Criticality in random simplicial complexes

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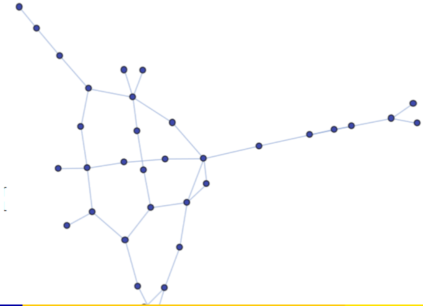
Random Graphs

1. Graphs generated from a random distribution
2. Most common model: $\mathbf{G}(n, p)$
   1. n is the number of nodes
   2. p is the independent probability that two nodes will have an edge between them
3. Why we care: can be used to model real world complex phenomena or as a null model



Phase transitions

1. Phase transitions occur at *a threshold*
2. Definition: a rapid and predicted change in a certain property of an object
3. A property can be anything physically observed
4. Can be one-sided or sharp
5. Why we care: criticality is all around us!

Journey through transitions: acyclic $\rightarrow$ cyclic graph

At $p = \frac{1}{n}$, the graph transitions away from acyclicity

1. Acyclic graph is a graph without cycles
2. Cycles are paths of length > 0 from a vertex back to itself

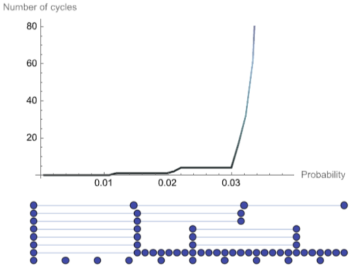


Figure: Acyclic graph and number of cycles with n = 60

Journey through transitions: emergence of giant component

At $p = \frac{1}{n}$, a giant connected component emerges

1. Coincides with the previous threshold...hmmm

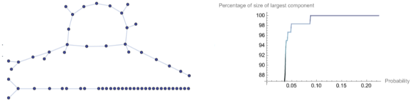


Figure: Size of largest component and visual with n = 60

Journey through transitions: emergence of giant component

At $p = \frac{1}{n}$, a giant connected component emerges

1. Coincides with the previous threshold...hmmm

Theorem

W.h.p $\mathbf{G}_{n,p}$ *consists of a unique giant component with* $(1 - \frac{x}{c} + o(1))n$ *vertices. Here* $0 < x < 1$ *is the unique solution of the equation* $x_1 e^{-x_1} = c_1 e^{-c_1}$ *The remaining components are of order at most* $O(\log n)$.

Giant component: proof sketch

Define constants β_0 and β_1 such that small components will have order $1 \leq k < \beta_0 \log n$ and giant components have order $k > \beta_1 n$. And $k_0 = \frac{1}{2\alpha} \log n, \alpha = c - 1 - \log c$

Lemma

The expected number of vertices within small tree components of order $1 \leq k \leq k_0$ is $\frac{nx}{c}$

Lemma

The number of vertices kX_k of small tree components with order: $k_0 < k \leq \beta_0 \log n$ is approximately $o(n)$.

Giant component: proof sketch

Lemma

The number of vertices kY_k of small connected components with order $1 < k \leq \beta_0 \log n$ is approximately $o(n)$.

The leading term in this summation of small components is $\frac{nx}{c}$. The rest of the vertices are in the giant component, it is distinct.

Journey through transitions: fully connected

At $p = \frac{\log n}{n}$, the graph becomes fully connected

1. There is a path from each vertex to every other vertex

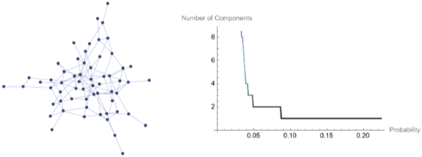


Figure: Number of components and visual with n = 60 as a function of p

Simplicial Complexes

1. an n-simplex, σ, is an n-dimensional shape
   1. they are simple shapes: points, lines, triangles, tetrahedron
2. a *simplicial complex* is a set of n-simplicies

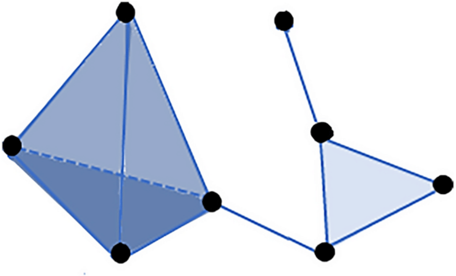


Figure: Image of 3d simplicial complex

Random Simplicial Complexes

Mashing the two together!

1. Linial-Meshulam Model: $Y_k(n, p)$
   1. k is the dimension
   2. k-1 simplicies (nodes) are connected by k-simplicies (edges)
2. $Y_1(k)$ is an Erdos Renyi graph
3. Interesting application as a null model: study patterns in brain structure in response to stimulus.

Crash course in homology

$H_d(X, G)$

1. Quantifies the d-dimensional holes
2. d is the dimension of the holes
3. X is the topological space
4. G is the group of integer coefficients for chains
5. A hole is a d-dimensional thing that resists compression in such dimension

Why homology in simplicial complexes?

1. $H_{d-1}(X, G)$ is the measure of connectivity
   1. think of a complete graph as a sphere that can be compressed
2. $H_d(X, G)$ is the measure of cyclicity (topology)
   1. Nontrivial holes are formed from cycles that do not have boundaries

Journey through transitions: acyclic $\rightarrow$ cyclic space

At $p = \frac{c_d}{n}, c_d = d + 1 - o(1)$ the simplicial complex is no longer acyclic.

1. Topologically, this means that $H_d(Y, G) = 0$
2. Interestingly, this coincides with another threshold...

Journey through transitions: giant shadow

At $p = \frac{c_d}{n}, c_d = d + 1 - o(1)$, a "giant" shadow $SH(X)$ emerges.

1. SH(X)is a set of d-simplicies that aren't in the set of X, but would create cycles if added
2. It is the dominant **higher-dimensional analogy** of the giant component, a measure of cyclicity and components

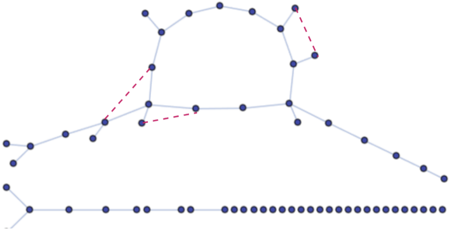


Figure: Some elements of the shadow visualized

Journey through transitions: fully connected simplicial complex

At $p = (1 + o(1))\frac{d \log n}{n}$, the simplicial complex becomes fully connected

1. Topologically, this means that $H_{d-1}(X, G) = 0$
2. Notice how similar this threshold is to the Erdos Renyi model threshold ($p = \frac{\log n}{n}$)