MEASURE THEORY AND LEBESGUE INTEGRATION

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Contents

Abstract. This expository paper offers a comprehensive introduction to the fundamental ideas of measure theory and Lebesgue integration, with an emphasis on mathematical rigor and conceptual clarity. We begin by motivating the need for a generalized notion of 'size' [11], leading to the construction of σ-algebras and measures. We then develop the theory of outer measures and use it to construct the Lebesgue measure on the real line. Building on this foundation, we explore measurable sets, measurable functions, and the properties that distinguish Lebesgue integration from the Riemann approach—particularly in handling discontinuous and non-classical functions. The aim is to provide a solid foundation for students encountering these ideas for the first time, while illustrating the elegance and necessity of measure-theoretic tools in modern analysis.

1. Introduction

The evolution of integration theory marks a significant turning point in modern analysis. While the Riemann integral served as a powerful foundational tool in 19th-century calculus, its limitations soon became apparent—especially when dealing with highly discontinuous functions, infinite domains, or limits under the integral sign. The classical framework struggled with many naturally arising functions in analysis, probability, and physics, prompting the need for a more general theory. This need was elegantly addressed by Henri Lebesgue, who in the early 20th century developed what is now known as Lebesgue integration—a formulation that underpins much of modern mathematics [6, 8].

To support this generalization, measure theory was developed as a way of extending the notion of length and area to more abstract sets. Rather than restricting integration to nicely

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behaved intervals, measure theory allows us to define a rigorous concept of "size" for a wide class of sets, even those with complicated or non-intuitive structure. Central to this theory is the notion of a σ-algebra, a collection of subsets closed under countable unions, intersections, and complements. This abstraction is critical for defining measurable sets, measurable functions, and ultimately, integrals that behave well under limits and transformations [4, 11].

The historical lineage of measure theory stretches beyond the formal developments of the 19th and 20th centuries. In fact, the intuitive idea of quantifying space and matter appears in ancient texts. Notably, the Indian mathematician Aryabhata, in his seminal work Āryabhaṭīyam (circa 499 CE), described formulas for computing areas and volumes of geometric figures such as circles, triangles, pyramids, and spheres [2]. Though not formalized in the modern sense, such work reflects early attempts to grasp the concept of magnitude—an idea that modern measure theory rigorously refines.

In this paper, we begin by motivating the construction of measures, starting with algebras and σ-algebras, then introducing outer measures to formally define Lebesgue measurable sets. We show how these ideas culminate in the construction of the Lebesgue measure on $\mathbb{R}$, which assigns to each interval its classical length and extends to much more complex sets via countable approximations [4, 6].

Following this, we define measurable functions—those functions for which the preimages of intervals are measurable sets—and explore how they interact with set operations and limit processes. This prepares us for the construction of the Lebesgue integral, which we introduce via simple functions (finite linear combinations of characteristic functions of measurable sets). We show how general functions can be approximated by increasing sequences of simple functions, enabling us to integrate functions that are not Riemann integrable, such as the characteristic function of the irrationals on [0,1] [5, 11].

To illustrate the contrast between Lebesgue and Riemann integration, we revisit classical examples and highlight the greater flexibility of the Lebesgue approach in handling discontinuities, infinite domains, and convergence of sequences of functions [1]. The Dirichlet function, for example, fails to be Riemann integrable but has a clear Lebesgue integral.

Throughout, the emphasis is on clarity and intuition. Each topic is introduced with motivation, supported by rigorous definitions and theorems, and supplemented with examples to illustrate key ideas. Our treatment is formal but pedagogically oriented, written for readers encountering these concepts for the first time in depth—especially students in advanced undergraduate or early graduate programs.

In conclusion, measure theory and Lebesgue integration provide an elegant and powerful framework that extends classical calculus to its modern form. These ideas are not just abstract constructs but essential tools for areas ranging from probability theory to functional analysis, ergodic theory, and quantum physics. Mastery of these foundational topics equips one with the language and logic underlying much of contemporary mathematics.

2. Measures and σ-algebras

In this section, we shall focus on developing our tools and dive into the field of mathematics named measure theory [11].

The intuition for measure is very simple; it is a generalization of geometrical measures (length, area, volume) and other common notions such as magnitude, mass, and the probability of events.

We wish to generalize the notion of a measure to be defined over abstract subsets (measure the ' length' of abstract subsets). Let us write a general measure μ as

$$\mu : \Sigma \to \mathbb{R} \cup \{-\infty, \infty\}$$

Where Σ is an abstract Set over which we wish to define the notion of a measure and $\mathbb{R} \cup \{-\infty, \infty\}$ is the extended Real number line, more will be said about it below. The extended real number system is obtained from the real number system $\mathbb{R}$ by adding two elements denoted as $+\infty$ and $-\infty$, which are respectively greater and lower than every real number.
As shall see below, the only subset of $\mathbb{R} \cup \{\infty, -\infty\} := [+\infty, -\infty]$ of interest to us is $[0, \infty]$. In this subset, we write the following counting rules [1]:-

- $x + \infty = \infty \ \forall x \in [0, \infty]$
- $x\infty = \infty \ \forall x \in [0, \infty]$

2.1. **σ-algebras.** Now, we shall take a short digression and recall the notion of an algebra. This will shortly be clear:-

Definition 2.1. Let X be an arbitrary set. A collection $\mathscr{A}$ of subsets of X is an *algebra* if

- $\phi, X \in \mathscr{A}$,
- For each set A that belongs to $\mathscr{A}$, A^c also belongs to $\mathscr{A}$,
- For each finite sequence $A_1, ...A_n$ of sets that belong to $\mathscr{A}$, the set $\cup_{i=1}^{N} A_i$
- For each finite sequence $A_1, ...A_n$ of sets that belong to $\mathscr{A}$, the set $\cap_{i=1}^{N} A_i$

Now, if X is a set, we wish to define a notion of measures for subsets of X and hence, $\Sigma \subseteq P(X)$. Now, we want to have a measure for the empty set ϕ and the whole set X. If we have a notion of measure for A, then we would also want a notion of the measure for the complement of A, A^c, a union and intersection of infinitely many sets which belong to Σ should also have a notion of a measure defined there. Combining all of these properties, we write the following important definition:-

Definition 2.2. $\Sigma \subseteq P(X)$ is called a σ - algebra if:-

- $\phi, X \in \Sigma$
- $A \in \Sigma$ implies $A^c \in \Sigma$
- $A_i \in \Sigma$ where $i \in \mathbb{N}$ implies $\cup_{i=1}^{\infty} A_i \in \Sigma$
- $A_i \in \Sigma$ where $i \in \mathbb{N}$ implies $\cap_{i=1}^{\infty} A_i \in \Sigma$

Note that the infinite union in the third and fourth criterion, as opposed to a finite union, is what distinguishes a σ-algebra from a general algebra. The reason why we define a σ-algebra and not just work with ordinary algebras is because that distinguishment is crucial for our purposes in particular, it is crucial for the countable additivity condition for a measure $A \in \Sigma$ is called a Σ **Measurable Set**.

Theorem 2.3. *All finite algebras are σ algebras.*

Proof. The infinite union of all subsets of a set is equivalent to a finite union of disjoint subsets of that set which is equivalent to finite unions of subsets of that set. Hence, all finite algebras are σ algebras since the finite union can be replaced by an infinite one. ■

To gain clarity, we illustrate the following trivial examples of Σ algebras [4]:-

- Let X be an set, and let Σ be the collection of all subsets of X. Then Σ is a σ-algebra on X.
- Let X be a set, and let $\Sigma = \{\phi, X\}$. Then Σ is a σ- algebra of X.
- Let X be an infinite set, and let Σ be the collection of all finite subsets of X . Then Σ does not contain X and is not closed under complementation; hence it is not a algebra (or a σ algebra) on X.
- Let X be an infinite set, and let Σ be the collection of all subsets A of X such that either A or A^c is finite. Then Σ is an algebra but is not closed under the formation of infinite unions and hence is not a σ-algebra.
- Let X be a set, and let Σ be the collection of all countable (i.e finite, or countably finite) subsets of X. Then Σ does not contain X and is not closed under complementation; hence it is not an algebra.
- Let Σ be the collection of all subsets of $\mathbb{R}$ that are unions of finitely many intervals of the form $(a, b]$, $(a, +\infty)$ or $(-\infty, b]$. It is easy to check that each set that belongs to Σ is the union of a finite disjoint collection of interval of the types listed above, and then to check that Σ is an algebra on $\mathbb{R}$ but not a σ-algebra.

