}$$

and similarly for the other five exclusions. Up to relabeling and symmetry, these six mystic pentagons form a distinct set on which S_6 can act.

8.2. **Action of S_6 on the Mystic Pentagons.** The symmetric group S_6 acts on the set of mystic pentagons by permuting the labels of the vertices. This gives a homomorphism:

$$\phi : S_6 \longrightarrow S_6,$$

where the codomain is the group of permutations of the six mystic pentagons.

However, this homomorphism is not the identity map, nor is it an inner automorphism — it is an example of a *nontrivial outer automorphism* of S_6. That is, the action of S_6 on the mystic pentagons induces an automorphism that does not arise from conjugation by an element of S_6.

8.3. **Consequences.** This construction demonstrates that:

$$\mathrm{Inn}(S_6) \subsetneq \mathrm{Aut}(S_6),$$

so that the quotient

$$\mathrm{Out}(S_6) := \mathrm{Aut}(S_6)/\mathrm{Inn}(S_6)$$

is nontrivial — in fact, $\mathrm{Out}(S_6) \cong \mathbb{Z}_2$. This is unique among symmetric groups: for all $n \neq 6$, we have $\mathrm{Out}(S_n) = \{e\}$. The symmetry in conjugacy class sizes and the special action on mystic pentagons only occurs when $n = 6$, making this construction an exception in group theory.

9. Outer Automorphism via Coset Action

Let us present a construction of the outer automorphism of S_6 by analyzing its action on cosets of a carefully chosen subgroup.

9.1. **Step 1: Subgroup $H \cong S_5 \subset S_6$.** Let S_6 be the symmetric group on 6 elements. Consider a subgroup $H \leq S_6$ isomorphic to S_5. For example, we may take H to be the stabilizer of a point in the natural action of S_6 on $\{1, 2, 3, 4, 5, 6\}$. That is,

$$H = \{\sigma \in S_6 \mid \sigma(6) = 6\} \cong S_5.$$

The index of this subgroup is 6, since there are 6 choices for the image of the fixed point under a general element of S_6. So:

$$[S_6 : H] = 6.$$

9.2. **Step 2: Permutation Representation via Cosets.** Let S_6 act on the left coset space S_6/H by left multiplication:

$$g \cdot (xH) := (gx)H \quad \text{for } g, x \in S_6.$$

This defines a group homomorphism:

$$\varphi : S_6 \to \mathrm{Sym}(S_6/H) \cong S_6.$$

Why is this isomorphic to S_6? Because the action is on a 6-element set, and so the group of permutations of that set is again S_6.

9.3. **Step 3: Kernel and Image.** The kernel of this homomorphism is the largest normal subgroup of S_6 contained in H. But since S_6 is simple modulo its alternating subgroup, and $A_6 \trianglelefteq S_6$ is of index 2, any normal subgroup must be either trivial, A_6, or S_6. Since $H \cong S_5$ and $A_6 \not\subset H$, the kernel is trivial. Hence, φ is injective.

Therefore, φ gives an embedding:

$$S_6 \hookrightarrow S_6.$$

9.4. **Step 4: Non-inner Automorphism.** Now, note that φ is a homomorphism from S_6 into itself, and by standard identification, we view it as an automorphism:

$$\varphi \in \mathrm{Aut}(S_6).$$

The key question is: is this automorphism inner?

To answer this, observe that the image $\varphi(S_6) \subset S_6$ has a subgroup isomorphic to S_5 — namely, the stabilizer of a point in the coset action — but this S_5 is not conjugate to the original point stabilizer in the standard action of S_6 on 6 letters.

More precisely, the subgroup $H \cong S_5$ does not lie in the same conjugacy class of subgroups of S_6 as the standard embedded copy of S_5. Thus, φ cannot be implemented by conjugation inside S_6. Therefore, φ is an outer automorphism.

9.5. **Conclusion.** This construction shows the existence of a non-inner automorphism of S_6. Hence,

$$\mathrm{Out}(S_6) = \mathrm{Aut}(S_6)/\mathrm{Inn}(S_6)$$

is nontrivial. In fact, it is of order 2, and S_6 is the only symmetric group S_n for which $\mathrm{Out}(S_n) \neq \{e\}$.

10. Overview of Constructions and Applications of the Outer Automorphism of S_6

The nature of the symmetric group S_6 is highlighted by the existence of a nontrivial outer automorphism. Several distinct constructions illustrate this phenomenon, each of which sheds light on different algebraic or geometric aspects of S_6:

- **1. Conjugacy Class Sizes:** By examining the sizes of conjugacy classes of elements in S_6, particularly the equality of sizes between single transpositions and triple disjoint transpositions, one can define a permutation of these classes that cannot arise from an inner automorphism. This combinatorial perspective offers a purely group-theoretic characterization of the outer automorphism.

- **2.** $\mathrm{PGL}_2(\mathbb{F}_5)$ **Action:** The projective linear group $\mathrm{PGL}_2(\mathbb{F}_5)$ acts sharply 3-transitively on 6 points, giving an embedding into S_6. This subgroup is isomorphic to S_5, but remains distinct from the standard one. The action on cosets yields the outer automorphism. This approach connects group theory with finite geometry and projective lines over finite fields.
- **3. Coset Action Representation:** We construct an automorphism by considering the action of S_6 on the coset space S_6/H, where $H \cong S_5$ is a subgroup of order 120. This leads to a homomorphism whose image is isomorphic to S_6, but not conjugate to the original copy. This blends representation theory and subgroup analysis to exhibit the outer automorphism concretely.
- **4. Mystic Pentagons / Geometric Construction:** The mystic pentagons comes from a combinatorial and geometric configuration related to S_5 which reveals symmetries that correspond to the outer automorphism. This viewpoint provides a visual and intuitive understanding of the concept, linking group theory to combinatorics and geometry.
- **5. Automorphisms of A_6:** The alternating group A_6 has an automorphism group strictly larger than itself, and since $A_6 \trianglelefteq S_6$, the automorphisms of A_6 extend to S_6. This approach utilizes structural properties of normal subgroups and simple groups.
- **6. Sylow Subgroups:** By analyzing the Sylow p-subgroups of S_6, we identify special embeddings of subgroups isomorphic to S_5 that are not conjugate within S_6. These embeddings lead to non-inner automorphisms which emphasizes the role of subgroup structure in understanding $\mathrm{Out}(S_6)$.

Takeaways.

- **Uniqueness in Group Theory:** S_6 is the only symmetric group with a nontrivial outer automorphism, making it an exceptional object of study and a classical exception in the classification of symmetric groups.
- **Connections to Finite Geometry and Algebraic Combinatorics:** The relationship with projective linear groups and combinatorial structures such as the mystic pentagons ties S_6 to geometric frameworks.

- **Role in Classification of Finite Simple Groups:** Understanding outer automorphisms of groups like A_6 and S_6 is crucial for the classification and characterization of finite simple groups and their automorphism groups.
- **Implications in Algebraic Topology and Galois Theory:** Outer automorphisms can correspond to symmetries of algebraic structures in topology and field extensions, linking abstract group properties to concrete algebraic objects.
- **Use in Mathematical Physics and Coding Theory:** Exceptional symmetries of S_6 appear in various physical models and error-correcting codes where symmetry groups govern system properties.

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