Traffic Flow Theory

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Euler Circle

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Intro

1. Fundamental concepts
2. Macroscopic models
   1. Lighthill-Whitham-Richards (LWR) model
   2. Payne-Whitham (PW) model
3. Microscopic models
   1. Car following models
   2. Cellular automata

What is Traffic Flow Theory?

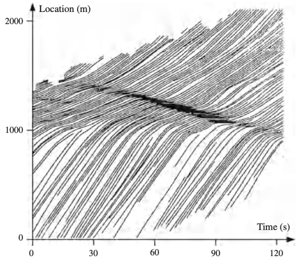


Figure 1: Time-space diagram

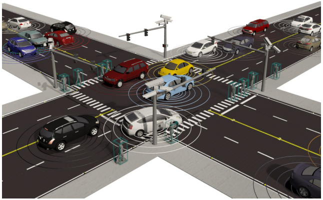
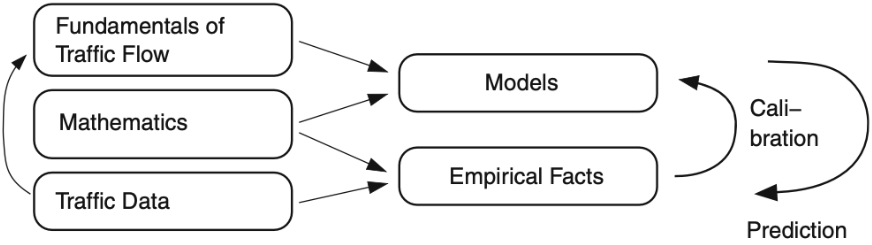


Figure 2: Simulation of a signalized intersection

Mathematical descriptions of the interaction between vehicles, operators and infrastructure.

What is Traffic Flow Theory?

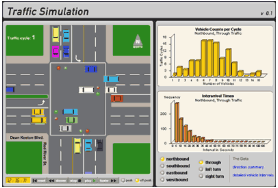
Traffic flow theory is a type of mathematical modelling that rely on large amount of empirical data and series of abstraction using mathematical equations.



Traffic Flow Modeling

Some applications of Traffic Flow Theory...

1. Development of traffic simulation software.
2. Operational analysis to optimized traffic flow.
3. Safety and emissions modelling.
4. Traffic assignment
5. Data analysis $\Rightarrow$ jam warning systems and dynamic navigation
6. Generating surrounding traffic for driving simulators
7. Autonomous driving



Brief History of Traffic Flow Theory

Traffic flow theory and modeling started in the 1930s, pioneered by the US-American Bruce D. Greenshields. However, since the 1990s, the field has gained considerable attraction.

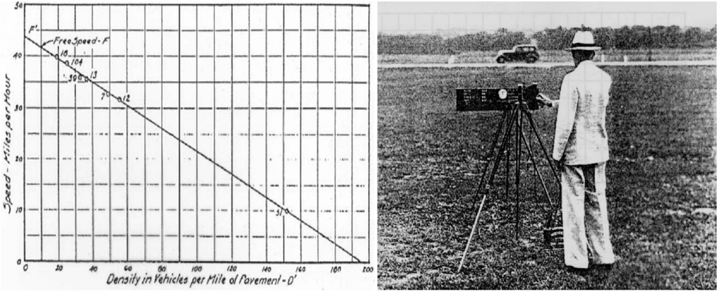
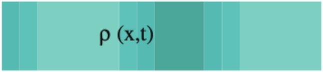


Figure 3: Traffic theory in the 1930s: Historical speed-density diagram and the experiment carried out by Bruce D. Greenshields.

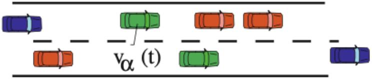
Model Categorization

Macroscopic Model



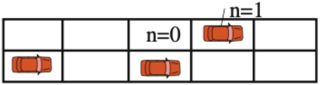
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V_e(\rho)) = 0$$

Microscopic Model



$$\frac{\mathrm{d}v_\alpha}{\mathrm{d}t} = a_\alpha(s_\alpha, v_\alpha, \Delta v_\alpha)$$