Now, given two sigma algebras, we wish to see if we can combine them to have a new sigma algebra. This would help use construct $\sigma - algebras$. In particular, we have the following result:-

Theorem 2.4. *If Σ_i denote the σ- algebras on X where $i \in I$(index set). Then, $\cap_{i\in I}\Sigma_i$ is also a σ- algebra on X.*

Proof. Let $\mathscr{C}$ be a nonempty collection of σ- algebras on X, and let $\mathscr{A}$ be the intersection of σ-algebras that belong to $\mathscr{C}$. Let us check that all the axioms of a σ algebra are indeed satisfied. X and ϕ both belong to $\mathscr{A}$. If$A \in \mathscr{A}$then, A belongs to all sigma algebras in $\mathscr{C}$ and thus, A^c also belongs to all sigma algebras and thus A^c belongs to $\mathscr{C}$. If $\{A_i\}$ is a sequence of sets that belong to $\mathscr{A}$ and hence to each σ- algebra in $\mathscr{C}$. Then $\cup_i A_i$ and $\cap_i A_i$ both belong to each sigma algebra in (C) and so to $\mathscr{A}$. ■

From the above theorem, we are motivated to define the following:-

Definition 2.5. For $M \subseteq P(X)$, there is a smallest σ algebra that contains M: $\sigma = \cap_{M\subseteq\Sigma}\Sigma$ this is known as the σ-algebra generated by M.

As an illustration of the σ-algebra generated by M; we consider the following example:-
For $X = \{a, b, c, d, e, f\}$ and $M = \{a, f, \{b, c, d, e\}\}$;
$\sigma(M) = \{\phi, X, a, f, \{b, c, d, e\}, \{a, f\}, \{a, b, c, d, e\}, \{b, c, d, e, f\}\}$.

Now, we would want to have σ- algebras be generated topologically (by open sets) and thus, we have the following definition [3]:-

Definition 2.6. Let $(X, \mathcal{T})$ be a topological space. The σ algebra generated by open sets is called the Borel σ-algebra $\mathscr{B}(X)$: $\mathscr{B}(X) = \sigma(T)$.

Motivated to study the generation of Borel σ- algebras [3], we have the following theorem:-

Theorem 2.7. *The σ-algebra $\mathscr{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$is generated by each of the following collection of sets.*

- *the collection of all closed subsets of $\mathbb{R}$;*

- *the collection of all sub-intervals of* $\mathbb{R}$ *of the form* $(-\infty, b]$*;*
- *the collection of all sub-intervals of* $\mathbb{R}$ *of the form* $(a, b]$*.*

Proof. Let $\mathscr{B}_1, \mathscr{B}_2$ and $\mathscr{B}_3$ be the σ- algebras generated by the collection of sets in the above theorem. We will show that $\mathscr{B}_3 \subseteq \mathscr{B}_2 \subseteq \mathscr{B}_1 \subseteq \mathscr{B}(\mathbb{R})$ and $\mathscr{B}(\mathbb{R}) \subseteq \mathscr{B}_3$; this will establish the theorem. Since $\mathscr{B}(\mathbb{R})$ includes the family of open subsets of $\mathbb{R}$ and is closed under completmentation, it includes the family of closed subsets of $\mathbb{R}$; thus it includes the σ-algebra generated by the closed subsets of $\mathbb{R}$, namely $\mathscr{B}_1$. The sets of the form $(-\infty, b]$ are closed and so belong to $\mathscr{B}_1$; Consequently, $\mathscr{B}_2 \subseteq \mathscr{B}_1$. Since $(a, b] = (-\infty, b] \cap (-\infty, a]^c$, each set of the form $(a, b]$ belongs to $\mathscr{B}_2$; thus $\mathscr{B}_3 \subseteq \mathscr{B}_2$. Finally, note that each open sub-interval of $\mathbb{R}$is the union of the sequence of sets of the form $(a, b]$ and that each open subset of $\mathbb{R}$ is the union of the sequence of open intervals . Thus each open subset of $\mathbb{R}$ belongs to $\mathscr{B}_3$, and so $\mathscr{B}(\mathbb{R}) \subseteq \mathscr{B}_3$. ∎

Putting everything together, we have a notion of the pair (X, Σ).

Definition 2.8. The pair (X, Σ) is called a *Measurable Space* and the members of Σ are called measurable sets.

2.2. **Measures.** Now, having studied some aspects of σ- algebras, we move on to our main and original goal of this section, to define measures. We want our measure to be non-negative (if this condition is dropped then, the measure is called a *signed measure*). We also want the empty set to have a zero measure and we would want to relate the union of abstract sets which belong to a σ- algebra with the addition operator (defined over $\mathbb{R}$ and the extended real umber line) between measures of different subsets of the σ-algebra. In particular, we have the following definition:-

Definition 2.9. Let (X, Σ) be a measurable space.
A map $\mu : \Sigma \to \mathbb{R} \cup \{+\infty, -\infty\}$. where μ satisfies the following properties:-

- For all $E \in \sigma$, $\mu(E) \geq 0$
- $\mu(\phi) = 0$
- For all countable collection $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ, $\mu(\cup_{k=1}^{\infty} E_k) = \Sigma_{k=1}^{\infty} \mu(E_k)$. This property is known as *Countable additivity*

is called a measure on a sigma algebra Σ

Note that if at least one set E has a finite measure, then the condition that $\mu(\phi) = 0$ is satisfied automatically since $\mu(E \cup \phi) = \mu(E) = \mu(E) + \mu(\phi)$ and thus, $\mu(\phi) = 0$.
Once again, putting everything together, we have the following:-

Definition 2.10. A triple (X, Σ, μ) is called a *Measure Space.*

Furthermore, measure theory has many applications in probability theory which we won't go into in this paper. This motivates us to define the following:-

Definition 2.11. A *probability measure* is a measure with a total measure one, that is, $\mu(X) = 1$.
A *Probability Space* is a measure space with a *probability measure.*

Now consider the following examples of measures [1, 4]:-

- Let X be an arbitrary set, and let Σ be a σ-algebra on X. Define a function $\mu : \Sigma \to [0, +\infty]$ be letting $\mu(A)$ be n if A is a finite set with n elements and$\mu(A)$ be

$+\infty$ if A is an infinite set. Then μ is a measure often called the *counting measure* on (X, Σ).
- Let X be a nonempty set, and let Σ be a σ-algebra on X. Let x be a member of X. Define a function $\delta_x : \Sigma \to [0, +\infty]$ by letting $\delta_x(A)$ be 1 if $x \in A$ and letting $\delta_x(A)$ be 0 if $x \notin A$. Then δ_x is a measure; it is called a *point mass* concentrated at x.
- Let X be an arbitrary set, and let Σ be an arbitrary σ-algebra on X. Define a function $\mu : \Sigma \to [0, +\infty]$ by letting $\mu(A)$ be $+\infty$ if $A \neq \phi$, and letting $\mu(A)$ be 0 if $A = \phi$. Then μ is a measure.

Having had basic notions of σ-algebras and measures, we move on to investigating certain properties of measures. The first property that we can read off the definition is monotones:-

Theorem 2.12. *If E_1 and E_2 are measurable sets with $E_1 \subseteq E_2$ then,*
$\mu_1(E_1) \leq \mu(E_2)$

Proof. Since $E_1 \subseteq E_2$, there exists a set F such that $E_2 = F \cup E_1$ and thus,
$\mu(E_2) = \mu(F) + \mu(E_1)$. Since measures are non-negative, we see that the following relation must hold true:- $\mu_1(E_1) \leq \mu(E_2)$ ■

Now, recall the third criterion for a measure in its definition. Let us see what happens we don't choose the set E_k to be disjoint. In particular, we have the following theorem:-

Theorem 2.13. *For any countable sequence E_1, E_2, E_3 of (not necessarily disjoint) measurable sets E_n in Σ:- $\mu(\cup_{i=1}^{\infty} E_i) \leq \Sigma_{i=1}^{\infty} E_i$. The property is known as countable sub-additivity*

Proof. Of the infinite collection $\{E_i\}$, we choose disjoint sets $\{E_{a_i}\}$ from them.
And thus, $\mu(\cup_{i=1}^{\infty} E_i) = \mu(\cup_{i=1}^{\infty} E_{a_i}) = \Sigma_{i=1}^{\infty} E_{a_i} \leq (\Sigma_{i=1}^{\infty} E_i)$. Hence, we proved the theorem we were seeking! ■

3. Outer Measures

Having defined the measures and investigating some of its properties, we now want to develop a procedure for constructing the measures. We saw earlier the conditions for countable additivity and countable subadditivity. Let us define a new quantity by relaxing the condition for countable additivity but instead having countable subadditivity [4]:-

Definition 3.1. Let X be a set, an *outer measure* on X is a function $\mu* : P(X) \to [0, +\infty]$ such that

- $\mu^*(\phi) = 0$
- if $A \subseteq B \subseteq X$ the $\mu(A) \leq \mu(B)$, and
- if $\{A_n\}$ is an infinite sequence of subsets of X, then $\mu^*(\cup_n A_n) \leq \Sigma_n \mu^*(A_n)$

and the notion of μ^*-measurable sets:-

Definition 3.2. Let X be a set, and let μ^*be an outer measure on X. A subset B of X is μ^* -measurable if $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$ holds for every subset A of X.

