Measure Theory and Lebesgue Integration

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Euler Circle

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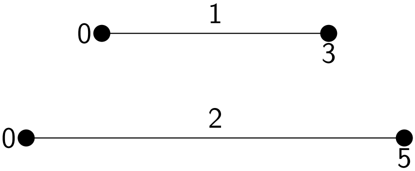
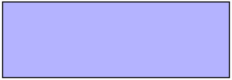
The notion of a measure



Want:- Assign a notion of "length"(Measure) for an abstract subset.
Familiar example:- Area of a square



Sets of interest

Definition 1

Now consider an abstract set X, we want to assign a notion of measure of a subset of the power set of X. This set (which we shall denote by Σ and call a σ-algebra) should be such that:-
(a) $\varnothing, X \in \Sigma$
(b) $A \in \Sigma$ implies $A^c \in \Sigma$
(c) $A_i \in \Sigma$ where $i \in \mathbb{N}$ implies $\cup_{i=1}^{\infty} A_i \in \Sigma$
(d) $A_i \in \Sigma$ where $i \in \mathbb{N}$ implies $\cap_{i=1}^{\infty} A_i \in \Sigma$

Definition 2

For $M \subseteq P(X)$, there is a smallest σ algebra that contains M:
$\sigma = \cap_{M \subseteq \Sigma} \Sigma$ this is known as the σ-algebra generated by M

Finally.. Measures

Let X be an abstract set and let Σ be a sigma-algebra corresponding to it. A map $\mu : \Sigma \to [+\infty, -\infty]$ is called a measure if μ satisfies the following properties:-

- **Non-negativity:-** For all $E \in \sigma$, $\mu(E) \geq 0$
- $\mu(\varnothing) = 0$
- **Countable additive**(or σ-additive):- For all countable collection $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ, $\mu(\cup_{k=1}^{\infty} E_k) = \Sigma_{k=1}^{\infty} \mu(E_k)$.

Countable Sub additivity

Theorem 3

*(**Countable sub additivity:-**) For any countable sequence E_1, E_2, E_3 of (not necessarily disjoint) measurable sets E_n in Σ:-*
$\mu(\cup_{i=1}^{\infty} E_i) \leq \Sigma_{i=1}^{\infty} \mu(E_i)$.

Proof.

Out of the infinite collection $\{E_i\}$, we choose disjoint sets $\{E_{a_i}\}$ out of them.
And thus, $\mu(\cup_{i=1}^{\infty} E_i) = \mu(\cup_{i=1}^{\infty} E_{a_i}) = \Sigma_{i=1}^{\infty} E_{a_i} \leq (\Sigma_{i=1}^{\infty} E_i)$. Hence, we proved the theorem we were seeking! ■

Outer Measures

Want:- Develop a procedure for constructing measures.

Definition 4

Let X be a set, an *outer measure* on X is a function $\mu* : P(X) \to [0, +\infty]$ such that
(a)$\mu^*(\varnothing) = 0$
(b) if $A \subseteq B \subseteq X$ the $\mu^*(A) \leq \mu^*(B)$, and
(c) if $\{A_n\}$ is an infinite sequence of subsets of X, then $\mu^*(\cup_n A_n) \leq \Sigma_n \mu^*(A_n)$.

Definition 5

Let X be a set, and let μ^*be an outer measure on X. A subset B of X is μ^*-measurable if $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$ holds for every subset A of X.

Constructing Measures

How does this connect to constructing measures?

It does so by the following theorem:-

Theorem 6

Let X be a set, let μ^* be an outer measure on X, and let $\mathscr{M}_{\mu^*}$ be the collection of all μ^*-measurable subsets on X. Then
(a) $\mathscr{M}_{\mu^*}$ is a σ algebra, and
(b) the restriction of μ^* to $\mathscr{M}_{\mu^*}$ is a measure on $\mathscr{M}_{\mu^*}$.

Lebesgue Measures

Let us illustrate the above procedure for and construct what is known as an Lebesgue Measure. We start with:-

Definition 7

For each A of $\mathbb{R}$, let $\mathscr{C}_A$ be the set of all infinite sequences $\{(a_i, b_i)\}$ of bounded open intervals such that $A \subseteq \cup_i (a_i, b_i)$. Then $\lambda^* : P(\mathbb{R}) \to [0, +\infty]$ is called the *Lebesgue outer measure* defined by $\lambda^*(A) = inf\{\Sigma_i (b_i - a_i) : \{(a_i, b_i)\} \in \mathscr{C}_A$

Now we proceed to the next step of our recipe, studying collections of μ^*-measurable subsets of X.

Definition 8

Let $(X, \mathcal{T})$ be a topological space. The σ algebra generated by open sets is called the *Borel σ-algebra* $B(X)$: $B(X) = \sigma(T)$. You can also prove that the Borel σ-algebra for reals is also generated by the collection of all sub intervals of $\mathbb{R}$ of the form$(-\infty, b]$.