Cellular Automaton (CA)



$$n_j(t+1) = F(\{n_k(t)\})$$

Fundamental Concepts

- **Flow**: Rate at which vehicles pass a fixed point, denoted q. [veh/hr]
- **Volume**: Another term for flow
- **Density**: Spatial concentration of vehicles at one point in time, denoted k. [veh/mi]
- **Speed**: Velocity of a single vehicle, denoted u. [mi/hr]
- **Time Headway**: Time between vehicle arrivals at a single point. [sec]
- **Space Headway**: Physical distance between vehicles (front bumper to front bumper). [ft]
- **Cumulative Count**: Total number of vehicles which have passed a given point since a given reference time, denoted N. [veh]

The flow q is the rate trajectories cross a horizontal line. The density k is the rate trajectories cross a vertical line.
What about time and space headway?

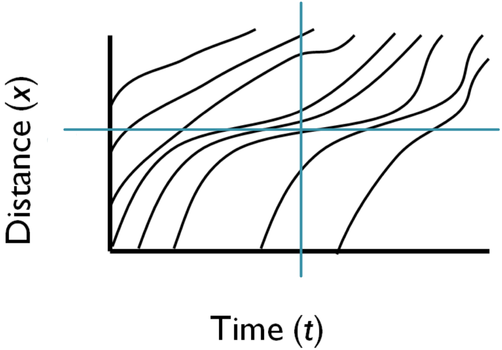


Figure 4: Trajectory diagram, also known as time-space diagram

Relating speed, flow and density

I sit at the side of the road for one hour, while cars drive by at 70 mi/hr. If the density is 10 veh/mi, how many vehicles pass by?

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700 vehicles.

This is a demonstration of the **fundamental relationship** between speed, flow, and density:

$$q = uk$$

If $q = 0$, either $k = 0$ or $u = 0$.
If $u = 0$, then the density is at its maximum value k_j, the jam density.
So, $q = 0$ if $k = 0$ or $k = k_j$.
The major assumption in the LWR model is that q is a function of k alone.(k determines u, so $Q(k) = U(k) \cdot k$.)
Therefore, $Q(k)$ must be a concave function with zeros at $k = 0$ and $k = k_j$.

Fundamental diagram

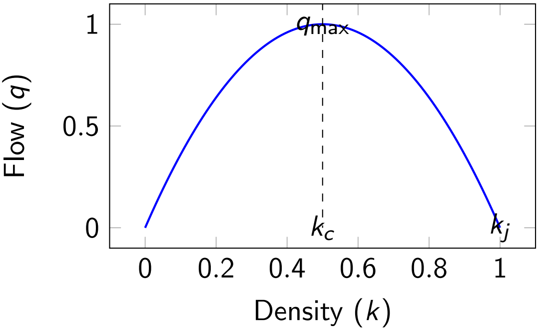


Figure 5: Fundamental Diagram based on Greenshield's relation (fig 3)

Greenshield's relation (1935): $u(k) = u_0 \left(1 - \frac{k}{k_j}\right)$

The fundamental diagram can be calibrated to data, resulting in different traffic flow models.

Fundamental diagram

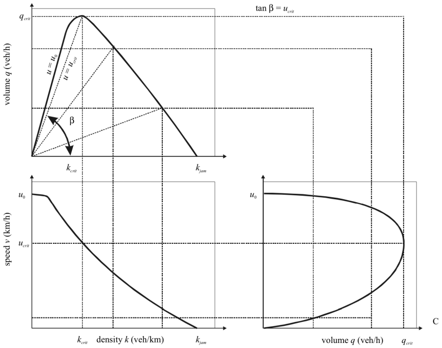
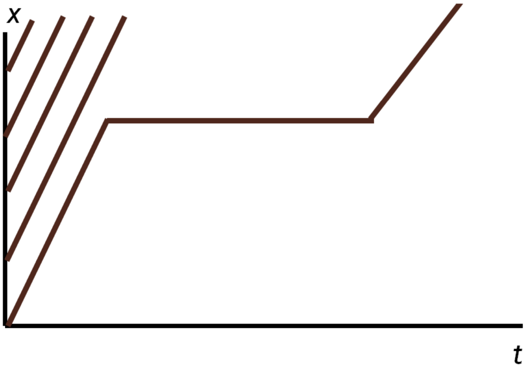
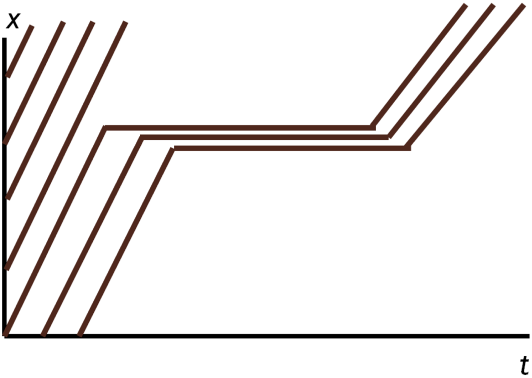