We illustrate some examples below:-

- Let X be an arbitrary set, and define μ^* on $P(X)$ by $\mu^*(A) = 0$ if $A = \phi$ and $\mu^*(A) = 1$otherwise. Then μ^* is an outer measure.

- Let X be an arbitrary set, and define μ^* on $P(X)$ by $\mu^*(A) = 0$ if A is countable and $\mu^*(A) = 1$ if A is uncountable . Then μ^* is an outer measure.
- Let X be an infinite set, and define μ^* on $P(X)$ by $\mu^*(A) = 0$ if A is finite, and $\mu^*(A) = 1$ if A is infinite. Then μ^* fails to be countably sub additive and so not an outer measure.

How does this connect to constructing measures? Well, if μ^* is an outer measure on X, let $\mathscr{M}_{\mu^*}$ be the collection of all μ^*-measurable subsets on X. Then, one can show that not only is $\mathscr{M}_{\mu^*}$ a σ algebra, but the restriction of μ^* to $\mathscr{M}_{\mu^*}$ is a measure on $\mathscr{M}_{\mu^*}$. In particular,

Theorem 3.3. *Let X be a set, let μ^* be an outer measure on X, and let $\mathscr{M}_{\mu^*}$ be the collection of all μ^*-measurable subsets on X. Then*

- *$\mathscr{M}_{\mu^*}$ is a σ algebra, and*
- *the restriction of μ^* to $\mathscr{M}_{\mu^*}$ is a measure on $\mathscr{M}_{\mu^*}$.*

Proof. We shall first show that $\mathscr{M}$ is an sigma algebra. Note that both X and the empty set are all μ^*-measurable subsets on X. The μ^* measurability of a set B implies the μ^*-measurability of any set B^c since $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$ and so, if $B \in \mathscr{M}_{\mu^*}$ then $B^c \in \mathscr{M}_{\mu^*}$. Now suppose that B_1 and B_2 are μ^*-measurable subsets of X; we will show that $B_1 \cup B_2$ is μ^*- measurable. For this, let A be an arbitrary subset of X. The μ^*-measurability of B_1 implies
$\mu^*(A\cap(B_1\cap B_2)) = \mu^*(A\cap(B_1\cap B_2)\cap B_1)+\mu^*(A\cap(B_1\cap B_2)\cap B_1^c) = \mu^*(A\cap B_1)+\mu^*(A\cap B_1^c\cap B_2)$.
Using this identity and the fact that $(B_1 \cup B_2)^c = B_1^c \cap B_2^c$ and appealing to the measurability of B_1 and B_2, we get
$\mu^*(A \cap (B_1 \cap B_2)) + \mu^*(A \cap (B_1 \cap B_2)^c) =$
$\mu^*(A \cap B_1) + \mu^*(A \cap B_1^c \cap B_2) + \mu^*(A \cap B_1^c \cap B_2^c) = \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c) = \mu^*(A)$.
Since A was an arbitrary subset of X, the set $B_1 \cup B_2$ must be measurable. Thus $\mathscr{M}_{\mu^*}$ is an algebra.
Suppose that$\{B_i\}$ is an infinite sequence of disjoint μ^*-measurable sets, by induction, $\mu^*(A) = \Sigma_{i=1}^{n}\mu^*(A \cap B_i) + \mu^*(A \cap (\cap_{i=1}^{n} B_i^c))$ since this is true for $i = 1$ by definition and the rest follows from induction since if this is true then, $\mu^*(A \cap (\cap_{i=1}^{n} B_i^c)) =$
$\mu^*(A \cap (\cap_{i=1}^{n} B_i^c) \cap B_{n+1}) + \mu^*(A \cap (\cap_{i=1}^{n} B_i^c) \cap B_{n+1}^c) = \mu^*(A \cap (\cap_{i=1}^{n+1} B_i^c)) + \mu^*(A \cap (\cap_{i=1}^{n+1} B_i^c))$
Which means,
$\mu^*(A) = \Sigma_{i=1}^{\infty}\mu^*(A \cap B_i) + \mu^*(A \cap (\cap_{i=1}^{\infty} B_i^c)) = \mu^*(A \cap (\cup_{i=1}^{\infty} B_i) + \mu^*(A \cap (\cap_{i=1}^{\infty} B_i^c))$
$\mu^*(A) \geq \mu^*(A \cap (\cup_{i=1}^{\infty} B_i) + \mu^*(A \cap (\cap_{i=1}^{\infty} B_i^c))$.
and thus, it follows that $\cup_{i=1}^{\infty} B_i$is μ^*-measurable. Similarly, we can do the same for intersections and prove that $\cap_{i=1}^{\infty} B_i$is μ^*-measurable. We have thus proved that $\mathscr{M}_{\mu^*}$ is a σ-algebra. Replacing A with $\cup_{i=1}^{\infty} B_i$, $\mu^*(\cup_{i=1}^{\infty} B_i) = \Sigma_{i=1}^{\infty}\mu^*(B_i) + 0$. Hence, we proved that the restriction of μ to $\mathscr{M}_{\mu^*}$ is a measure on $\mathscr{M}_{\mu^*}$

■

Now, we turn into a very important example of an outer measure which we call a *Lebesgue outer measure* in $\mathbb{R}$

Definition 3.4. For each A of $\mathbb{R}$, let $\mathscr{C}_A$ be the set of all infinite sequences $\{(a_i, b_i)\}$ of bounded open intervals such that $A \subseteq \cup_i(a_i, b_i)$. Then $\lambda^* : P(\mathbb{R}) \to [0, +\infty]$ is called the *Lebesgue outer measure* defined by $\lambda^*(A) = inf\{\Sigma_i(b_i - a_i) : \{(a_i, b_i)\} \in \mathscr{C}_A$

Now let us prove that this is indeed an outer measure:-

Theorem 3.5. *Lebesgue outer measure on* $\mathbb{R}$ *is an outer measure and assigns to each sub-interval of* $\mathbb{R}$ *its length.*

Proof. We begin by verifying λ^* is an outer measure. The relation $\lambda^*(\phi) = 0$ holds, since for every positive number ϵ there is a sequence $\{(a_i, b_i)\}$ of open intervals (whose union necessarily includes ϕ) such that$\sum_i (b_i - a_i) < \epsilon$. For the monotonicity of λ^*, note that if $A \subseteq B$, then each sequence $\{(a_i, b_i)\}$ of open intervals that cover B also covers A, and so $\lambda^*(A) \leq \lambda^*(B)$. Now consider the countable sub-additivity of λ^*. Let $\{A_n\}_{n=1}^{\infty}$. If $\Sigma_n \lambda^*(A_n) = +\infty$, then $\lambda^*(\cup_n A_n) \leq \Sigma_n \lambda^*(A_n)$ certainly holds. So, suppose that $\Sigma_n \lambda^*(A_n) < +\infty$ and let ϵ be an arbitrary positive number. For each n choose a sequence $\{(a_i, b_i)\}_{i=1}^{\infty}$ that covers A_n and satisfies $\sum_{i=1}^{\infty}(b_{n,i} - a_{n,i}) < \lambda^*(A_n) + \frac{\epsilon}{2^n}$ If we combine these sequences to one sequence$\{a_i, b_j\}$ then the combined sequence satisfies $\cup_n A_n \subseteq \cup_j (a_j, b_j)$ and $\sum_j (b_j - a_j) < \sum_n (\lambda^*(A_n) + \frac{\epsilon}{2^n}) = \sum_n \lambda^*(A_n) + \epsilon$. Now we compute the Lebesgue outer measure of the sub-intervals of $\mathbb{R}$. First consider a closed bounded interval $[a, b]$. Now, $\lambda^*[a, b] \leq b - a$ since the RHS is the minimum of what the Lebesgue outer measure can produce since that interval will always be covered. We turn to the reverse inequality. Let $\{(a_i, b_i)\}$ be a sequence of bounded open intervals whose union includes $[a, b]$. Since $[a, b]$ is compact, there is a positive integer n such that$[a, b] \subseteq \cup_{i=1}^{n}(a_i, b_i)$. Now, $b - a = \sum_{i=1}^{n}(b_i - a_i)$ and thus, $b - a = \sum_{i=1}^{\infty}(b_i - a_i)$. Since $\{(a_i, b_i)\}$ was an arbitrary sequence whose union includes$[a, b]$, it follows that $b - a \leq \lambda^*([a, b])$. Thus, $\lambda^*([a, b]) = b - a$ The outer measure of an arbitrary bounded interval is its length, since such an interval I includes and is included in closed bounded intervals of length arbitrarily close to the length of I. Finally an unbounded interval has an infinite Lesbegue open measure. ■