Now, we have the following theorem:-

Theorem 9

Every Borel subset of $\mathbb{R}$ is Lebesgue measurable.

We shall denote the Lebesgue outer measures restricted to $\mathscr{M}_{\lambda^*}$ as λ and call it the Lebesgue measure (since, it is indeed a measure). Note that λ is translationally invariant.

Measurable Functions

Theorem 10

Let (X, Σ) be a measurable space, and let A be a subset of X that belongs to Σ. For a function $f : A \to [-\infty, +\infty]$ the conditions
(a)for each real number t the set $\{x \in A : f(x) \leq t\}$ belongs to Σ
(b)for each real number t the set $\{x \in A : f(x) < t\}$ belongs to Σ
(c)for each real number t the set $\{x \in A : f(x) \geq t\}$ belongs to Σ
(d)for each real number t the set $\{x \in A : f(x) > t\}$ belongs to Σ
are equivalent.

A function $f : A \to [-\infty, +\infty]$ is *measurable with respect to* Σ if it satisfies one and hence all conditions of the above theorem.

Integration- Riemann is not the full story

$$f(x) = \begin{cases} 1 & \text{if x rational} \\ 0, & \text{if x irrational} \end{cases}$$

Not Riemann Integrable!

The function ξ_E defined by:-

$$\xi_E = \begin{cases} 1 & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$$

is called a **characteristic function** on E. Let ϕ function which vanishes outside of a measurable set(simple function):- **Canonical representation of** ϕ:- $\phi = \sum_{i=1}^{n} a_i \xi_{A_i}$ where $A_i = \{x|\phi(x) = a_i\}$.We define the integral ϕ by $\int \phi dx = \sum_{i=1}^{n} a_i \mu(A_i)$.

Lebesgue Integration

Definition 11

If f is a bounded measurable function defined on a measurable set E with $\mu(E)$ finite , we define the Lebesgue integral of f over E by
$\int_E f(x)dx = inf \int_E \psi(x)$
for all simple functions $\psi \geq f$. and
$\int_E f(x)dx = sup \int_E \phi(x)$ for all simple functions $\phi \leq f$

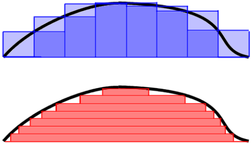


Figure 1: Riemann and Lebesgue integration. The blue one represents Riemann integration and the the red one Lebesgue.

Basic Properties of Lebesgue integration

Theorem 12

If f and g are bounded measurable functions defined on the set E of finite measure then,

(a) $\int_E af = a\int_E f$

(b)$\int_E (f+g) = \int_E f + \int_E g$

(c)If $f \leq g$ then, $\int_E f \leq \int_E g$

(d) If $f = g$ then, $\int_E f = \int_E g$

(e)If $f = g$ then, $\int_E f = \int_E g$

(f) If $A \leq f(x) \leq B$, the $A\mu(E) \leq \int_E f \leq B\mu(E)$

(g)If A and B are disjoint measurable set of finite measure, then $\int_{A\cup B} f = \int_A f + \int_B f$

3 Beasts- 1. Monotone Convergence Theorem

In real analysis:-

Theorem 13

Let $< x_n >$ be an increasing real sequence which is bounded above. Then $< x_n >$ converges to its supremum.

In measure theory and integration theory:-

Theorem 14

Let (X, Σ, μ) be a measurable space. Let $u : X \to \mathbb{R}$ be a positive Σ-measurable function. Let $< u_n >_{n \in \mathbb{N}}$ be a sequence of positiveΣ measurable functions $u_n : X \to \mathbb{R}$ such that $u_i(x) \leq u_j(x)$ for all $i \leq j$ and: $u(x) = lim_{n \to \infty} u_n(x)$ holds for all $x \in X$. Then, $\int u(x) d\mu = \lim_{n \to \infty} \int u_n(x) d\mu$.

3 Beasts- 2. Fatou's lemma

Theorem 15

Let (X, Σ, μ) *be a measure space and* $\{f_n : X \to [0, +\infty]\}$ *a sequence of nonnegative measurable functions. Then the function* $\lim_{n\to\infty} \inf f_n$ *is measurable and* $\int_X \lim_{n\to\infty} \inf f_n \leq \lim_{n\to\infty} \inf \int_X f_n$.

3 Beasts-3. Dominated Convergence Theorem

Theorem 16

Suppose $f_n : \mathbb{R} \to [-\infty, +\infty]$ *are Lebesgue measurable functions such that the point wise limit* $f(x) = \lim_{n\to\infty} f_n(x)$ *exists. Assume there is an integrable* $g : \mathbb{R} \to [0, \infty]$ *with* $|f_n(x)| \leq |g_n(x)|$ *for each* $x \in \mathbb{R}$*. Then* f *is integrable as is* f_n *for each* n*, and*
$\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} f_n d\mu = \int_{\mathbb{R}} f d\mu$