Figure 6: Three interrelated forms of the fundamental diagram

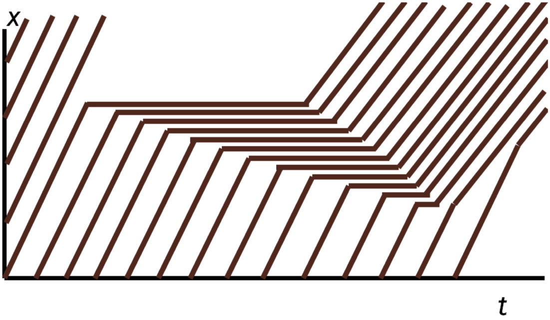
Shockwave



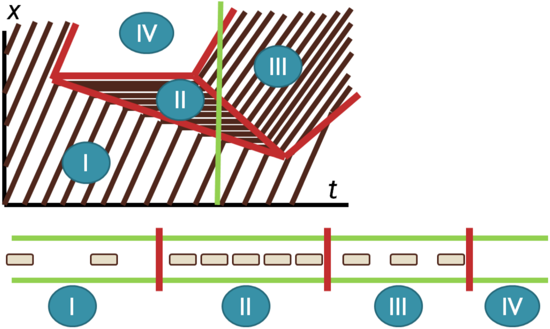
Shockwave



Shockwave



Shockwave



Deriving Lighthill-Whitham-Richards Model

In general, the values of q, k, and u can vary with x and t, subject to the fundamental relationship and fundamental diagram. Denote these as $q(x,t)$, $k(x,t)$, and $u(x,t)$.

We can also define cumulative counts N as a function of x and t. While actual vehicle trajectories are discrete, we can "smooth" them so $N(x,t)$ is continuous.

Flow and density are related to the cumulative counts $N(x,t)$ as follows:

$$q(x,t) = \frac{\partial N(x,t)}{\partial t}, \tag{0.1}$$

$$k(x,t) = -\frac{\partial N(x,t)}{\partial x}. \tag{0.2}$$

Deriving Lighthill-Whitham-Richards Model

If N is twice continuously differentiable, we have the mixed partial derivatives equality:

$$\frac{\partial^2 N}{\partial x \partial t} = \frac{\partial^2 N}{\partial t \partial x}. \tag{0.3}$$

By taking the partial derivative of $q(x, t)$ with respect to x and taking the partial derivative of $k(x, t)$ with respect to t, we can combine those results to get:

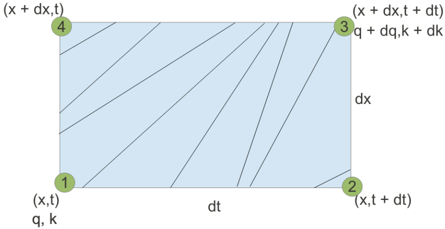
$$\frac{\partial q}{\partial x} + \frac{\partial k}{\partial t} = \frac{\partial^2 N}{\partial x \partial t} - \frac{\partial^2 N}{\partial t \partial x} = 0. \tag{0.4}$$

Therefore,

$$\frac{\partial q}{\partial x} + \frac{\partial k}{\partial t} = 0. \tag{0.5}$$

This is one way to derive the conservation equation. This equation holds everywhere except at shockwaves.