We denote the *collection of Lebesgue measurable subsets of* $\mathbb{R}$ by $\mathscr{M}_{\lambda^*}$

Theorem 3.6. *Every Borel subset of* $\mathbb{R}$ *is Lebesgue measurable*

Proof. $\mathscr{B}(\mathbb{R})$ is the smallest σ-algebra that contains each interval of the form $(-\infty, b]$. Now, if every interval of the form $(-\infty, b]$ is Lebesgue measurable the, the statement of the theorem follows. Now, let B be such an interval and A be any subset of $\mathbb{R}$. So, we need only check that $\lambda^*(A) \geq \lambda^*(A \cap B) + \lambda^*(A \cap B^c)$ since the reverse inequality follows automatically by the countable sub-additivity condition. Let ϵ be an arbitrary positive number and let $\{(a_n, b_n)\}$ be a sequence of open intervals that cover A and satisfy $\sum_{i=1}^{\infty}(b_n - a_n) < \lambda^*(A) + \epsilon$. Then for each n the sets$(a_n, b_n) \cap B$ and $(a_n, b_n) \cap B^c$ are disjoint intervals whose union is (a_n, b_n), and so
$b_n - a_n = \lambda^*((a_n, b_n)) = \lambda^*((a_n, b_n) \cap B) + \lambda^*((a_n, b_n), \cap B^c)$. Since the sequence $\{(a_n, b_n) \cap B\}$ covers $A \cap B$ and the sequence $(a_n, b_n) \cap B^c\}$ covers $A \cap B^c$, from the countable sub-additivity of λ^*, we have:-
$\lambda^*(A \cap B) + \lambda^*(A \cap B^c) \leq \sum_n \lambda^*((a_n, b_n) \cap B) + \sum_n \lambda^*((a_n, b_n) \cap B^c) = \sum_n (b_n - a_n) < \lambda^*(A) + \epsilon$
However, ϵ was arbitrary and so the Lebesgue measurability of B follow.Thus, the collection $\mathscr{M}_{\lambda^*}$ of Lesbegue measurable sets is a σ-algebra on $\mathbb{R}$ all intervals of the form$(-\inf, b]$. However, $\mathscr{B}(\mathbb{R})$ is the smallest σ-algebra that contains all these intervals and so $\mathscr{B}(\mathbb{R}) \subseteq \mathscr{M}_{\lambda*}$

■

We shall denote the Lebesgue outer measures restricted to $\mathscr{M}_{\lambda^*}$ as λ and call it the Lebesgue measure (since, it is indeed a measure). Now, we shall study the various properties of Lebesgue measures. Now let us investigate Lebesgue measures of sets in terms of Lebesgue measures of its subsets and vica-versa:-

Theorem 3.7. *Let A be a Lebesgue measurable subset of* $\mathbb{R}$. *Then*

- $\lambda(A) = inf\{\lambda^(U) : U is\ open\ and\ A \subseteq U\}$
- $\lambda(A) = sup\{\lambda(K) : K is\ compact\ and\ K \subseteq A\}$

Proof. Note that the monotonicity of λ implies that
$\lambda(A) \leq inf\{\lambda(U) : U$ is open and $A \subseteq U\}$
and
$\lambda(A) \geq inf\{\lambda(K) : K$ is compact and $K \subseteq A\}$
. Hence, we need only prove reverse inequalities.Let ϵ be an arbitrary positive number.Then, there is a sequence$\{R_i\}$ of open intervals such that $A \subseteq \cup_i R_i$ and $\sum_i length(R_i) < \lambda(A) + \epsilon$. Let U be a union of these intervals. Then U is open, $A \subseteq U$, and $\lambda(U) = \sum_i \lambda(R_i) = \sum_i length(R_i) < \lambda(A) + \epsilon$. Since ϵ is arbitrary the first part is proved. We turn to part (b) and deal first with the case where A is bounded. Let C be a closed and bounded set that includes A, and let ϵ be an arbitrary positive number. Use part (a) to choose an open set U that includes $C - A$ and satisfies$\lambda(U) < \lambda(C - A) + \epsilon$.
Let $K = C - U$. Then K is a closed and bounded subset of A;furthermore, $C \subseteq K \cup U$ and so $\lambda(C) \leq \lambda(K) + \lambda(U)$. Now $\lambda(C - A) = \lambda(C) - \lambda(A)$ and thus, $\lambda(A) - \epsilon < \lambda(K)$. Since ϵ was arbitrary, part (b) is proved in the case where A is bounded.
Finally, consider the case where A is not bounded. Suppose b is any real number lass than $\lambda(A)$; we will produce a compact subset K of A such that $b < \lambda(K)$. Let $\{A_J\}$ be an increasing sequence of bounded measurable subsets of A such that $A = \cup_j A_j$.Now, $\lambda(A) = lim_j \lambda(A_j)$ and so we can choose j_o such that $\lambda(A_{j_0}) > b$. Now apply$\lambda(A_{j_0}) - \epsilon < \lambda(K)$. Since ϵ; this gives a compact subset K of A_{j_0} such that $\lambda(K) > b$. Since b was an arbitrary number less than $\lambda(A)$, the proof is complete. ■

Furthermore, we have the following translational symmetry property:-

Theorem 3.8. *Lebesgue outer measure on* $\mathbb{R}$ *is translational invariant, in the sense that if* $x \in \mathbb{R}$ *and* $A \subseteq \mathbb{R}$, *then* $\lambda^*(A) = \lambda^*(A + x)$. *Furthermore, a subset B of* $\mathbb{R}$*is Lebesgue measurable if and only if* $B + x$ *is Lebesgue measurable.*

Proof. For all$\{a, b\} \in \mathbb{R}$, $\lambda^*(\{a, b\}) = b - a = (b + x) - (a + x) = \lambda^*(\{a + x, b + x\})$. The second assertion follows from the first and the condition for Lebesgue measurable sets. ■

Theorem 3.9. *There is a subset of* $\mathbb{R}$, *and in fact of the interval* $(0, 1)$, *that is not Lebesgue measurable*

Proof. Define a relation $\cong$ on $\mathbb{R}$ by letting $x \cong y$ hold if and only if $x - y$ is rational. Note that; $\cong$ is reflexive ($x \cong x$ since 0 is rational), symmetric ($x \cong y$ implies $y \cong x$ since if p is rational then, $-p$ is also rational) and transitive (if $x \cong y$ and $y \cong z$ then, $x \cong z$, this is true because,if $x - y = p$ and $y - z = q$ where p and q are rational then, $p + q = x - z$'and since the sum of two rationals is always a rational, $x \cong z$).And hence, $\cong$ is an equivalence class. Note that each equivalence class under $\cong$ has the form $\mathbb{Q} + x$ for some x and so is dense in $\mathbb{R}$. Since these equivalence classes are disjoint , and since each intersects the interval $(0, 1)$, we can use the axiom of choice to form a subset E of $(0, 1)$ that contains exactly one element from each equivalence class. We will prove that E is not Lebesgue measurable. ■

Let $\{r_n\}$ be an enumeration of rational numbers in the interval $(-1, 1)$, and for each n and let $E_n = E + r_n$. We will check that
(a) the sets E_m are disjoint,
(b)$\cup_n E_n$ is included in the interval (-1,2), and,
the interval $(0, 1)$is included in $\cup_n E_n$.
To check (a), note that if $E_m \cap E_n \neq \phi$, then there are elements e and e' of E such that$e + r_m = e' + r_n$; it follows that $e \sim e'$ and hence$e = e'$ and $m = n$. Thus, (a) is proved. Assertion (b) follows from the inclusion $E \subseteq (0, 1)$ and the fact that each term of the sequence $\{r_n\}$ belongs to $(-1, 1)$. Now consider the assertion (c). Let x be an arbitrary member of $(0, 1)$ and let e be the member of E that satisfies $x \sim e$. Then $x - e$ is a rational and belongs to $(-1, 1)$ and so has the form r_n for some n. Hence $x \in E_n$ and assertion (c) is proved.
Suppose that the set E_n is Lebesgue measurable. Then, for each n, the set E_n is measurable , and so the property (a) above implies that $\lambda(\cup_n E_n) = \sum_n \lambda(E_n)$; furthermore the translational invariance of λ implies that $\lambda(E_n) = \lambda(E)$ holds for each n. Hence, if $\lambda(E) = 0$, then $\lambda(\cup_n E_n) = 0$, contracting the assertion (c) above, while if $\lambda(E) \neq 0$, then $\lambda(\cup_n E_n) = +\infty$ contracting the assertion (b) above. Thus, by proof by contradiction, E is not Lebesgue measurable.

4. Measurable Functions

In this section we shall extend the notion of measurability to functions rather than just sets and define the notion of a measurable functions. In particular, we shall begin with the following observation [4]:-

Theorem 4.1. *Let (X, Σ) be a measurable space, and let A be a subset of X that belongs to Σ. For a function $f : A \to [-\infty, +\infty]$ the conditions*

- *for each real number t the set $\{x \in A : f(x) \leq t\}$ belongs to Σ*
- *for each real number t the set $\{x \in A : f(x) < t\}$ belongs to Σ*
- *for each real number t the set $\{x \in A : f(x) \geq t\}$ belongs to Σ*
- *for each real number t the set $\{x \in A : f(x) > t\}$ belongs to Σ are equivalent.*

Proof. The identity $\{x \in A : f(x) < t\} = \cup_n \{x \in A : f(x) \leq t - 1/n\}$ implies that each of the sets appearing in the second condition is a union of sequence of sets appearing in the first condition; hence the first condition implies the second condition . The sets appearing in the third condition can be expressed by those appearing in the second condition by the means of the identity $\{x \in A : f(x) \geq t\} = A - \{x \in A : f(x) < t\}$;
thus the second condition implies the third condition . Similarly by
$\{x \in A : f(x) > t\} = \cup_n \{x \in A : f(x) \geq t - 1/n\}$, we see that the third condition implies the fourth condition. ■

A function $f : A \to [-\infty, +\infty]$ is *measurable with respect to* Σ if it satisfies one and hence all conditions of the above theorem. To illustrate this, we shall consider the following examples:-

- Let $f : \mathbb{R} \to \mathbb{R}$ be continuous . Then for each real number t the set $\{x : \mathbb{R}^d : f(x) < t\}$ is open and so is a Borel set . Thus f is Borel measurable.
- Let I be a sub-interval of $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be decreasing . Then for each real number t the set $\{x \in I : f(x) < t\}$ is a Borel set(it is either an interval, a set consisting of only one point, or the empty set).Thus f is Borel measurable.

Now, we further explore the properties of measurable functions and how a measurable space (a σ algebra) can be "made" out of two measurable functions:-

Theorem 4.2. *Let (X,Σ) be a measurable space, let A be a subset of X that belongs to Σ, and let f and g be* $[-\infty, +\infty]$*-valued functions on A. Then the sets* $\{x \in A : f(x) \leq g(x)\}$*, and* $\{x \in A : f(x) = g(x)\}$ *belong to* Σ.

Proof. Note that the inequality $f(x) < g(x)$ holds if and only if there is a rational number r such that $f(x) < r < g(x)$. Thus,
$\{x \in A : f(x) < g(x)\} = \cup_{r \in \mathbb{Q}}(\{x \in A : f(x) < r\} \cap \{x \in A : r > g(x)\}$
and so $\{x \in A : f(x) < g(x)\}$, as the union of a countable collection of sets that belong to Σ, itself belongs to Σ. The set $\{x \in A : g(x) < f(x)\}$ likewise belongs to Σ. This and the identity $\{x \in A : f(x) \leq g(x)\} = A - \{x \in A : g(x) < f(x)\}$
imply that $\{x \in A : f(x) \leq g(x)\}$ belong to Σ. Finally $\{x \in A : f(x) = g(x)\}$ is the difference of $\{x \in A : f(x) \leq g(x)\}$ and $\{x \in A : f(x) < g(x)\}$ and so belongs to Σ. ■

Let f and g be $[-\infty, +\infty]$- valued functions having a common domain. The *maximum* and *minimum* of f and g, written $f \vee g$ and $f \wedge g$, are functions from A to $[-\infty, +\infty]$ defined by
$(f \vee g)(x) = max(f(x), g(x))$ and
$(f \wedge g)(x) = min(f(x), g(x))$
If $\{f_n\}$ is a sequence of $[-\infty, +\infty]$- valued functions on A, then $sup_n f_n : A \to [-\infty, +\infty]$ is defined by $(sup f_n)_n(x) = sup\{f_n(x) : n = 1, 2, ...\}$.
$\inf_n f_n$, lim $sup_n f_n$, lim $\inf_n f_n$ and $lim_n f_n$ are defined in analogous ways.
Now that we have defined these $\wedge$ and $\vee$ operations , we move on to the next reasonable thing to do- check whether the measurability of a function is preserved by these operations. In particular, we have the following theorems:-

Theorem 4.3. *Let* (X, Σ) *be a measurable space, let A be a subset of X that belongs to* Σ, *and let f and g be*$[-\infty, +\infty]$*- valued measurable functions on A. Then* $f \vee g$ *and* $f \wedge g$ *are measurable.*

Proof. The measurability of $f \vee g$ follows form the identity
$\{x \in A : (f \vee g)(x) \leq t\} = \{x \in A : f(x) \leq t\} \cap \{x \in A : g(x) \leq t\}$,
and the measurability of $f \wedge g$ follows from the identity
$\{x \in A : (f \wedge g)(x) \leq t\} = \{x \in A : f(x) \leq t\} \cup \{x \in A : g(x) \leq t\}$ ■

Theorem 4.4. *Let* (X, Σ) *be a measurable space, let A be a subset of X that belongs to* Σ, *and let* f_n *be a sequence of* $[-\infty, +\infty]$*-valued measurable functions on A. Then*

- *the functions* $sup_n f_n$ *and* $inf_n f_n$ *are measurable*
- *the functions* $limsup_n f_n$ *and* $liminf_n f_n$ *are measurable,*
- *the functions* $lim_n f_n$ *(whose domain is* $\{x \in A$:$limsup_n f_n$ =$lim\ inf_n f_n$

Proof. The measurability of $sup_n f_n$ and $\inf_n f_n$ follows from the identities:-
$\{x \in A : sup_n f_n(x) \leq t\} = \bigcap_n \{x \in A : f_n(x) \leq t\}$ and
$\{x \in A : \inf_n f_n(x) < t\} = \bigcup_n \{x \in A : f_n(x) < t\}$
For each positive integer k define the functions g_k and h_k by $g_k = sup_{n \geq k} f_n$ and
$h_k = inf_{n \geq k} f_n$. Part (a) of the proposition implies the first that each g_k is measurable and that each h_k is measurable and then that $inf_k g_k$ and $sup_k h_k$ are measurable. Since $limsup_n f_n$ and $liminf_n f_n$ are equal to $inf_k g_k$ and $sup_k g_k$, they too are measurable . Let A_0 be the domain of $lim_n f_n$. Then A_0 is equal to $\{x \in A : \mathrm{lim} sup_n f_n(x) = \mathrm{liminf}_n f_n(x)\}$ which

belong to Σ. Since
$\{x \in A : lim_n f_n(x) \le t\} = A_0 \cap \{x \in A : lim_n sup f_n(x) \le t\}$,
the measurability of $lim_n f_n$ follows. ■

Similarly, now we check whether scalar multiplication and addition preserves measurability:-

Theorem 4.5. *Let (X, Σ) be a measurable space , let A be a subset of X that belongs to Σ, let f and g be $[0, +\infty]$-valued measurable-functions on A, and let α be a nonnegative real number . Then αf and $f + g$ are measurable.*