Another derivation of the conservation law



- $N(x, t) = N_0$
- $N(x, t + dt) = N_0 + q\, dt$
- $N(x + dx, t + dt) = N_0 + q\, dt - (k + dk)\, dx$
- $N(x + dx, t) = N_0 + q\, dt - (k + dk)\, dx - (q + dq)\, dt$
- $N(x, t) = N_0 + q\, dt - (k + dk)\, dx - (q + dq)\, dt + k\, dx$

Therefore,

$$\frac{\partial q}{\partial x} + \frac{\partial k}{\partial t} = 0.$$

Solving LWR problem

In the language of differential equations, the "solution" to the LWR problem is to find the functions $k(x,t)$ and $q(x,t)$ such that:

1. Conservation is satisfied: $\frac{\partial q}{\partial x} + \frac{\partial k}{\partial t} = 0$,
2. The fundamental diagram is satisfied: $q(x,t) = Q(k(x,t))$,
3. Any boundary conditions (typically value of $N(x,t)$ are satisfied.

Luckily, these partial differential equations (PDEs) can usually be solved without too much difficulty.

Characteristics

The fundamental diagram implies q is a function of k. Therefore, the conservation law

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0$$

can be rewritten as

$$\frac{\partial k}{\partial t} + \frac{dq}{dk}\frac{\partial k}{\partial x} = 0,$$

which is a PDE in $k(x, t)$ alone.
We solve this PDE by looking for characteristics, straight lines along which $k(x, t)$ is constant:$dx = \frac{dq}{dk}dt$
Moving in this direction,

$$dk = -\frac{dq}{dk}\frac{\partial k}{\partial x}dt + \frac{\partial k}{\partial x}\frac{dq}{dk}dt = 0,$$

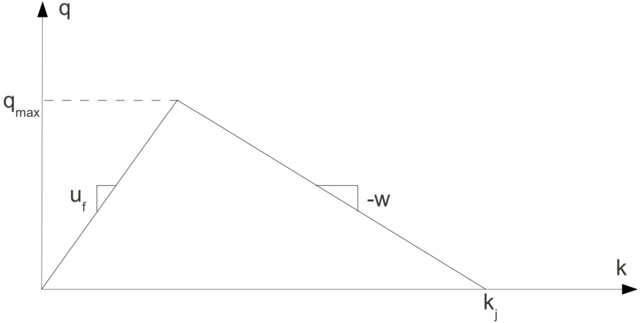
so $k(x, t)$ is constant along lines with slope $\frac{dq}{dk}$, known as the wave speed.

Characteristics

Therefore, knowing $k(x,t)$ at any point determines $k(x,t)$ along lines with slope $\frac{dq(x,t)}{dk}$, except where there are shockwaves. In uncongested parts ($\frac{dq}{dk} > 0$), uncongested states propagate downstream. In congested parts ($\frac{dq}{dk} < 0$), congested states propagate upstream.

In other words, where there is no congestion, upstream conditions prevails and vice verse.

Newell's method



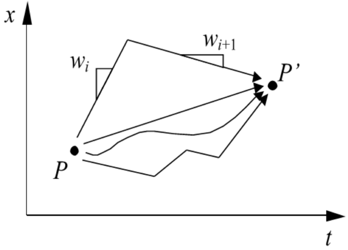
Newell's Method:Outline

- We want to calculate $k(x, t)$ or $N(x, t)$ at some point (x, t).
- Either this point is congested or uncongested.
  - If congested, the wave speed is $-w$, so past conditions downstream will determine $k(x, t)$ and $N(x, t)$ here.
  - If uncongested, the wave speed is U_f, so past conditions downstream will determine $k(x, t)$ and $N(x, t)$ here.
- Of these two possibilities, the correct solution is the one corresponding to the lowest $N(x, t)$ value.
  - If upstream conditions prevail, the $N(x, t)$ value based on the uncongested wave speed will be lower.
  - If downstream conditions prevail, the $N(x, t)$ value based on the congested wave speed will be lower.
- The major tool in this methods:

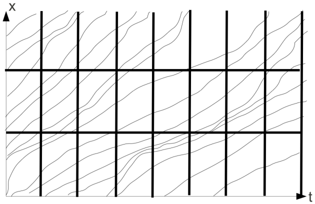
$$N(x_2, t_2) - N(x_1, t_1) = \int_C (q\, dt - k\, dx)$$

Other methods to solve LWR

In 2005, Daganzo proposed a variational approach which generalizes Newell's method. The main idea in Daganzo's method is to expand the set of paths considered, we will now consider any "valid path".



The cell transmission model (CTM) is a discrete approximation to the LWR model. A roadway link is divided into "cells," and we track the number of vehicles in each cell at discrete points in time.