Proof. For the measurability of αf, note that if $\alpha = 0$, then αf is identically 0 and hence measurable, while if $\alpha > 0$, then for each t the set $\{x \in A : \alpha f(x) < t\}$ is equal to $\{x \in A : \alpha f(x) < t/\alpha\}$ and so belongs to Σ.
We turn to $(f + g)(x)$. Now $(f + g)(x) < t$ holds if an only if there is a rational number r such that $f(x) < r$ and $g(x) < t - r$. Thus
$\{x \in A : (f + g)(x) < t\} = \bigcup_{r \in \mathbb{Q}}(\{x \in A : f(x) < r\} \bigcap \{x \in A : (f + g)(x) < t - r\}$
and so $\{x \in A : (f + g)(x) < t\}$, as the union of countable collection of sets that belong to Σ, belong to Σ. The measurability of $f + g$ follows. ■

And now, we consider a wider range of operators and a wider class of measurable functions:-

Theorem 4.6. *Let (X, Σ) be a measurable space, let A be a subset of X that belongs to Σ, let f and g be a measurable- real valued functions on A, and let α be a real number. Then α, $f + g$,$f - g$, f/g (where the domain of $\{x \in A : g(x) \neq 0\}$) are measurable*

Proof. The measurability αf and $f + g$ can be verified by modifying the proof of the theorem above (note that if $\alpha < 0$, then $\{x \in A : \alpha f(x) < t\} = \{x \in A : f(x) > t/\alpha\}$. The measurability of $f + g$ follows from that of $f + (-1)g$. We turn to the proof of measurable functions and begin by showing if $h : A \to \mathbb{R}$ is measurable, then h^2 is measurable. For this note that if $t \le 0$, then
$\{x \in A : h^2(x) < t\} = \phi$ while if $t > 0$, then
$\{x \in A : h^2(x) < t\} = \{x \in A : \sqrt{-t} < h^2(x) < \sqrt{t}\}$; the measurability of h^2 follows. Hence if f and g are measurable, then f^2 and g^2 and $(f + g)^2$ are measurable, and the measurability of fg follows from the identity $fg = \frac{(f+g)^2 - f^2 - g^2}{2}$.
Let $A_0 = \{x \in A : g(x) \neq 0\}$, so that A_0 is the domain of f/g. A_0 belongs ot Σ. Since for each t the set $\{x \in A_0 : (f/g)(x) < t\}$ is the union of
$\{x \in A : g(x) > 0\} \cap \{x \in A : f(x) < tg(x)\}$
and
$\{x \in A : g(x) < 0\} \cap \{x \in A : f(x) > tg(x)\}$
the measurability of f/g follows. ■

Motivated by the definition of $\wedge$ and $\vee$ operators, we write the following definition down:-

Definition 4.7. Let A be a set, and let f bean extended real-valued function on A. The *positive part* f^+ and the negative part f^- of f are extended real valued functions defined by
$f^+ = max(f(x), 0) = f \vee 0$ and
$f^- = -min(f(x), 0) = (-f) \wedge 0$

Investigating measurability more, we have the following theorem:-

Theorem 4.8. *Let (X, Σ) be a measurable space, let A be a subset of X that belongs to Σ, and let f be a $[0, +\infty]$-valued measurable function on A. Then there is a sequence $\{f_n\}$ of simple $[0, +\infty)$-valued measurable functions on A that satisfy*
$f_1(x) \leq f_2(x) \leq ...$
and
$f(x) = lim_n f_n(x)$
at each x in A

Proof. For each positive integer n and for $k = 1, 2, ...n2^n$ let
$A_{n,k} = \{x \in A : \frac{k-1}{2^n} < f(x) < \frac{k}{2^n}\}$. The measurability of f implies that each $A_{n,k}$ belongs to Σ. Define the sequence $\{f_n\}$ of functions from A to $\mathbb{R}$ by requiring f_n to have the value $\frac{(k-1)}{2^n}$ at each point in $A_{n,k}$ (for $k = 1, 2, ...n2^n$) and to have value n at each point in $A - \cup_k A_{n,k}$. The functions defined above are simple and measurable, and they do satisfy the above conditions at each x in A. ∎

5. Integration

5.1. **Riemann Integration.** Now, we proceed to the second part of the paper namely-Integration [5, 8, 11]. We shall first have a quick overview of Riemann integration after which we shall dice into Lebesgue integration

Definition 5.1. The lower Riemann sum of f(x) corresponding to the dissection
$\Delta = x_{j+1} - x_j$ for all $j \in \mathbb{N}$ is defined as the following sum:-
$s(f, \Delta) = \sum_{n=1}^{j} \Delta \inf_{x \in \Delta} f(x)$
and the upper Riemann sum is given by:-
$S(f, \Delta) = \sum_{n=1}^{j} \Delta \sup_{x \in \Delta} f(x)$

The upper Riemann integral is always at least as large as the lower one, if the two are equal, we say that f is Riemann integrable and call this common value the Riemann integral of f. Now, we shall relate the Riemann integral with step functions:-

Definition 5.2. A **step function** is a function ψ that has the form
$\psi(x) = c_i, x_{i-1} < x < x_i$ for some subdivision of $[a, b]$ and some set of constants c_i

The integral of $\psi(x)$ is defined by
$s(\psi, \Delta) = \sum_{i=1}^{n} c_i \Delta$
With this in mind, we see that $s(f, \Delta) = inf s(\psi, \Delta)$
for all step functions $\psi(x) \geq f(x)$.
Similarly, $s(f, \Delta) = sup s(\phi, \Delta)$
for all step functions $\phi(x) \leq f(x)$.
Now, to sketch the shortcomings of the Riemann integral, we consider the following example:-
If

$$f(x) = \begin{cases} 1 & \text{if x rational} \\ 0, & \text{if x irrational} \end{cases}$$

then, $S(f, \Delta) = b - a$ and $s(f, \Delta) = 0$
Thus, we see that $f(x)$ is not integrable in the Riemann sense.

5.2. **Lebesgue Integral.** Now, we would like a function which is 1 in a measurable set and zero elsewhere to be integrable and have its integral the measure of the set.
The function ξ_E defined by:-

$$\xi_E = \begin{cases} 1 & \text{if x} \in \text{E} \\ 0, & \text{if x} \notin \text{E} \end{cases}$$

is called a **characteristic function** on E. A linear combination $\phi(x) = \sum_{i=1}^{n} a_i \xi_E(x)$ is called a **simple** function if the sets E_i are called measurable. This representation of ϕ is not unique. If ϕ is a simple function and $[a_1, a_2, ..., a_n]$ the set of non-zero values of ϕ, then $\phi = \sum_{i=1}^{n} a_i \xi_{A_i}$ where $A_i = \{x|\phi(x) = a_i\}$. **This representation for ϕ is called the canonical representation and** is characterized by the fact that the $A_i's$ are disjoint and the a_i are distinct and nonzero.
If ϕ vanishes outside a set of finite measure , we define the integral ϕ by
$\int \phi dx = \sum_{i=1}^{n} a_i \mu(A_i)$. In particular, we have the following theorem:-

Theorem 5.3. *If $E_1, E_2, ...E_n$ are disjoint measurable subset of E then every linear combination $\phi = \sum_{i=1}^{n} c_i \xi_{E_i}$*
With real coefficients $c_1, c_2, ..., c_n$ is a simple function and $\int \phi = \sum_{i=1}^{n} c_i \mu(E_i)$

Proof. It is clear that ϕ is a simple function. Let $a_1, a_2, ..., a_n$ denote the non-zero real number in $\phi(E)$. For each $j = 1, 2, ...n$.
Let $A_j = \cup_{c_i = a_j} E_i$.
Then we have $A_j = \phi^{-1}(a_j) = \{x|\phi(x) = a_j\}$ and the canonical representation
$\phi = \sum_{j=1}^{n} a_j \xi_{A_j}$. Consequently, we obtain
$\int \phi = \sum_{j=1}^{n} a_j \mu(A_j) = \sum_{j=1}^{n} a_j \mu(\cup_{c_i=a_j} E_i) = \sum_{j=1}^{n} a_j \sum_{c_i=a_j}^{n} \mu(E_i) = \sum_{j=1}^{n} c_j \mu(E_j)$
This completes the proof! ∎

Now, we turn over to investigate certain properties of this integral, in particular, we have the following property of linearity and monotoneness of integrals of simple functions that we would expect:-

Theorem 5.4. *Let ϕ and ψ be simple functions that vanish outside of a set of finite measure . Then $\int(a\phi + b\psi) = a\int \phi + b\int \psi$ and if, $\phi \geq \psi$ then, $\int \phi \geq \int \psi$*

Proof. Let $\{A_i\}$ and $\{B_i\}$ be the sets which occur in the canonical representation of ϕ and ψ. Let A_0 and B_0 be the sets where ϕ and ψ are zero. Then the set E_k is obtained by taking all the intersections $A_i \cap B_k$ form a finite disjoint collection of measurable sets, and we write
$\phi = \sum_{k=1}^{N} a_k \xi_{E_k}$ and $\psi = \sum_{k=1}^{N} b_k \xi_{E_k}$
and so, $a\phi + b\psi = a\sum_{k=1}^{N} a_k \xi_{E_k} + b\sum_{k=1}^{N} b_k \xi_{E_k} = \sum_{k=1}^{N}(aa_k + bb_k)\xi_{E_k}$
Therefore,$\int a\phi + b\psi = \sum_{k=1}^{N}(aa_k + bb_k)\mu(E_k) = a\sum_{k=1}^{N}(a_k)\mu(E_k) + b\sum_{k=1}^{N}(b_k)\mu(E_k) =$ $a\int \phi + b\int \psi$.
And hence, the first statement of the theorem is proved! To prove the second statement, we note that $\int \phi - \int \psi = \int \phi - \psi \geq 0$ since the integral of a simple function which is greater than or equal to 0 is non negative.

∎

In what follows, we shall also use the notion of **almost everywhere**, a property is said to hold almost everywhere if it holds everywhere except where the measure of the set under

consideration is 0.
Now we want to have a notion for Integrability like when we had for Riemann Integrability, if the upper and lower Riemann integral are equal. This motivates us to have the following theorem:-

Theorem 5.5. *Let f be defined and bounded on a measurable set E with $\mu(E)$ finite. In order that $inf_{f\leq\psi}\int_E \psi(x)dx = sup_{f\geq\phi}\int_E \phi(x)dx$. For all simple functionsϕ and ψ, it is necessary and sufficient thatf be measurable.*

Proof. Let f be bounded by M and suppose that f is measurable . Then the sets
$E_k = \{x|\frac{KM}{n} \geq f(x) > \frac{(K-1)M}{N}\}$,
are measurable, disjoint and have union E. This $\sum_{k=-n}^{k=+n} \mu E_k = \mu E$.
The simple functions defined by $\psi_n(x) = \frac{M}{n}\sum_{k=-n}^{n} k\xi_{E_k}(x)$ and
$\psi_n(x) = \frac{M}{n}\sum_{k=-n}^{n}(k-1)\xi_{E_k}(x)$
satisfy $\phi_n(x) \leq f(x) \leq \psi_n(x)$.
Now, $inf\int_E \psi(x)dx \leq \int_E \psi_n(x)dx = \frac{M}{n}\sum_{k=-n}^{n} k\mu(E_k)$ and
$sup\int_E \phi(x)dx \geq \int_E \phi_n(x)dx = \frac{M}{n}\sum_{k=-n}^{n}(k-1)\mu(E_k)$.
Hence,$0 \leq inf\int_E \psi(x)dx - sup\int_E \phi(x)dx \leq \frac{M}{n}\sum_{k=-n}^{n}\mu(E_k) = \frac{M}{n}\mu(E_)$. Since n is arbitrary,$inf\int_E \psi(x)dx - sup\int_E \phi(x)dx = 0$
and the condition is sufficient.
Suppose now that $inf_{\psi\geq f}\int_E \psi(x)dx = sup_{f\geq\phi(x)}\int_E \phi(x)dx$. Then given n simple functionsϕ_n and $\psi_n$such that $\phi_n \leq f(x) \leq \psi_n$. And $\int \psi_n dx - \int \phi_n dx < \frac{1}{n}$.
Then the functions $\psi^* = inf\psi_n$ and $\phi^* = sup\phi_n$ are measurable and $\phi^*(x) \leq f(x) \leq \psi^*(x)$.
Now the set $\Delta = \{x|\phi^*(x) < \psi(x)$ is the union of the sets $\Delta_v = \{x|\phi^*(x) < \psi^*(x) - \frac{1}{v}\}$.
But each Δ_v is contained in the set$\{x|\phi_n(x) < \psi_n(x) - \frac{1}{v}\}$ and this latter set has a measure less than $\frac{v}{n}$. Since n is arbitrary, $\mu(\Delta_v) = 0$ and hence $\mu(\Delta) = 0$. Thus $\phi^* = \psi^*$, and $\phi^* = f$. Thus f is measurable and the condition is also necessary. ∎

Definition 5.6. If f is a bounded measurable function defined on a measurable set E with $\mu(E)$ finite , we define the Lebesgue integral of f over E by
$\int_E f(x)dx = inf\int_E \psi(x)$
for all simple functions$\psi \geq f$. By the previous theorem, this may also be defined as
$\int_E f(x)dx = sup\int_E \phi(x)$

Now, having this notion of Lebesgue integration, in order to get most out of it, we compare it with that of Riemann Integration. Notable, we have the following main differences:-

- The most obvious difference is that in Lebesgue's definition, we divide up the interval into subsets while in the case of Riemann we divide it into sub-intervals.
- In both Riemann and Lebesgue's definitions, we have upper and lower sums which tend to limits. In the Riemann case, the two integrals are not necessarily the same and the function is integrable only if they are same.In the Lebesgue case, the two integrals are not necessarily the same, their equality being the consequence of the assumption that the function is measurable .
- Lebesgue's definitions more general than Riemann. We know if the function is R-integrable then it is Lebesgue integrable also, but the converse need not be true. For example, the characteristic function of the set of irrational points has Lebesgue integral but not R-integrable .

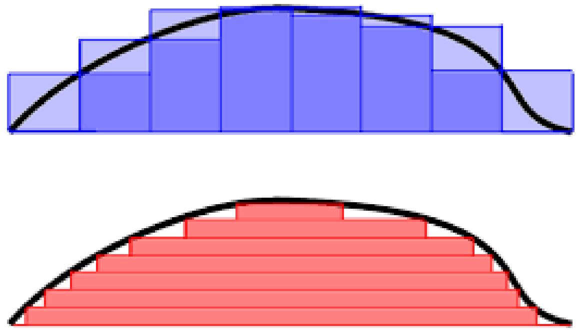

Figure 1. Riemann and Lebesgue integration. The blue one represents Riemann integration and the the red one Lebesgue.

Now let us establish some basic properties of Lebesgue integrals:-

Theorem 5.7. *If f and g are bounded measurable functions defined on the set E of finite measure then,*

- $\int_E af = a\int_E f$
- $\int_E (f+g) = \int_E f + \int_E g$
- *If $f \leq g$ then, $\int_E f \leq \int_E g$*
- *If $f = g$ then, $\int_E f = \int_E g$*
- *If $f = g$ then, $\int_E f = \int_E g$*
- *If $A \leq f(x) \leq B$, the $A\mu(E) \leq \int_E f \leq B\mu(E)$*
- *If A and B are disjoint measurable set of finite measure, then $\int_{A\cup B} f = \int_A f + \int_B f$.*

Proof. We know that if ψ is a simple function then so is ψ. Hence $\int_E af = inf_{\psi \geq f} \int_E a\psi = ainf_{\psi \geq f} \int_E \psi = a\int_E f$ which proves the first statement. To prove the second statement, let ϵ denote any positive real number. These are simple functions $\phi \leq f, \psi \geq f, \xi \leq g$ and $\eta \geq g$ satisfying
$\int_E \phi(x)dx < \int_E f - \epsilon$ and $\int_E \psi(x)dx > \int_E f + \epsilon$,
$\int_E \xi(x)dx > \int_E g - \epsilon$ and $\int_E \eta(x)dx < \int_E g + \epsilon$,
Since these hold for every $\epsilon > 0$, we have $\int_E (f+g) = \int_E f + \int_E g$. To prove the third statement,from the second statement, we see that it suffices to establish that if $f \leq g$ then, $\int_E g - f \geq 0$. Now from the condition of the problem, $g - f \geq 0$ and thus, for all simple functions $\psi \geq g - f \geq 0$, $\int_E \psi \geq 0$ therefore, $\int_E g - f = inf_{\psi \geq g-f} \int_E \psi(x) \geq 0$ which establishes the third statement. Similarly, we can show that $\int_E (f-g) = inf_{\psi \geq (f-g)} \int_E \psi(x) \leq 0$ and thus, the fourth statement follows. To prove the fifth statement, we are given that $A \leq f(x) \leq B$. Applying the third statement, we see that$\int_E fdx \leq \int_E Bdx = B\int_E dx = B\mu(E)$. That is, $\int_E f \leq B\mu(E)$; similarly we can also prove that $A\mu(E) \leq \int_E f$. We know that $\xi_{A\cup B} = \xi_A + \xi_B$ which proves the fourth statement. ■