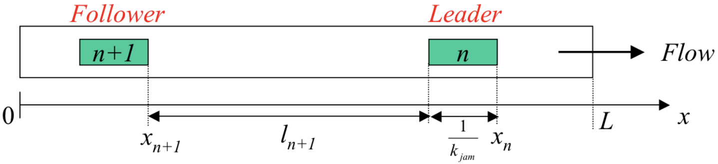
3	3	3	2	2	1	2	3
3	4	4	4	2	2	4	5
4	4	4	4	5	4	3	1

Towards microscopic models

Any car-following model must make assumptions about how users will behave. As a starting point, let's assume that drivers want to simultaneously:

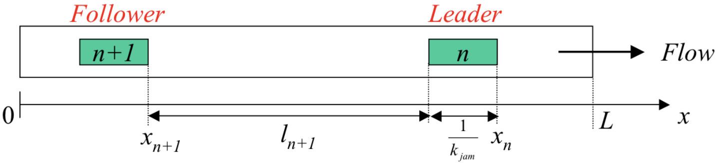
1. Keep up with the vehicle in front of them
2. Avoid collisions

Car-following model



- Space headway: $x_n(t) - x_{n+1}(t) = l_{n+1} + \frac{1}{k_j}$
- Speed: $\frac{dx_n(t)}{dt} = \dot{x}_n(t)$
- Acceleration: $\ddot{x}_n(t)$
- $\dot{x}_n(t) - \dot{x}_{n+1}(t) = \dot{l}_{n+1}(t)$
- car following regime: $\dot{l}_{n+1}(t)$ is below a certain threshold.

Simple car-following model



A simple car-following model can be expressed as:

$$\ddot{x}_n(t+T) = \dot{x}_n(t) - \dot{x}_{n+1}(t) = a\dot{l}_{n+1}(t) \tag{0.6}$$

where a is the sensitivity factor ($a \approx 0.37s^-1$), and T is the reaction time ($T \approx 1.5s$).

Does it have a relationship with macroscopic models?

Is it realistic?

From Microscopic Models To Macroscopic Models

Simple car-following model: $\ddot{x}_n(t) = \dot{x}_n(t) - \dot{x}_{n+1}(t)(T + 0)$
Fundamental diagram: $q = q_{max}(1 - \frac{k}{k_j})$

Proof of "equivalence".

Integrating this equation over time t, we get:

$$\dot{x}_n(t) - \dot{x}_n(0) = \int_0^t \ddot{x}_n(t)\,dt = \int_0^t a\dot{l}_{n+1}(t).$$

By definition, $\dot{x}_n(t) = u_n(t)$ and $\dot{x}_n(0) = u_n(0)$. If, $l_{n+1}(t) = u_{n+1}(t) = 0$ then $u = a(\frac{1}{k} - \frac{1}{k_j})$. Therefore, we conclude that the simple car-following model and the fundamental diagram are equivalent.

■

Non-linear Car linear Car-following Models

$$\ddot{x}_n(t+T) = a_0 \frac{\dot{x}_n(t) - \dot{x}_{n+1}(t)}{(x_n(t) - x_{n+1}(t))^{1.5}}$$

If $T = 0$, the corresponding fundamental diagram is:

$$q = u_{max} k (1 - (\frac{k}{k_j})^{0.5})$$

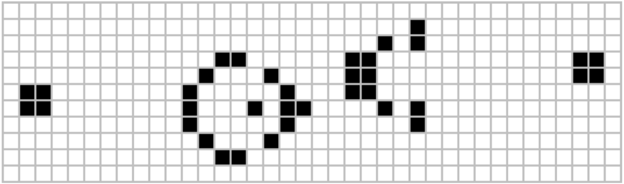
Flow Models Derived from Car Flow Models Derived from Car-Following Models

$$\ddot{x}_{n+1}(t+T) = a_0 \dot{x}_{n+1}^m (t+T) \frac{\dot{x}_n(t) - \dot{x}_{n+1}(t)}{\left(x_n(t) - x_{n+1}(t)\right)^l}$$

l	m	Flow vs. Density
0	0	$q = q_m \left(1 - \frac{k}{k_{jam}}\right)$
1	0	$q = u_c k \ln\left(\frac{k_{jam}}{k}\right)$
1.5	0	$q = u_{\max} k \left[1 - \left(\frac{k}{k_{jam}}\right)^{0.5}\right]$
2	0	$q = u_{\max} \left(1 - \frac{k}{k_{jam}}\right)$
2	1	$q = u_{\max} k \exp\left(1 - \frac{k}{k_{jam}}\right)$
3	1	$q = u_{\max} k \exp\left[-\frac{1}{2}\left(\frac{k}{k_{jam}}\right)^2\right]$

Cellular Automatan

Cellular automata were developed by John von Neumann and Stanislaw Ulam in the 1940s, and have been applied to simulate computer processors, seashell patterns, neurons, fluid dynamics, and are the traffic model used in TRANSIMS. Kai Nagel pioneered the application to traffic flow.



Celluar automata are defined on a discrete grid of cells, at a discrete set of times.

- Each cell exists in one of a finite number of states
- Moving from one timestep to the next, the state of each cell is updated based on the state of nearby cells.

Cellular Automaton in Traffic Modelling

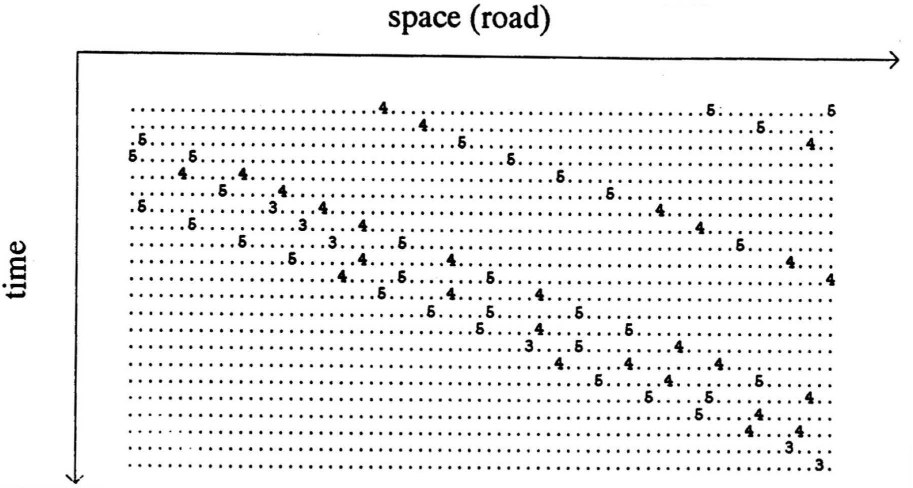
Consider a one-lane roadway, which is represented with a one-dimensional line of cells, every cell at maximum one vehicles. The state of a cell is either "empty" (if there is no vehicle present), or a non-negative integer v expressing the vehicle's speed.

The system is governed by the following rules, all four of which are applied to each vehicle in the stated order:

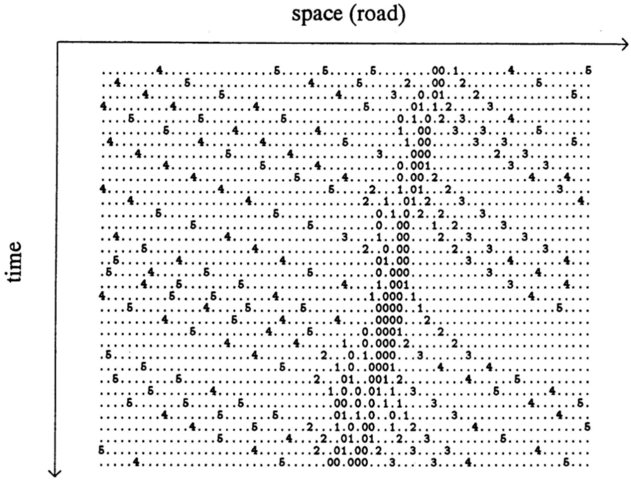
- Acceleration: If the velocity v is less than V_{max}, and the distance to the next car ahead is greater than $v + 1$, the speed increases by 1.
- Car-following: If the distance to the next vehicle is j and $j \leq v$, the speed decreases to $j - 1$.
- Stochastic: If the velocity is positive, it decreases by 1 with probability p
- Motion: The vehicle advances v cells.

These steps are perfromed in parallel for each vehicle.

Example of Cellular Automata



Example of Cellular Automata



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