5.3. **Fatou's lemma, Monotone Convergence Theorem and the Dominated Convergence Theorem.** Now, we turn our attention into 3 profound theorems in measure theory which highlight the advantage of Lebesgue integration over Riemann integration. We start with the *Monotone Convergence Theorem.*

Recall the Monotone Convergence theorem in real analysis [9]:-

Theorem 5.8. *Let $< x_n >$ be an increasing real sequence which is bounded above. Then $< x_n >$ converges to its supremum.*

Proof. Let $B = sup x_n$. We need to show that $x_n \to B$ as $n \to \infty$. For all $\epsilon > 0$, $B - \epsilon$ is not an upper bound. Thus, $\exists N \in \mathbb{N} : x_N > B - \epsilon$.; but $< x_n >$ is increasing. Hence, $\forall n > N : x_n \geq x_N > B - \epsilon$. Now, B is till an upper bound for $< x_n >$, then: $\forall n > N : B - \epsilon < x_n \leq B$ which means, $\forall n > N : B - \epsilon < x_n < B + \epsilon$ finally, $\forall n > N : |x_n - B| < \epsilon$. Hence, $< x_n >$ indeed converges to its supremum. ∎

We now proceed to the monotone convergence theorem in the framework of measure theory and Lebesgue integration [10]:-

Theorem 5.9. *Let (X, Σ, μ) be a measurable space. Let $u : X \to \mathbb{R}$ be a positive Σ-measurable function. Let $< u_n >_{n\in\mathbb{N}}$ be a sequence of positiveΣ measurable functions $u_n : X \to \mathbb{R}$ such that $u_i(x) \leq u_j(x)$ for all $i \leq j$ and: $u(x) = lim_{n\to\infty} u_n(x)$ holds for all $x \in X$ except when $\mu(x) = 0$. Then, $\int u(x)d\mu = \lim_{n\to\infty} \int u_n(x)d\mu$.*

Proof. Now since $u_i(x) \leq u_j(x)$ for all $i \leq j$, it follows that $\int u_i d\mu \leq \int u_j d\mu$. and hence, the the integral of the sequence$< u_n >$ of positive Σ measurable functions is an increasing sequence. Applying the monotone convergence theorem for real analysis, $\lim_{n\to\infty} \int u_n(x)d\mu$ exists and $\int u(x)d\mu \geq \lim_{n\to\infty} \int u_n(x)d\mu$. To prove the equality, we know prove the reverse inequality. Let ϕ be a simple function $\phi \leq u$ and, $E_n = \{x : f_n(x) \geq \phi(x)\}$ since u_n is an increasing sequence, $E_n \subseteq E_{n+1}$ and $E_n \subseteq X$. Now, $\lim_{n\to\infty} \int_{E_n} \phi d\mu = \int_X \phi d\mu = \int \phi d\mu$. Now, $\int_{E_n} \phi d\mu \leq \int_{E_n} u d\mu$ taking limit, $n \to \infty$, $\int_X \phi d\mu \leq \lim_{n\to\infty} \int u_n d\mu$ taking the supremum of ϕ, we get $\int u(x)d\mu \leq \lim_{n\to\infty} \int u_n(x)d\mu$. We have thus obtained the result we desired. ∎

We know proceed to the second main result of this subsection, namely, Fatou's lemma [7].

Theorem 5.10. *Let (X, Σ, μ) be a measure space and $\{f_n : X \to [0, +\infty]\}$ a sequence of nonnegative measurable functions. Then the function $\lim_{n\to\infty} inf f_n$ is measurable and $\int_X \lim_{n\to\infty} inf f_n \leq \lim_{n\to\infty} \inf \int_X f_n$.*

Proof. For each $k \in \mathbb{N}$, let $g_k = \inf_{n\geq k} f_n$ and define $h = \lim_{k\to\infty} g_k = \lim_{k\to\infty} \inf_{n\geq k} f_n = \lim_{n\to\infty} \inf f_n$. Note that $\int g_k \leq \int f_n$ for all $n \geq k$, and thus; $\int g_k \leq \inf_{n\geq k} \int f_n$. By the monotone convergence theorem ,$\int_X \lim_{n\to\infty} \inf f_n = \int h = \lim_{k\to\infty} \int g_k \leq \lim_{k\to\infty} \inf_{n\geq k} \int f_n = \lim_{n\to\infty} \inf \int f_n$. And thus, we have arrived at the desired result. ∎

And now, we finally turn our attention into the dominated convergence theorem [12]:-

Theorem 5.11. *Suppose $f_n : \mathbb{R} \to [-\infty, +\infty]$ are Lebesgue measurable functions such that the point wise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists.Assume there is an integrable $g : \mathbb{R} \to [0, \infty]$ with $|f_n(x)| \leq |g_n(x)|$ for each $x \in \mathbb{R}$. Then f is integrable as is f_n for each n, and $\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} f_n d\mu = \int_{\mathbb{R}} f d\mu$.*

Proof. Since $|f_n(x)| \leq |g_n(x)|$ and g is integrable, $\int_{\mathbb{R}} |f_n| d\mu \leq \int_{\mathbb{R}} g d\mu < \infty$. So f_n is integrable. Let $h_n = g - f_n$, so that $h_n \geq 0$. By Fatou's lemma, $\lim_{n\to\infty} \inf \int_{\mathbb{R}} (g - f_n) d\mu > \int_{\mathbb{R}} \lim_{n\to\infty} \inf (g - f_n) d\mu$, and that gives, $\lim_{n\to\infty} \inf (\int_{\mathbb{R}} g d\mu - \int_{\mathbb{R}} f_n d\mu) = \int_{\mathbb{R}} g d\mu - \lim_{n\to\infty} \sup \int_{\mathbb{R}} g d\mu \geq \int_{\mathbb{R}} f d\mu - \int_{\mathbb{R}} f_n d\mu$ or,

$\lim_{n\to\infty}\sup\int_{\mathbb{R}} f_n d\mu \leq \int_{\mathbb{R}} f d\mu$. Repeat this Fatou's lemma argument with $g + f_n$ instead of $g - f_n$, we get $\lim_{n\to\infty}\inf\int_{\mathbb{R}}(g + f_n)d\mu > \int_{\mathbb{R}}\lim_{n\to\infty}\inf(g + f_n)d\mu$, and that gives $\lim_{n\to\infty}\inf(\int_{\mathbb{R}} gd\mu + \int_{\mathbb{R}} f_n d\mu) = \int_{\mathbb{R}} gd\mu + \lim_{n\to\infty}\inf\int_{\mathbb{R}} f_n d\mu \geq \int_{\mathbb{R}} gd\mu + \int_{\mathbb{R}} f_n d\mu$ or, $\lim_{n\to\infty}\inf\int_{\mathbb{R}} f_n d\mu \geq \int_{\mathbb{R}} f d\mu$. Finally, we have $\int_{\mathbb{R}} f d\mu \leq \lim_{n\to\infty}\inf\int f_n d\mu \leq \lim_{n\to\infty}\sup\int_{\mathbb{R}} f_n d\mu \leq \int f_n d\mu$ and that gives the result because if $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty}$, it implies that $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$. ∎

In summary, the monotone convergence theorem, Fatou's lemma and the dominated convergence theorem are fundamental results in integration theory and measure theory. The Monotone convergence theorem in measure theory, motivated by that in real analysis,is used when dealing with increasing sequence of functions particularity when you can establish monotonicity and point wise convergence.This theorem indicates the power of Lebesgue integration. Fatou's lemma is particularly used when dealing with sequence of non-negative functions and results in an inequality which has its application in mathematics like measure theory,probability theory . The dominated convergence theorem provides the conditions under which the limit of the integral of a sequence is equal to the integral of the limit of the sequence, this theorem is useful across areas of mathematics like probability theory, functional analysis and the study of partial differential equations.

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